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Elementary Linear Algebra with Applications

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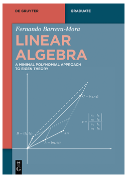
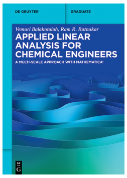

Fernando Barrera-Mora
LINEAR ALGEBRA
A MINIMAL POLYNOMIAL APPROACH TO EIGEN THEORY

Linear Algebra
A Minimal Polynomial Approach to Eigen Theory
Fernando Barrera-Mora, 2023
ISBN 978-3-11-113589-2, e-ISBN (PDF) 978-3-11-113591-5,
e-ISBN (EPUB) 978-3-11-113614-1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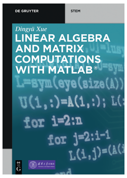
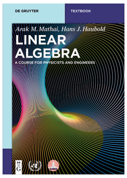
A MULTI-SCALE APPROACH WITH MATHEMATICA

Applied Linear Analysis for Chemical Engineers
A Multi-scale Approach with Mathematica®
Vemuri Balakotaiah, Ram R. Ratnakar, 2023
ISBN 978-3-11-073969-5, e-ISBN (PDF) 978-3-11-073970-1,
e-ISBN (EPUB) 978-3-11-073978-7


Dingyü Xue
LINEAR ALGEBRA AND MATRIX COMPUTATIONS WITH MATLAB

Linear Algebra and Matrix Computations with MATLAB®
Dingyü Xue, 2020, in Cooperation with Tsinghua University Press
ISBN 978-3-11-066363-1; e-ISBN 978-3-11-066699-1,
e-ISBN (EPUB) 978-3-11-066371-6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LINEAR ALGEBRA
A COURSE FOR PHYSICISTS AND ENGINEERS

Linear Algebra
A Course for Physicists and Engineers
Arak M. Mathai, Hans J. Haubold, 2017
ISBN: 978-3-11-056235-4; e-ISBN 978-3-11-056250-7,
e-ISBN (EPUB) 978-3-11-056259-0

Journal für die reine und angewandte Mathematik

Journal für die reine und angewandte Mathematik
Tobias Colding, Ana Caraiani, Jun-Muk Hwang, Olivier Schiffmann,
Jakob Stix, Gábor Székelyhidi (Eds.), since 1826
Print ISSN: 0075-4102; Online ISSN: 1435-5345

George Nakos

Elementary Linear Algebra with Applications

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Mathematics Subject Classification 2020
15A03, 15A04, 15A06, 15A09, 15A15, 15A18

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ISBN 978-3-11-133179-9
e-ISBN (PDF) 978-3-11-133185-0
e-ISBN (EPUB) 978-3-11-133195-9

Library of Congress Control Number: 2024930010

Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available on the Internet at http://dnb.dnb.de.

© 2024 Walter de Gruyter GmbH, Berlin/Boston
Cover image: Sphere and Ellipsoid/George Nakos
Typesetting: VTeX UAB, Lithuania
Printing and binding: CPI books GmbH, Leck

www.degruyter.com

To: Debra, Constantine, David,
my mother, and the memory of my father.

Preface

> Mathematics is the music of reason.
>
> J. J. Sylvester, English mathematician (1814–1897).

Mathematics

Since ancient times mathematics has served both as a tool of science and as an intellectual endeavor. According to Aristotle, *the mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful* (Metaphysica).

Mathematicians have always contributed to the advancement of science. In 1994, the Nobel Prize in Economic Science was awarded to a mathematician, *John Nash*, for his work on game theory. Science has, in turn, contributed to the advancement of mathematics. For example, modern physics theories have paved the way to new interesting mathematics. Mathematical physicist *Sir Roger Penrose* shared the 2020 Nobel Prize in Physics (Figure 1).

Figure 1: Sir Roger Penrose, 2020 Nobel Prize in Physics.
Image by Cirone-Musi, Festival della Scienza, CC BY-SA 2.0,
https://en.wikipedia.org/wiki/Roger_Penrose.
Roger Penrose is a British mathematician, physicist, and philosopher. He the Emeritus Rouse Ball Professor of Mathematics in the University of Oxford. He was born in 1931, in Colchester, Essex, UK. He is known for his work on singularities, black holes, and Penrose tiles. He was awarded the Nobel Prize in Physics in 2020 for the discovery that black hole formation is predicted by the general theory of relativity.

Linear algebra

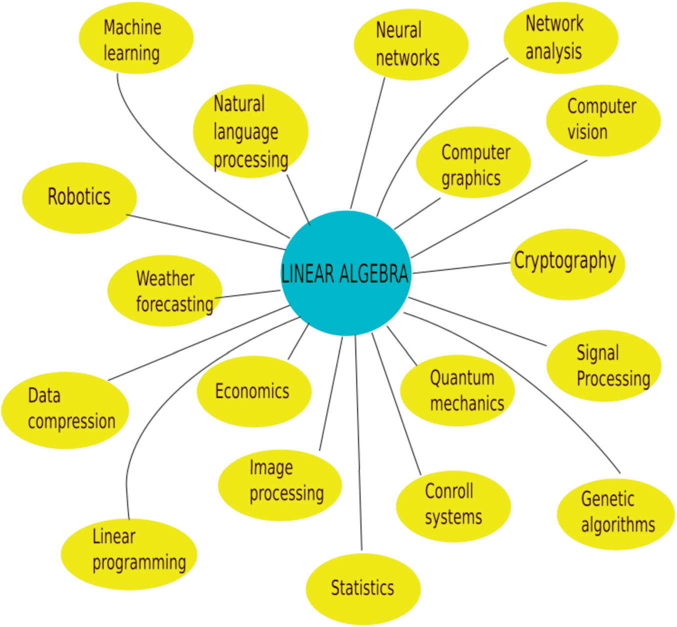
Linear algebra is the most useful mathematics course for the student of science, engineering, or mathematics [1].

The material is foundational for higher courses in many areas that use mathematics as a tool. In addition to the many applications, the student is introduced to rigorous mathematical reasoning. This adds confidence to the validity of the results that will be used in applications and also in further development of mathematical theory.

Linear algebra has a history of thousands of years, yet it is in the frontier of modern mathematics and its applications. The student will see a third century BCE linear system from an ancient Chinese text alongside modern applications to computer graphics, to the NFL ratings of quarterbacks, to image recognition, to weather modeling, and many more.

https://doi.org/10.1515/9783111331850-201

Linear algebra covers a broad spectrum of real-life applications. Some areas of science that rely on linear algebra are illustrated in Figure 2.


Machine learning
Neural networks
Network analysis
Natural language processing
Computer vision
Computer graphics
Robotics
Cryptography
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Weather forecasting
Signal Processing
Data compression
Economics
Quantum mechanics
Image processing
Conroll systems
Genetic algorithms
Linear programming
Statistics

Figure 2: Some applications of linear algebra.

Main goals

This text includes the basic mathematical theory, a variety of applications, numerical methods, and projects. It is designed for maximum flexibility. A good understanding of the mathematical theory is our primary goal. Just as important is the study of some of the many applications that use linear algebra.

Level and style

The book was written primarily for a one-semester course at the sophomore or junior level for students of mathematics or science. However, there is enough material for a complete two-semester course, if desired.

The material largely agrees with the recommendations in the *Summary Report of the Linear Algebra Study Group* of the NSF-sponsored workshop held in August 1990 at the College of William and Mary.

The style of the writing is simple and direct. Lengthy discussions are avoided. The course takes the reader from the particular to the general in small steps. Carefully chosen examples build understanding. The basic concepts are repeated in a *spiral learning* approach.

Linear algebra can be mastered by a careful step-by-step study, the same way as one acquires a complex skill, such as learning how to play an instrument.

Examples and exercises

There is a great number of examples and exercises. The examples are carefully chosen and presented in sufficient detail to cover the material thoroughly. They can be used in *class*, in *self-study*, or in *group-study*. The instructor can choose which examples to discuss depending on what needs to be emphasized. There is a variety of exercises: some are routine, many are computational, some extend the theory, many are on applications, and a few are on lesser known topics. Each exercise section has some *true* or *false* exercises. These types of exercises test the student's understanding and often highlight some of the "common errors". Some of the exercises relate linear algebra to other areas of mathematics.

Variety of applications

There is a wide variety of applications interlaced throughout the text. Chapters end with one or more applications sections. The applications can be used to any extent that an instructor or a self-taught student wishes. They can also be used for student group study.

There are applications to physics, engineering, mechanics, chemistry, economics, business, sociology, psychology, and the Google search engine. There are also applications to mathematics in the areas of graph theory, analytic geometry, fractal geometry, coding theory, wavelet theory, dynamical systems, and solutions of polynomial systems.

Some of the applications are new in book form. These include *Tesselations in Weather Models* and the *Object-image Equations* in image recognition.

Emphasis on geometry

There has been a substantial effort to emphasize the geometric understanding of the material. Traditionally, linear algebra books lack geometric insight. With hundreds of figures, an effort is made to illuminate the basic concepts both geometrically and algebraically. As an example of this approach, there is an early introduction to dot products, orthogonality, lines, and planes.

Emphasis on orthogonality

One of the most applicable parts of linear algebra comes from inner products and the concept of orthogonal vectors and functions. We emphasize orthogonality, the method of *Least squares and curve fitting*, the *QR* and *SVD* factorizations, and we introduce the *Fourier series*. As an important modern application, we discuss some basic *Wavelet theory*.

Numerical methods

Sections on numerical methods show what can go wrong when we deal with real-life problems. Many of these problems involve large-scale calculations that require reliability and precision. The numerical sections are independent of the basic material. Sample methods include *iterative solutions*, *numerical computations of eigenvalues*, and *least squares solutions*.

New emphasis on determinants

In recent years, texts in linear algebra tend to minimize the role of determinants. The perception was that determinants have mostly theoretical use and that they are overshadowed by Gaussian elimination in numerical calculations. This trend is changing. Determinants are becoming again increasingly important. In form of resultants, they are used to solve polynomial systems. They are also used in computer graphics: transforming 2D/3D objects, in quantum mechanics: describing the wave function of a particle, in economics: calculating the stability of a market or the stability of a financial portfolio, in engineering: calculating stress and strain in structural analysis, etc.

Complex numbers

Complex numbers are important in mathematics, physics, engineering, and other areas. Some of the examples and exercises in this book help the student to get acquainted with complex number arithmetic, matrices with complex entries, and linear systems with complex coefficients. We also discuss inner products with complex numbers. These are used in physics, especially in quantum mechanics.

Miniprojects and special topics

Each chapter has a miniprojects section. The projects are fairly short and relatively simple. Some are like lengthy exercises, some extend the basic theory, and some emphasize a particular application.

There are a few special topics sections designed to introduce interesting recent problems that can be solved with the help of linear algebra.

Technology-aided problems and answers with Mathematica® MATLAB®, and Maple™

Each chapter ends with a set of problems that can be solved by the use of technology. These problem sets help with familiarization of technology and aim at exploration and discovery. Most problems are solved by using the mathematical packages **Mathematica**® (Wolfram), **MATLAB**® (MathWorks), and **Maple**™ (MapleSoft and Cybernet Systems).

Multiple choices of setting up a course

This book is structured so that there is maximum flexibility in choice of material. One may choose what to cover out of a variety of applications, theory, and numerical methods. Most sections that do not cover basic material are independent from one another. Likewise, there is a variety of exercises to choose from.

Overview of chapters

Chapter 1, Linear systems

Chapter 1 is about systems of linear equations. Section 1.1 introduces linear systems and discusses Gauss elimination, which is a major solution method of these systems. It also discusses geometrical representations of solutions. Section 1.2 goes deeper into Gauss elimination and then discusses some of the main properties of the solution sets. The section ends with the Gauss–Jordan elimination method, which is a variant of Gauss elimination.

Section 1.3 offers a variety of applications: the demand function, balancing of chemical reactions, heat conduction, traffic flows, and statics and weight balancing. Section 1.4 covers numerical methods for solving linear systems.

Chapter 2, Vectors

Chapter 2 is about vectors, matrices, and some their properties. Section 2.1 introduces matrices and vectors along with addition, scalar multiplication, and linear combinations. Section 2.2 discusses matrix transformations, their linearity, and their relation to linear systems. Section 2.3 is about the span of vectors, and Section 2.4 is on the linear independence of vectors. Section 2.5 is on the dot product and its properties. In addition, there is also a discussion about lines and hyperplanes in the Euclidean space.

Sections 2.6–2.8 discuss a variety of applications of vectors. Such applications include plane transformations used in computer graphics, averaging, discrete dynamical systems, population models, and the topic of tessellations in weather models.

Chapter 3, Matrices

This chapter is on matrices. The basic material is covered in Sections 3.1–3.3. Section 3.1 is on matrix multiplication. Section 3.2 introduces the inverse of a matrix and a method of its computation. The section ends with a brief application to the stiffness of an elastic beam. Section 3.3 is on elementary matrices, including a justification of the matrix inversion algorithm and characterizations of invertible matrices in different ways. Section 3.4 is about the very useful LU factorization of any size matrix. The last subsection discusses the case where row interchanges are necessary. Section 3.5 discusses block and sparse matrices.

Sections 3.6–3.8 discuss a variety of applications. The topics include stochastic matrices, Markov chains, the Leontief input–output models, graph theory, adjacency matrices and walks for digraphs and dominance graphs. The miniprojects section introduces the Hill cipher in cryptology.

Chapter 4, Vector spaces

This chapter is about general vector spaces. The student is well prepared by now for transitioning from $\mathbf{R}^n$ to the more general vector space.

Section 4.1 includes the definitions of a vector space and a vector subspace as well as several examples. Section 4.2 is about the span and linear independence. These concepts are already familiar. What needs to be stressed here is how to rely more on the definition of linear independence rather than on the direct use of matrix row reduction. Section 4.3 is on the fundamental concepts of basis and dimension. Several theorems connect dimension, linear independence, spanning sets, and bases together. Section 4.4 is on coordinates and change of bases. Section 4.5 covers the null space, and Section 4.6 covers the column space and the row space. It also introduces the concept of rank and nullity as well as the rank theorem. It then discusses the connection between rank and linear systems. Section 4.7 offers a deeper application of vector spaces to coding theory, in particular, to the Hamming (7,4)-code.

Chapter 5, Linear transformations

This chapter is about linear transformations between vector spaces. Linear transformations generalize matrix linear transformations.

Section 5.1 defines linear transformations and discusses several examples. Section 5.2 is about the kernel and the range of a linear transformation. The concept of isomorphism is also covered here. Section 5.3 discusses the matrix of a transformation with respect to chosen bases. Section 5.4 examines the set of linear transformations as a vector space. Section 5.5 offers a first glimpse to fractals.

Chapter 6, Determinants

Determinants are introduced in Section 6.1 by cofactor expansion. This is not the mathematical method of choice, but it is direct, and it works with the students. Section 6.2 is on the basic properties of determinants. Computing determinants by correct row reduction is a point that should be emphasized here. Section 6.3 is on the adjoint and Cramer's rule. Section 6.4 discusses how to define and compute determinants by permutations.

Section 6.5 has applications of determinants to equations of geometric objects, elimination theory, Sylvester resultant, and solutions of polynomial systems. Section 6.6 is an essay on a recent and not yet widely known application of determinants to the deduction of the *object-image equations* of six points in image recognition.

Chapter 7, Eigenvalues

This chapter is devoted to eigenvalues and eigenvectors. These are very important topics in linear algebra. Section 7.1 discusses the definitions of eigenvectors and eigenvalues and offers several examples. It also discusses eigenvalues of linear transformations. Section 7.2 is on diagonalization of matrices and linear transformations. The section ends with the computation of powers of diagonalizable matrices.

Section 7.3 is a continuation of the study on dynamical systems, examination of their trajectories and long-term behavior by using eigenvalues. Section 7.4 includes more dynamical systems plus Markov chains, probability vectors, and limits of stochastic matrices. Section 7.5 discusses as a special topic the Google PageRank Algorithm.

Section 7.6 belongs to numerical linear algebra. It discusses a variety of important methods of approximating eigenvectors and eigenvalues. In the miniprojects section a proof of the Cayley–Hamilton theorem is outlined.

Chapter 8, Orthogonality and least squares

This chapter is on dot and inner products. The concept of orthogonality dominates this chapter.

Section 8.1 is on orthogonality and orthogonal matrices. This material is important and should be learned well. Section 8.2 offers a detailed discussion on orthogonal projections and highlights the Gram–Schmidt process. Section 8.3 is on the very useful QR factorization. Section 8.4 on least squares is also important. In practice, most linear systems are inconsistent, and the method of least squares is used to best fit solutions. Section 8.5 is about inner product spaces. An inner product is a generalization of the dot product of $\mathbf{R}^n$ to a general vector space. Section 8.6 studies inner products over complex vector spaces. It also defines and studies the important unitary matrices along with Hermitian and skew-Hermitian matrices. Section 8.7 revisits least squares, this time in the context of curve fitting and fitting of continuous functions.

Sections 8.8 is a special topic on the uncovering of a formula that NFL uses to rate quarterbacks. This is based on a paper by Roger W. Johnson, and it is an illustration of the power of least squares.

Chapter 9, Quadratic forms, SVD, wavelets

This chapter is about deeper mathematical applications of orthogonality.

Section 9.1 is on how to diagonalize a symmetric matrix by using orthogonal matrices. The important Spectral Theorem is proved by using the Schur Decomposition Theorem. Section 9.2 is on quadratic forms and conic sections. This is an application of orthogonal diagonalization of symmetric matrices. Section 9.3 is on the numerically important Singular Value Decomposition (SVD) and on the pseudoinverse.

Section 9.4 is a special topic on the important application of SVD to image compression. Section 9.5 is an exposition of some of the basic ideas of Fourier series and Fourier polynomials. Section 9.6 is a simple introduction to wavelets. These last two sections illustrate one more time the importance of orthogonal sets and their properties.

Appendix A

This appendix consists of an introduction to complex numbers, their arithmetic, basic properties, and polar decomposition.

Appendix B

This appendix offers a proof of the uniqueness of the reduced row echelon form of a matrix.

Answers to selected exercises

There are numerical answers to selected odd-numbered exercises. Answers that require either proofs or graphs are not included.

Lesson plan suggestions

This text is designed for maximum flexibility and for a variety of uses. By default it is a textbook **to be used in a class**. However, it may also be used for **self-study**, for **group-study**, as well as for **reference**. Here we offer a few ideas on using this text to teach a first course in linear algebra.

While there is enough material for a two-semester course, it will typically be used for one-semester courses. These would consist of some standard material plus a flavor of choice such as emphasis on either theory, or applications, or projects, or technology.

For a one-semester 15-week course, with three contact hours per week, we assume that 36–38 lessons are devoted to new material.

We think of the following sections of 28 lessons as the basis of any variant of the course:

1.1–1.2, 2.1–2.5, 3.1–3.3, 4.1–4.6, 5.1–5.3, 6.1–6.3, 7.1–7.2, 8.1–8.4.

For the remaining 10 lessons, we outline a few ideas to assist instructors design a syllabus.

- If **theoretical emphasis** is a strong component in the course, then one may consider covering Sections 2.6, 2.7, 3.4, 3.7, 3.8, 4.7, 6.5, 6.6. Another choice would be to cover more advanced sections such as 6.4, 6.5, 7.6, 8.5, 8.6, 9.1, 9.2, 9.3. Choices like these will add up to eight additional lessons. Any mixing of these sections or a few extra ones may add to these options.
- If **numerical linear algebra** is a strong component in the course, then some basic sections stand out: 1.4, 3.4, 3.5, 3.6, 7.3, 7.4, 7.5, 9.3, etc.
- If **student collaboration and problem solving** is a strong component in the course, then one may consider covering several or all *miniproject* sections. This may add up to eight additional lessons.

- If **technology and student collaboration** is important, then one may consider covering several or all *technology-aided problems* sections. This may add up to eight additional lessons. Here, it may be a good idea to have students work in teams on a subset of the available problems. The problems in these sections are not restricted to specific software.
- If **applications of linear algebra** are important, then one may consider covering several of the many applications sections available, for example, 1.3, 2.6, 2.7, 3.6, 3.7, 4.7, 7.3, 7.4, 8.8, etc. There is certainly enough material for many other choices.
- If **historical research** is important, then one may use some of the dozens of historical remarks as starting topics for some focused historical research. For example, one may assign biographies and some explanation of the works of certain mathematicians such as Emmy Noether, Joseph Fourier, etc.

There are several other possible setups. For example, one may consider a course that is strong both in *numerical linear algebra and the use of technology*, or a course that emphasizes the *computational aspects of linear algebra*.

If a curriculum is set up so as to allow for a two-semester course, then one could design a course that covers all the basic theory, a wide variety of applications and numerics, as well as some use of technology.

For example, the first four chapters consist of 34 sections, which could be covered in one semester. This would include numerics, miniprojects, special topics, and technology. The remaining chapters consisting of 41 sections may be used for the second semester. Chapter 5 could be restricted to minimum coverage if there is not enough time to finish all chapters.

Advanced books on linear algebra

There are many interesting books to read on advanced linear algebra, once elementary linear algebra has been mastered.

Here is a list of celebrated authors and texts that have greatly benefited students and teachers alike, including the author of this text. These texts by the following authors are referenced in the bibliography:

- T. M. Apostol [2]
- D. S. Bernstein [3]
- F. R. Gantmacher [4, 5, 6]
- G. H. Golub and C. F. Van Loan [7]
- W. H. Greub [8]
- R. A. Horn and C. R. Johnson [9]
- A. Jeffrey [10]

- S. Lang [11]
- W. Nef [12]
- G. E. Shilov [13]
- G. Strang [14]

Acknowledgment

I am most grateful to my family for the love, support, inspiration, and encouragement. I dedicate this effort to my family.

I would like to thank my many students from the U.S. Naval Academy and the Johns Hopkins University. They all motivated me to try to become a better teacher. My colleagues in both institutions have been supportive of my work and I thank them for that. In particular, I would like to thank my coauthor in an earlier project, David W. Joyner, Professor Emeritus of Mathematics, U.S. Naval Academy.

I would like to thank my Johns Hopkins professors J. Michael Boardman, W. Stephen Wilson, Jean-Pierre Meyer, and Jack Morava for teaching me algebraic topology and for instilling in me the love of mathematical research and teaching.

I would like to thank De Gruyter Academic Publishing for the opportunity to publish with them, for their warm support, and for expert typesetting and copy editing.

Finally, I would like to thank the reviewers who read part of the manuscipt and offered several valuable suggestions.

https://doi.org/10.1515/9783111331850-202

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1 Linear systems

> I am so grateful to everyone who likes linear algebra and sees its importance. So many universities now appreciate how beautiful it is and how valuable it is.
>
> Gilbert Strang, American mathematician and educator (Figure 1.1).

Figure 1.1: William Gilbert Strang.
Self-published work This file is licensed under the Creative Commons Attribution-Share Alike 4.0 International license.
William Gilbert Strang (born in 1934) is an American mathematician known for his contributions to numerical linear algebra, finite element theory, the calculus of variations, and wavelets. He is a renowned mathematics educator who has authored several influential textbooks on linear algebra.

Introduction

Linear systems, vectors, and matrices are the three pillars of linear algebra.

In this chapter, we discuss linear systems. These are sets of linear equations. A linear equation is the simplest form of a mathematical equation we can imagine. The unknowns are multiplied by given numbers and added together to yield another given number. The unknowns are in the first power; there are no products or powers or fractions or any other functions in them.

Linear systems have been studied for centuries. The third century BCE Chinese text *Nine Chapters of Mathematical Art* devotes its eighth chapter (titled *Fangcheng*) to eighteen word problems that result in linear systems, which are then solved (Figure 1.2). The first problem is the following [15, 16].

Suppose that H stands for “the number of units of high-quality rice straws”, likewise, M for “mid-quality”, and L for “low-quality”. Find H, M, and L given that:

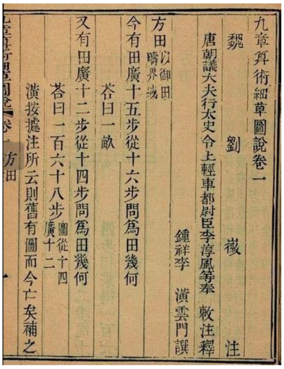
- 3 bundles of H, 2 bundles of M, and 1 bundle of L produce 39 units of rice.
- 2 bundles of H, 3 bundles of M, and 1 bundle of L produce 34 units of rice.
- 1 bundle of H, 2 bundles of M, and 3 bundles of L produce 26 units of rice.

The above information leads to the following system of linear equations:

$$\begin{aligned} 3H + 2M + L &= 39, \\ 2H + 3M + L &= 34, \\ H + 2M + 3L &= 26, \end{aligned}$$

which needs to be solved for H, M, and L.

https://doi.org/10.1515/9783111331850-001



九章筭術細草圖說卷一

魏　劉　徽　注

唐朝議大夫行太史令上輕車都尉臣李淳風等奉　敕注釋

鍾祥李　潢雲門譔

方田（以御田疇界域）

今有田廣十五步從十六步問爲田幾何

荅曰一畝

又有田廣十二步從十四步問爲田幾何

荅曰一百六十八步（圖從十四廣十二）

潢按據注所云則舊有圖而今亡矣補之

方田

Figure 1.2: A page of **The Nine Chapters on Mathematical Art** (1820 edition). (http://pmgs.kongfz.com/detail/1_158470/, Public Domain, https://commons.wikimedia.org/w/index.php?curid=22913440.) The “Nine Chapters on the Mathematical Art” is an influential ancient Chinese mathematical treatise, which is a compilation of mathematical problems and problem solving techniques that were practiced and refined over several centuries. It is believed to have been written and edited over a long period, from around the tenth century BCE to the CE second century, during China’s Han dynasty.

Another interesting old problem resulting in solving a linear system is the *Archimedes Cattle Problem*. This is discussed in Section 1.5.

Modern uses of linear systems are discussed in Section 1.3 with applications to economics (the demand function), chemistry (balancing of chemical reactions), physics and engineering (electrical networks, heat conduction, statics and weight balancing), and traffic flow problems.

1.1 Introduction to linear systems

We introduce linear systems and study a method of solution called *elimination*, or *Gauss elimination*. This section introduces the method, and Section 1.2 offers a more in-depth study of it.

1.1.1 Linear equations

The first equations encountered in school algebra are linear equations. A *linear equation* in *unknowns* or *variables* $x_1, \dots, x_n$ is an equation that can be written in the *standard form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b. \tag{1.1}$$

The a_is are the *coefficients*, and b is the *constant term*. These are real numbers and, on occasion, complex numbers. The first variable with nonzero coefficient of a linear equation is called the *leading variable*. The remaining variables are called *free variables*. An equation that is not linear is called *nonlinear*.

Example 1.1.1.

1. The equation

$$x_1 + x_2 + 4x_3 - 6x_4 - 1 = x_1 - x_2 + 2$$

is linear, because it can be written in the form (1.1) as

$$0x_1 + 2x_2 + 4x_3 - 6x_4 = 3.$$

The leading variable is x_2. The free variables are x_1, x_3, and x_4.

2. The following equations are linear:

$$x_1 + 2x_2 - \sqrt{5}x_3 - x_4 = 0, \quad x - 4y + 9z = \tan(4), \quad F = \frac{9}{5}C + 32.$$

3. The following equations are nonlinear due to x_1^2, $\frac{1}{x_2}$, and $\sin(x_1)$:

$$x_1^2 - x_2 = 7, \quad \frac{x_1}{x_2} - 3x_3 = 2, \quad \sin(x_1) + x_2 = 0.$$

1.1.2 Definition of linear system

A linear system is a set of linear equations. The following systems are linear:

$$\begin{aligned} x_1 + x_2 &= 5, & 3x + 2y + z &= 39, & y_1 + y_2 + y_3 &= -2, \\ x_1 - 2x_2 &= 6, & 2x + 3y + z &= 34, & y_1 - 2y_2 + 7y_3 &= 6. \\ -3x_1 + x_2 &= 1, & x + 2y + 3z &= 26, & & \end{aligned}$$

Definition 1.1.2. A *linear system* of m equations in n *unknowns*, or *variables*, $x_1, \dots, x_n$, is a set of m linear equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{1.2}$$

The a_{ij}s are the *coefficients*, and the b_is are the *constant terms*. If all constant terms are zero, then the system is called *homogeneous*. The homogeneous system that has the same coefficients as system (1.2) is said to be *associated* with (1.2). If $m = n$, then the system is called a *square system*.

Example 1.1.3. The system

$$\begin{aligned} x_1 + 2x_2 \phantom{{}- 2x_3} &= -3, \\ 2x_1 + 3x_2 - 2x_3 &= -10, \\ -x_1 \phantom{{}+ 3x_2} + 6x_3 &= 9 \end{aligned} \tag{1.3}$$

is linear square with coefficients 1, 2, 0, 2, 3, −2, −1, 0, 6, constant terms −3, −10, 9, and associated homogeneous system

$$\begin{aligned} x_1 + 2x_2 \phantom{{}-2x_3} &= 0, \\ 2x_1 + 3x_2 - 2x_3 &= 0, \\ -x_1 \phantom{{}+3x_2} + 6x_3 &= 0. \end{aligned}$$

A rectangular arrangement of elements from a set is called a *matrix* over that set. These elements are the *entries* of the matrix. A matrix has *rows*, numbered top to bottom, and *columns*, numbered left to right. The entry at the intersection of the ith row and jth column is the *(i,j) entry*. A matrix with m rows and n columns has *size $m \times n$* (pronounced “m by n”). A matrix with only one column is also called a *vector*. Usually, the matrices in this text have entries that are real numbers and, sometimes, complex numbers.

The matrix whose rows consist of the coefficients and constant terms of each equation of a linear system is called the *augmented matrix* of the system. The augmented matrix of System (1.3) is

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & 3 & -2 & -10 \\ -1 & 0 & 6 & 9 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 2 & 0 & : & -3 \\ 2 & 3 & -2 & : & -10 \\ -1 & 0 & 6 & : & 9 \end{bmatrix}.$$

The second notation indicates that the last column consists of constant terms. The matrix with entries the coefficients of the system is the *coefficient matrix*. The vector of all constant terms is the *vector of constants*. The coefficient matrix and the vector of constants of System (1.3) are

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -2 \\ -1 & 0 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ -10 \\ 9 \end{bmatrix}.$$

A linear system with coefficient matrix A and vector of constants $\mathbf{b}$ is abbreviated by

$$[A : \mathbf{b}]. \tag{1.4}$$

1.1.3 Solution of linear system

Definition 1.1.4. A sequence $r_1, \dots, r_n$ of scalars is a *(particular) solution* of System (1.2), if all equations are satisfied when we substitute $x_1 = r_1, \dots, x_n = r_n$. The set of all possible solutions is the *solution set*. Any generic element of the solution set is called the *general solution*. If a system has solutions, then it is called *consistent*; otherwise, it is called *incon-*

sistent. Two linear systems with the same solution sets are called *equivalent*. A solution that consists of zeros only is called the *trivial solution*.

A homogeneous system has always the trivial solution as one of its solutions. Thus, a homogeneous system is consistent.

Example 1.1.5. The substitutions $x_1 = -15$, $x_2 = 6$, and $x_3 = -1$ satisfy all equations of System (1.3), making $-15, 6, -1$ a particular solution of that system.

1.1.4 Geometry of solutions in two variables

For numbers a, b, c, all planar points (x_1, x_2) that satisfy the equation

$$ax_1 + bx_2 = c$$

are the points of a straight line, provided that not both a and b are zero. Hence, solution sets of linear systems in two variables are, in general, intersections of several straight lines. These intersections can be one point, a line, or the empty set, as illustrated in Figure 1.3 for systems of two linear equations.

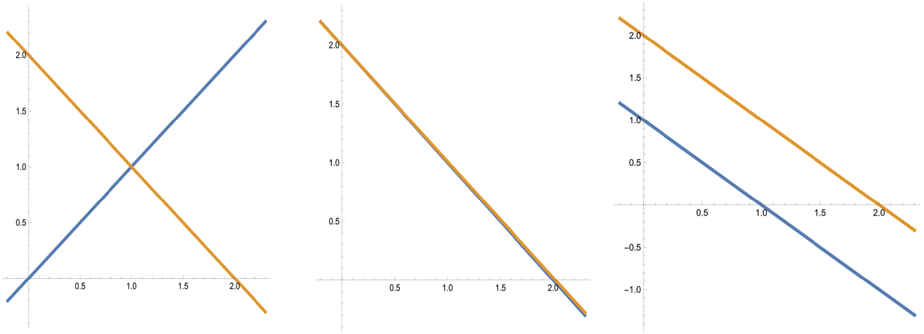


Figure 1.3: (a) Exactly one solution. (b) Infinitely many solutions. (c) No solutions.

1.1.5 Back-substitution

The easiest systems to solve are those in triangular form. A system is in *echelon form* or in *triangular form*, if the leading variable in each equation occurs to the right of the leading variable of the equation above it.

To solve such systems we first solve for the leading variable of the last equation, then substitute the value found into the equation above it, and repeat. This method is called *back-substitution*.

Example 1.1.6. Solve the echelon form linear system by back-substitution.

$$\begin{aligned} x_1 + 5x_2 + x_3 &= -4, \\ -2x_2 + 4x_3 &= 14, \\ 3x_3 &= 9. \end{aligned}$$

Solution. Going from the bottom up, the last equation yields $x_3 = 3$, the second $x_2 = -1$, and the first $x_1 = -2$. Hence, the only solution is $x_1 = -2$, $x_2 = -1$, $x_3 = 3$.

In Example 1.1.6, all variables were leading variables. This need not be always the case. The next example has free variables. In such cases the leading variables are computed in terms of the free variables. The free variables then can take on any values. These values are called *parameters*.

Example 1.1.7. Solve the system

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 + 2x_5 - x_6 &= 1, \\ -x_3 \phantom{{}-x_4} + x_5 \phantom{{}-x_6} &= 1, \\ -x_5 + x_6 &= 3. \end{aligned} \tag{1.5}$$

Solution. We solve for the leading variables x_5, x_3, x_1 in terms of the free variables x_6, x_4, x_2, which can take on any values, say $x_6 = r$, $x_4 = s$, $x_2 = t$. By back-substitution we get the general solution

$$\begin{aligned} x_1 &= -2r + s + t + 11, \\ x_2 &= t, \\ x_3 &= r - 4, \\ x_4 &= s, \\ x_5 &= r - 3, \\ x_6 &= r, \end{aligned} \qquad \text{for all } r, s, t \in \mathbf{R} \tag{1.6}$$

The parameters are r, s, and t. The solution set is a *three-parameter* infinite set. To obtain particular solutions, we let the parameters take on specific values.

1.1.6 Introduction to Gauss elimination

The basic idea for solving general linear systems is to *eliminate* unknowns in a way that will result in an equivalent system in echelon form and then use back-substitution. This is done by performing appropriate combinations of *elementary equation operations*. These are: (a) adding to an equation a multiple of another, (b) multiplying an equation by a nonzero scalar, and (c) switching two equations. It is more economical to perform these operations to the augmented matrix of the system.

Definition 1.1.8. The *elementary row operations* of any matrix are

Elimination: *Add a constant multiple of one row to another.* $\boxed{R_i + cR_j \to R_i}$

Scaling: *Multiply a row by a nonzero constant.* $\boxed{cR_i \to R_i}$

Interchange: *Interchange two rows.* $\boxed{R_i \leftrightarrow R_j}$

Example 1.1.9. Solve the system by elimination:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & & & = & -3, \\ 2x_1 & + & 3x_2 & - & 2x_3 & = & -10, \\ -x_1 & & & + & 6x_3 & = & 9. \end{array}$$

Solution. We have

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & & & = & -3, \\ 2x_1 & + & 3x_2 & - & 2x_3 & = & -10, \\ -x_1 & & & + & 6x_3 & = & 9, \end{array} \quad \text{or} \quad \left[\begin{array}{rrrcr} 1 & 2 & 0 & : & -3 \\ 2 & 3 & -2 & : & -10 \\ -1 & 0 & 6 & : & 9 \end{array}\right].$$

Multiplying the first equation by −2 and adding to the second equation will eliminate x_1 from the second equation ($R_2 - 2R_1 \to R_2$). Adding the first equation to the third one will also eliminate x_1 from the third equation ($R_3 + R_1 \to R_3$):

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & & & = & -3, \\ & & -x_2 & - & 2x_3 & = & -4, \\ & & 2x_2 & + & 6x_3 & = & 6, \end{array} \quad \text{or} \quad \left[\begin{array}{rrrcr} 1 & 2 & 0 & : & -3 \\ 0 & -1 & -2 & : & -4 \\ 0 & 2 & 6 & : & 6 \end{array}\right].$$

To eliminate x_2 from the third equation, we perform $R_3 + 2R_2 \to R_3$:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & & & = & -3, \\ & & -x_2 & - & 2x_3 & = & -4, \\ & & & & 2x_3 & = & -2, \end{array} \quad \text{or} \quad \left[\begin{array}{rrrcr} 1 & 2 & 0 & : & -3 \\ 0 & -1 & -2 & : & -4 \\ 0 & 0 & 2 & : & -2 \end{array}\right].$$

The system is now in echelon form. Starting at the bottom, we work upwards to eliminate unknowns *above* the leading variable of each equation (back-substitution). To eliminate x_3 from the second equation, we perform $R_2 + R_3 \to R_2$:

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & -3, \\ & & -x_2 & = & -6, \\ & & 2x_3 & = & -2, \end{array} \quad \text{or} \quad \left[\begin{array}{rrrcr} 1 & 2 & 0 & : & -3 \\ 0 & -1 & 0 & : & -6 \\ 0 & 0 & 2 & : & -2 \end{array}\right].$$

To eliminate x_2 from the first equation, we perform $R_1 + 2R_2 \to R_1$:

$$\begin{array}{rcl} x_1 \qquad\qquad & = & -15, \\ -x_2 \qquad & = & -6, \\ 2x_3 & = & -2, \end{array} \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 & 0 & : & -15 \\ 0 & -1 & 0 & : & -6 \\ 0 & 0 & 2 & : & -2 \end{bmatrix}.$$

Finally, we scale $(1/2)R_3 \to R_3$ and $(-1)R_2 \to R_2$ to get

$$\begin{array}{rcl} x_1 \qquad\qquad & = & -15, \\ x_2 \qquad & = & 6, \\ x_3 & = & -1, \end{array} \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 & 0 & : & -15 \\ 0 & 1 & 0 & : & 6 \\ 0 & 0 & 1 & : & -1 \end{bmatrix}.$$

Hence, $x_1 = -15$, $x_2 = 6$, and $x_3 = -1$ is the only solution of the system.

1.1.7 Geometry of solutions in three variables

It is known that for given numbers a, b, c, d, all space points (x_1, x_2, x_3) satisfying the equation

$$ax_1 + bx_2 + cx_3 = d$$

are all the points of a plane, provided that not all of a, b, and c are zero. So, solution sets of linear systems in three variables are, in general, intersections of planes. These intersections can be a point, a line, a plane, or the empty set.

Example 1.1.10 (One solution). Find the intersection of the three planes

$$x_1 + 3x_2 - x_3 = 4, \quad -2\,x_1 + x_2 + 3\,x_3 = 9, \quad 4\,x_1 + 2\,x_2 + x_3 = 11.$$

Solution. Elimination on the augmented matrix of the system yields $\left[\begin{smallmatrix} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{smallmatrix}\right]$. Hence the intersection is the point $P(1,2,3)$ (Figure 1.4).

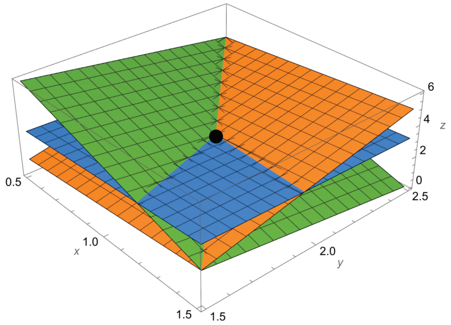


Figure 1.4: Three planes intersecting at one point.

Example 1.1.11 (Infinitely many solutions). Find the intersection of the three planes

$$x_1 + 2x_2 - x_3 = 4, \quad 2x_1 + 5x_2 + 2x_3 = 9, \quad x_1 + 4x_2 + 7x_3 = 6.$$

Solution. Elimination on the augmented matrix yields $\begin{bmatrix} 1 & 0 & -9 & : & 2 \\ 0 & 1 & 4 & : & 1 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$, which represents the system $x_1 - 9x_3 = 2$, $x_2 + 4x_3 = 1$. We solve by back-substitution to get

$$x_1 = 9r + 2, \quad x_2 = -4r + 1, \quad x_3 = r \quad \text{for all } r \in \mathbf{R}$$

These are the parametric equations of a straight line. This line is the common intersection of the three planes (Figure 1.5).

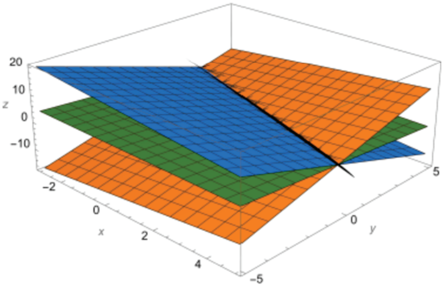


Figure 1.5: The three planes have a common line.

Example 1.1.12 (No solutions). Find the intersection of the three planes

$$x_2 - 2x_3 = -5, \quad 2x_1 - x_2 + x_3 = -2, \quad 4x_1 - x_2 = -4.$$

Solution. The augmented matrix of the system reduces to $\begin{bmatrix} 2 & -1 & 1 & : & -2 \\ 0 & 1 & -2 & : & -5 \\ 0 & 0 & 0 & : & 5 \end{bmatrix}$. The last row corresponds to the false equation $0x_1 + 0x_3 + 0x_3 = 5$. Hence, the system is inconsistent. Therefore, the planes do not have a common intersection.

The three cases of an inconsistent three-variable system representing three planes are graphically illustrated in Figure 1.6.

> Our examination of the geometry of solutions suggests that a linear system has either exactly one solution, or infinitely many solutions, or no solutions. This is true and is proved in Section 1.2.

1.1.8 Linear systems with complex numbers

The same elimination method applies to solving linear systems of complex numbers. A review of complex numbers is in Appendix A.

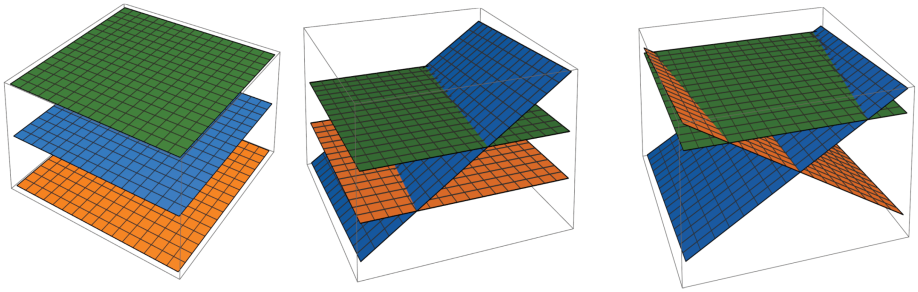


Figure 1.6: No common intersection. No solutions.

Example 1.1.13. Solve the linear system

$$\begin{aligned} -2z_1 \quad + z_2 &= -3i, \\ (2-2i)z_1 + iz_2 &= \ \ 5. \end{aligned}$$

Solution. We use elimination on the augmented matrix to get by the indicated operations:

$$\left[\begin{array}{cccc} -2 & 1 & : & -3\,i \\ 2-2i & i & : & 5 \end{array}\right] \boxed{(1-i)\,R_1 + R_2 \to R_2} \left[\begin{array}{cccc} -2 & 1 & : & -3\,i \\ 0 & 1 & : & 2-3\,i \end{array}\right]$$

$$\boxed{R_1 - R_2 \to R_1} \left[\begin{array}{cccc} -2 & 0 & : & -2 \\ 0 & 1 & : & 2-3\,i \end{array}\right] \boxed{\left(-\frac{1}{2}\right) R_1 \to R_1}$$

$$\left[\begin{array}{cccc} 1 & 0 & : & 1 \\ 0 & 1 & : & 2-3\,i \end{array}\right].$$

Therefore $z_1 = 1$ and $z_2 = 2 - 3\,i$.

1.1.9 Interchanges in terms eliminations and scalings

The elementary row operation of interchanging two rows can be obtained by a finite sequence of the other two elementary row operations, i. e., eliminations and scalings! Find a sequence of eliminations and scalings (but no interchanges) that will convert

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} c & d \\ a & b \end{bmatrix}.$$

Exercises 1.1

Linear equations

1. Identify each equation as linear or nonlinear. If an equation is linear, then classify it as homogeneous or nonhomogeneous. For each linear equation, find, if possible, the general solution and two particular solutions.

(a) $2x_1 + 3x_2 = x_3 - 5x_2$;
(b) $x_1x_2 + x_3 = x_1 - x_2$;
(c) $x_1 + x_2 + x_3 = 1 + x_1 - x_4 + x_5$.

2. Which of the points $P(2,-3,0)$, $Q(2,-3,-1)$, and $R(2,-3,-7)$ is in the plane with equation $x_1 - x_2 + x_3 = -2$?

3. Find all the values of a such that each of the following equations has (i) exactly one solution, (ii) infinitely many solutions, (iii) no solutions:
(a) $a^2x_1 - 2a^2 = 4ax_1 + a$;
(b) $ax_1 - a^2x_2 = 1 + ax_3$.

4. Find the solution set of each equation:
(a) $0x + 0y = 0$;
(b) $0x + 0y = 1$;
(c) $0x + 0y + 0z = 0$;
(d) $0x + 0y + 0z = 1$.

5. *True or False?* The equation expressing the relation between x_1 and x_2 is linear, where (x_1, x_2) is a point on a given
(a) circle;
(b) straight line;
(c) parabola;
(d) hyperbola.

Linear systems

6. First, rewrite the linear system in standard form:

$$\begin{aligned} 2x_1 + 4x_3 + 1 &= 0, \\ 2x_3 + 2x_4 - 2 &= x_1, \\ -2x_1 - x_3 + 3x_4 &= -3, \\ x_2 + x_3 + x_5 &= x_4 + 4. \end{aligned}$$

Then, find
(a) the coefficient matrix;
(b) the vector of constants;
(c) the augmented matrix;
(d) the associated homogeneous system.

7. Find the intersection of the straight lines $-3x_1 + 2x_2 = 5$ and $2x_1 + x_2 = 20$ shown in Figure 1.7.

8. Use a linear system to find the equation of the line passing through the points $(1,-2)$ and $(-5,6)$.

9. Use back-substitution to solve the system

$$\begin{aligned} x_1 + 2x_2 + x_3 + x_5 &= -1, \\ -2x_3 + 4x_6 &= 2, \\ 4x_4 - 2x_5 &= 0. \end{aligned}$$

10. Use back-substitution to solve the associated homogeneous system of the system in Exercise 9.

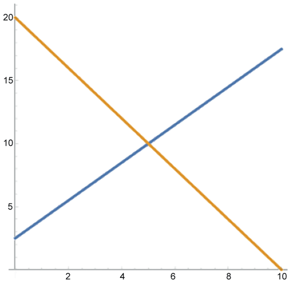


Figure 1.7: Intersection of lines.

11. Let *M* be the matrix

$$\begin{bmatrix} 1 & -1 & 1 & -5 & 6 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 2 & 0 & 0 \end{bmatrix}.$$

(a) Write a system whose augmented matrix is *M*.
(b) Write the associated homogeneous system with the linear system of part (a).
(c) Apply one elementary row operation to *M* so that the resulting matrix corresponds to a system in echelon form.

12. Find the general solution of the system whose augmented matrix is matrix *M* defined in Exercise 11.

13. Find the general solution of the associated homogeneous system with the system of Exercise 12.

14. Without actually solving the systems, prove that they are equivalent.

$$\begin{aligned} x_1 - x_2 + x_3 &= 1, \\ 2x_1 + 2x_2 - 3x_3 &= -2, \\ -3x_1 + 4x_2 + 4x_3 &= -1; \end{aligned} \qquad \begin{aligned} 4x_1 - 4x_2 + 4x_3 &= 4, \\ 2x_1 + 2x_2 - 3x_3 &= -2, \\ 5x_2 + 2x_3 &= -2. \end{aligned}$$

In Exercises 15–18, find the consistent systems and compute their general solution.

15.
$$\begin{aligned} -x_1 + x_2 - x_3 &= 1, \\ -2x_1 + x_2 + 3x_3 &= 10, \\ 3x_1 + x_2 + 2x_3 &= 3. \end{aligned}$$

16. (a) For $b = 9$; (b) For $b = 10$.

$$\begin{aligned} 3x_1 + x_2 + 3x_3 &= 15, \\ -x_1 + 3x_2 - x_3 &= -5, \\ 2x_1 + 4x_2 + 2x_3 &= b. \end{aligned}$$

17.
$$\begin{aligned} x + 3y + z - w &= 0, \\ 3x + y + 3z \phantom{{}- 2w} &= -2, \\ 2x + 6y + 2z - 2w &= 2. \end{aligned}$$

18. (a) $\begin{aligned} x_1 + x_2 &= 1, \\ x_2 + x_3 &= 1, \\ x_3 + x_4 &= 1, \\ x_1 + x_4 &= 1; \end{aligned}$ (b) $\begin{aligned} x + y &= 1, \\ y + z &= 1, \\ z + w &= 1, \\ y + w &= 1. \end{aligned}$

19. Solve the ancient Chinese system mentioned in the introduction of this chapter.

In Exercises 20–21, solve the systems with the given augmented matrices.

20. $\begin{bmatrix} -1 & 2 & 0 & : & -6 \\ 3 & -2 & -1 & : & 10 \\ 3 & 2 & 2 & : & -14 \end{bmatrix}$,

21. $\begin{bmatrix} -1 & -2 & -1 & : & 1 \\ -1 & -1 & 1 & : & 4 \\ 1 & 1 & -1 & : & 4 \end{bmatrix}$.

22. Find the intersection of the three planes $3x_1 + 2x_2 - x_3 = 4$, $-x_1 + 3x_2 + 2x_3 = 1$, $x_1 + x_2 + x_3 = 1$, shown in Figure 1.8.

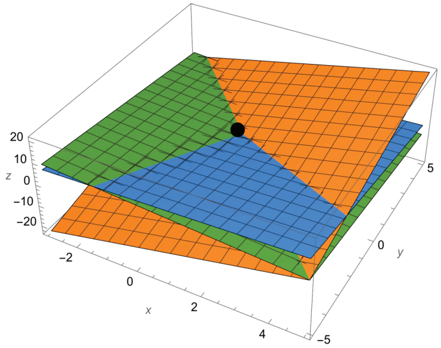


Figure 1.8: Intersection of planes.

23. *True or False?* Explain. Linear systems in three variables

(a) can have exactly three solutions;

(b) with two equations always have infinitely many solutions;

(c) with one equation can have no solutions;

(d) that are homogeneous always have infinitely many solutions.

24. *True or False?* Explain. The operation is an elementary row operation:

(a) $R_2 + 3R_9 \longrightarrow R_9$;

(b) $R_2 + 3R_9 \longrightarrow R_2$.

25. Solve the nonlinear system for the angle θ given in radians:

$$2\sin\theta + \sqrt{2}\tan\theta = 2\sqrt{2}, \quad 4\sin\theta - 3\sqrt{2}\tan\theta = -\sqrt{2}.$$

26. Find a relation between k_1 and k_2 that makes the following system consistent:

$$\begin{aligned} x_1 - x_2 - x_3 &= k_1, \\ x_1 + x_2 + x_3 &= k_2, \\ 2x_2 + 2x_3 &= 0. \end{aligned}$$

27. Consider the homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0, \\ a_{21}x_1 + a_{22}x_2 &= 0. \end{aligned}$$

(a) Prove that if $x_1 = r_1, x_2 = r_2$ is a solution of this system, then for any scalar c, $x_1 = cr_1, x_2 = cr_2$ is also a solution.

(b) Prove that if $x_1 = r_1, x_2 = r_2$ and $x_1 = s_1, x_2 = s_2$ are two solutions, then $x_1 = r_1 + s_1, x_2 = r_2 + s_2$ is also a solution.

Exercises 28 and 29 use the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= c_1, \\ a_{21}x_1 + a_{22}x_2 &= c_2. \end{aligned} \tag{1.7}$$

28. Assume that $a_{11}a_{22} - a_{12}a_{21} \neq 0$ in (1.7). Prove the following:

(a) The system has exactly one solution. Find this solution.

(b) The associated homogeneous system has only the trivial solution.

29. Assume that $a_{11}a_{22} - a_{12}a_{21} = 0$ in (1.7). Prove the following:

(a) The system has either infinitely many solutions or no solutions.

(b) The associated homogeneous system has nontrivial solutions.

30. *True or False?* The system is consistent for all choices of c_1 and c_2:

(a) $2x_1 - 11x_2 = c_1, 8x_1 + 12x_2 = c_2$;

(b) $-x_1 + 3x_2 = c_1, 44x_1 - 132x_2 = c_2$.

31. Solve the *nonlinear* system by using a linear system. Your answer of eight points is the intersection of three surfaces, a sphere, and two circular hyperboloids, shown in Figure 1.9.

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 4, \\ x_1^2 - x_2^2 + x_3^2 &= 2, \\ x_1^2 + x_2^2 - x_3^2 &= 2. \end{aligned}$$

32. Solve the nonlinear systems and find geometric interpretations of the equations and the solutions:

(a) $x + y = 1, x^2 + y^2 = 5$;

(b) $x^2 + y^2 = 3, x^2 - y^2 = 1$.

Complex linear systems

In Exercises 33–34, solve each of the systems.

33.

$$\begin{aligned} 2iz_1 + 4z_2 &= 6i, \\ -iz_1 + (1 - i)\, z_2 &= 1. \end{aligned}$$

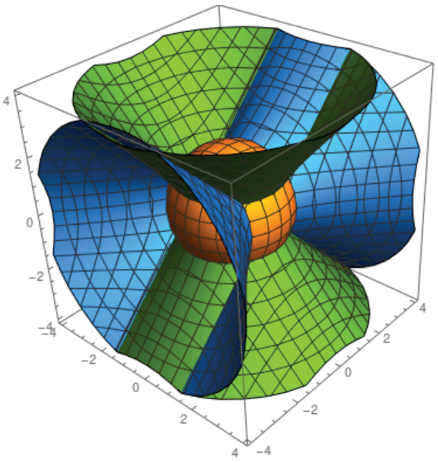


Figure 1.9: Intersection of surfaces.

34.
$$\begin{aligned} -iz_1 + 3iz_2 + z_3 &= 1 + 6i, \\ 2z_1 + (1+i)\,z_2 &= 4 + 4i, \\ 4z_1 + (1-i)\,z_2 + iz_3 &= 5 + 2i. \end{aligned}$$

Mathematical Applications

35. Find the equation of the parabola with vertical axis in the xy-plane passing through the points $P(1,4)$, $Q(-1,6)$, $R(2,9)$. Do so by seeking a function of the form $y(x) = ax^2 + bx + c$, where a, b, c are unknown coefficients (Figure 1.10).

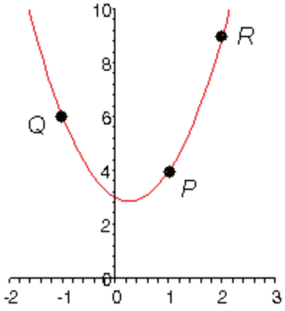


Figure 1.10: Parabola through Q, P, R.

36. (*Law of cosines*) Obtain the law of cosines by solving the linear system in unknowns $\cos\alpha$, $\cos\beta$, $\cos\gamma$ (Figure 1.11):

$$\begin{aligned} c\cos\beta + b\cos\gamma &= a, \\ c\cos\alpha + a\cos\gamma &= b, \\ a\cos\beta + b\cos\alpha &= c. \end{aligned}$$

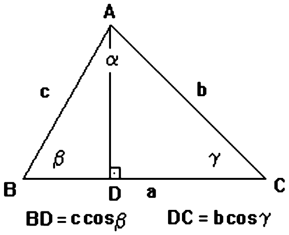


Figure 1.11: Law of cosines.

37. (*Partial fractions*) Find constants A, B, and C such that

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

1.2 Gauss elimination

This section is devoted to a detailed study of *Gauss elimination*, which was introduced in Section 1.1 to solve linear systems. A variant of this method, called *Gauss–Jordan elimination*, is also discussed. The material here is important to all remaining chapters.

Although the solution algorithm for small linear systems was apparently known since ancient times, it was Gauss who first used it in a systematic way to solve large-size linear systems that appeared in his least squares method (Figure 1.12).

Figure 1.12: Karl Friedrich Gauss.
By Christian Albrecht Jensen – 1840.
Public domain, via Wikimedia Commons.
Karl Friedrich Gauss (1777–1855) ranks with Archimedes, Euler, and Newton as one of the greatest mathematicians of all time. Born in Germany, a child prodigy, at 18 he constructed a 17–sided regular polygon by a ruler and compass, solving a two-thousand-year-old problem. He wrote *Disquisitiones Arithmeticae*, a masterpiece in number theory, and proved the *Fundamental Theorem of Algebra*. Besides pure mathematics, Gauss also contributed to geodesy, astronomy, and physics.

1.2.1 Matrices in echelon form

A *zero row* (*zero column*) of a matrix is a row (column) that consists entirely of zeros. The first nonzero entry of a nonzero row is called a *leading entry*. If a leading entry is 1, then it is called a *leading 1*.

Definition 1.2.1. Consider the following conditions on a matrix A.

1. If there are any zero rows, then all of them are at the bottom of the matrix.
2. The leading entry of each nonzero row, after the first, occurs to the right of the leading entry of the previous row.
3. The leading entry in any nonzero row is 1.
4. All entries in the column above and below a leading 1 are zero.

If A satisfies the first two conditions, then it is said to be in *(row) echelon form*. If it satisfies all four conditions, it is said to be in *reduced (row) echelon form* or *RREF* (Figure 1.13).

$$\begin{bmatrix} * & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ 0 & * & \times & \times & \times & \times & \times & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & * & \times & \times & \times & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & * & \times & \times & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \times & 0 & 0 & \times & \times & 0 & \times & \times & \times & 0 \\ 0 & 1 & \times & 0 & 0 & \times & \times & 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & 1 & 0 & \times & \times & 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & 1 & \times & \times & 0 & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \times & \times & \times & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 1.13: (a) Row echelon form. (b) Reduced row echelon form.

Example 1.2.2. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 7 & 0 & 9 & 0 \\ 0 & 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Matrices A, B, D, F, G, H are in echelon form, A, B, D, F are in reduced echelon form, G and H are not in reduced echelon form, and C and E are not in echelon form.

The size $n \times n$ matrix I_n that has 1s along the upper left to lower right diagonal line and 0s elsewhere is called an *identity matrix*.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \dots, \quad I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Note that all I_n are in reduced row echelon form.

1.2.2 The Gauss elimination algorithm

In Section 1.1, we solved linear systems by reducing the augmented matrix of the system into echelon form, so that the corresponding system was in echelon form. In general, reducing *any* matrix to reduced echelon form can be done as follows.

Algorithm 1.2.3 (Gauss elimination). To reduce any matrix to reduced row echelon form, apply the following steps.

1. Find the leftmost nonzero column.
2. If the first row has a zero in the column of Step 1, then interchange it with one that has a nonzero entry in the same column.
3. Obtain zeros below the leading entry by adding suitable multiples of the top row to the rows below that.
4. Cover the top row and repeat the same process starting with Step 1 applied to the leftover submatrix. Repeat this process with the rest of the rows *until the matrix is in echelon form.*
5. Starting with the last nonzero row work upward: For each row, obtain a leading 1 and introduce zeros above it by adding suitable multiples to the corresponding rows.

The nonzero entries in Step 2 are called *pivots*, and the positions where they occur are called *pivot positions*. The columns with pivots are called *pivot columns*.

Example 1.2.4. Apply Gauss elimination to find a reduced echelon form of the matrix

$$\begin{bmatrix} 0 & 3 & -6 & -4 & -3 \\ -1 & 3 & -10 & -4 & -4 \\ 4 & -9 & 34 & 0 & 1 \\ 2 & -6 & 20 & 8 & 8 \end{bmatrix}.$$

Solution.

STEP 1. *Find the leftmost nonzero column.* This is the first column here.

STEP 2. *If the first row has a zero in the column of Step 1, then interchange it with one that has a nonzero entry in the same column:*

$$\boxed{R_1 \leftrightarrow R_2} \quad \begin{bmatrix} \boxed{-1} & 3 & -10 & -4 & -4 \\ 0 & 3 & -6 & -4 & -3 \\ 4 & -9 & 34 & 0 & 1 \\ 2 & -6 & 20 & 8 & 8 \end{bmatrix}.$$

The pivot now is –1, at pivot position $(1, 1)$.

STEP 3. *Obtain zeros below the leading entry by adding suitable multiples of the top row to the rows below that:*

$$\boxed{\begin{array}{c} R_3 + 4R_1 \to R_3 \\ R_4 + 2R_1 \to R_4 \end{array}} \quad \begin{bmatrix} -1 & 3 & -10 & -4 & -4 \\ 0 & 3 & -6 & -4 & -3 \\ 0 & 3 & -6 & -16 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

STEP 4. *Cover the top row and repeat the same process starting with Step 1 applied to the leftover submatrix:*

$$\left[\begin{array}{ccccc} -1 & 3 & -10 & -4 & -4 \\ \hline \multicolumn{1}{|c}{0} & 3 & -6 & -4 & \multicolumn{1}{c|}{-3} \\ \multicolumn{1}{|c}{0} & 3 & -6 & -16 & \multicolumn{1}{c|}{-15} \\ \multicolumn{1}{|c}{0} & 0 & 0 & 0 & \multicolumn{1}{c|}{0} \\ \hline \end{array}\right] \xrightarrow{\text{Step 1}} \left[\begin{array}{c|cccc|} -1 & 3 & -10 & -4 & -4 \\ \cline{2-5} 0 & 3 & -6 & -4 & -3 \\ 0 & 3 & -6 & -16 & -15 \\ 0 & 0 & 0 & 0 & 0 \\ \cline{2-5} \end{array}\right].$$

Repeat this process with the rest of the rows, until the matrix is in echelon form. The next pivot is 3, at position $(2, 2)$:

$$\begin{bmatrix} -1 & 3 & -10 & -4 & -4 \\ 0 & \boxed{3} & -6 & -4 & -3 \\ 0 & 3 & -6 & -16 & -15 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boxed{R_3 - R_2 \to R_3} \begin{bmatrix} -1 & 3 & -10 & -4 & -4 \\ 0 & 3 & -6 & -4 & -3 \\ 0 & 0 & 0 & \boxed{-12} & -12 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

STEP 5. *Starting with the last nonzero row work upward, for each row, obtain a leading 1 and introduce zeros above it by adding suitable multiples to the corresponding rows:*

$$\boxed{(-1/12)R_3 \to R_3} \begin{bmatrix} -1 & 3 & -10 & -4 & -4 \\ 0 & 3 & -6 & -4 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boxed{\begin{array}{c} R_2 + 4R_3 \to R_2 \\ R_1 + 4R_3 \to R_1 \end{array}}.$$

$$\begin{bmatrix} -1 & 3 & -10 & 0 & 0 \\ 0 & 3 & -6 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boxed{(1/3)R_2 \to R_2} \begin{bmatrix} -1 & 3 & -10 & 0 & 0 \\ 0 & 1 & -2 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \boxed{R_1 - 3R_2 \to R_1}$$

$$\begin{bmatrix} -1 & 0 & -4 & 0 & -1 \\ 0 & 1 & -2 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \boxed{(-1)R_1 \to R_1} \quad \begin{bmatrix} 1 & 0 & 4 & 0 & 1 \\ 0 & 1 & -2 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last matrix is in reduced row echelon form. The pivot positions are $(1, 1)$, $(2, 2)$, and $(3, 4)$. The pivot columns are columns 1, 2, and 4.

The first four steps of Algorithm 1.2.3 are the *forward pass*. At this stage the matrix is already in echelon form. Step 5 is the *backward pass* or *back-substitution* (Figure 1.14).

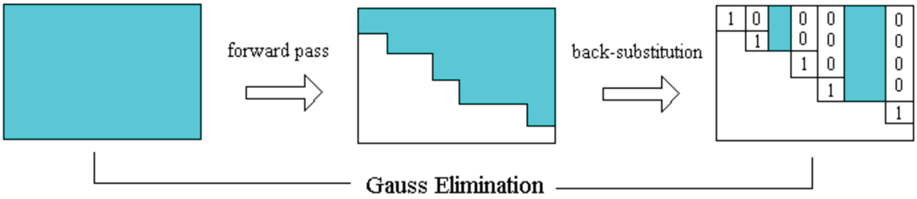


Figure 1.14: The two stages of Gauss elimination.

Avoid changing the order of steps in the algorithm, or using nonelementary operations, such as $cR_i + dR_j \to R_i$ or $cR_i + R_j \to R_i$ with $c \neq 1$. *The row that gets replaced should not be multiplied by anything.*

Definition 1.2.5. Two matrices are *(row) equivalent* if one can be obtained from the other by a finite sequence of elementary row operations. If A and B are row equivalent, then we write

$$A \sim B.$$

Exercise 7 states that elementary row operations are *reversible*. This means that if an elementary row operation is used to produce matrix B from matrix A, then there is another elementary row operation that will reverse this effect and will transform B back to A. So it makes sense to say that two matrices are row equivalent, without specifying which matrix is first.

Gauss elimination produces row equivalent matrices. After Step 4, each of these matrices is in echelon form, called an *echelon form of A*. A matrix can be row equivalent to several echelon form matrices, but only to one reduced row echelon form. This is expressed in the following theorem, which is proved in Appendix B.

Theorem 1.2.6 (Uniqueness of reduced row echelon form). *Every matrix is row equivalent to one and only one matrix in reduced row echelon form.*

Note that in any echelon form of a matrix A, the leading entries occur at the same columns. This follows from the uniqueness of the reduced echelon form and the fact that after Step 4 the positions of the leading entries do not change. We conclude the following:

> Pivot positions and pivot columns do not change with row reduction. Therefore, row equivalent matrices have the same pivot column positions and pivot positions.

!

1.2.3 Solution algorithm for linear systems

The solution process of linear systems introduced in Section 1.1 is described now as follows.

Algorithm 1.2.7 (Solution of linear system). To solve any linear system, use the following steps.

1. Apply Gauss elimination to the augmented matrix of the system (forward pass). If during any stage of this process it is found that the last column is a pivot column, then stop. In this case the system is inconsistent. Otherwise, continue with Step 2.
2. Complete Gauss elimination to reduced row echelon form. Write the system whose augmented matrix is the reduced echelon form matrix, ignoring any zero equations.
3. Separate the variables of the reduced system into leading and free (if any). Write the free variables as parameters. Solve the leading variables in terms of the parameters and/or numbers.

Example 1.2.8 (General solution of linear system). Find the general solution of the system

$$\begin{aligned} 3x_2 - 6x_3 - 4x_4 - 3x_5 &= -5, \\ -x_1 + 3x_2 - 10x_3 - 4x_4 - 4x_5 &= -2, \\ 2x_1 - 6x_2 + 20x_3 + 2x_4 + 8x_5 &= -8. \end{aligned}$$

Solution. Gauss elimination on the augmented matrix of the system yields

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 1 & -3 \\ 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

Therefore the original system reduces to the equivalent system

$$\begin{aligned} x_1 \quad +4\,x_3 \quad +x_5 &= -3, \\ x_2 \; -2x_3 \quad -x_5 &= 1, \\ x_4 \quad &= 2. \end{aligned}$$

We use parameters for the free variables, and we solve for the leading variables to get the two-parameter infinite set:

$$\begin{aligned} x_1 &= -4s - r - 3, \\ x_2 &= 2s + r + 1, \\ x_3 &= s, && \text{for any } r, s \in \mathbf{R}. \\ x_4 &= 2, \\ x_5 &= r \end{aligned}$$

1.2.4 The Gauss–Jordan elimination algorithm

An interesting variant of the Gauss elimination occurs if during the forward pass, we first produce leading 1s and then zeros below and *above* them. Hence, at the end of the forward pass the matrix is already in reduced row echelon form. This is known as *Gauss–Jordan elimination.*[1]

Example 1.2.9. Find the reduced echelon form of A by using Gauss–Jordan elimination.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -2 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Solution. First, we scale the second row to get a leading 1. Then we obtain zeros below and *above* this leading 1. We repeat with the third row:

$$A \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & \boxed{1} & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

1.2.5 Existence and uniqueness of solutions

If during elimination of the augmented matrix of a linear system a row of the form

$$[\, 0\; 0\; 0\; \cdots\; 0 \,:\, c \,]\,, \qquad c \neq 0,$$

---

[1] **Wilhelm Jordan** (1842–1899), German engineer. He wrote the popular *Pocket Book of Practical Geometry*. According to Gewirtz, Sitomer, and Tucker, “he devised the pivot reduction algorithm, known as Gauss–Jordan elimination, for geodetic reasons”.

is found, then the system is inconsistent. This is because such a row corresponds to the impossible equation

$$0x_1 + \cdots + 0x_n = c\,, \quad c \neq 0.$$

In this case the reduction is abandoned, and the system is declared as inconsistent. If, however, the reduction is carried on until an echelon form is reached, then the last column is seen to be a pivot column. If the last column is not a pivot column, then the last nonzero equation has a leading variable, which can be solved for, and then back-substitution yields a solution. Hence, the system in this case is consistent. We have proved the following theorem.

Theorem 1.2.10 (Criterion for consistent system). *A linear system is consistent if and only the last column of its augmented matrix is not a pivot column.*

Now suppose that a linear system is consistent. If there are free variables, then the system has infinitely many solutions determined by the parameters of the free variables. If there are no free variables, then all variables are leading, and each leading variable is a unique constant. Hence, in this case, there is exactly one solution. We have just proved the following claim made in Section 1.1.

Theorem 1.2.11. *For a given linear system, only one of the following is true: the system has*

1. *exactly one solution;*
2. *infinitely many solutions;*
3. *no solutions.*

A consistent system has exactly one solution if and only if it has no free variables. This is the same as saying that all the variables are leading variables, in which case each should be a unique constant. Now the leading variables correspond to pivot columns. So the existence of a unique solution is equivalent to requiring that all columns of the augmented matrix except for the last one to be pivot columns. This argument proves the following theorem.

Theorem 1.2.12 (Uniqueness of solutions). *For a* consistent *linear system, the following statements are equivalent (Figure* 1.15*):*

1. *The system has exactly one solution;*
2. *The system has no free variables;*
3. *Each column of the augmented matrix other than the last one is a pivot column, and the last column is not a pivot column.*

The mere presence of free variables does not guarantee infinitely many solutions, because the system may still be inconsistent. (e. g., $x + y + z = 1$, $x + y + z = 2$.)

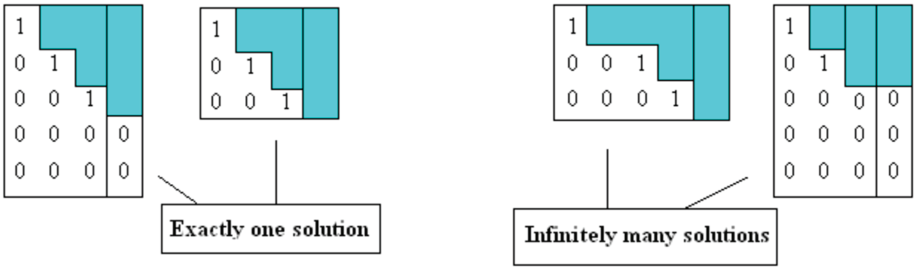


Figure 1.15: (a) Exactly one solution. (b) Infinitely many solutions.

Example 1.2.13. Discuss the solutions of the systems whose augmented matrices reduce to the given echelon form matrices:

$$\begin{bmatrix} 1 & a & b & d & g \\ 0 & 2 & c & e & h \\ 0 & 0 & 3 & f & i \\ 0 & 0 & 0 & 4 & j \end{bmatrix}, \quad \begin{bmatrix} 1 & a & b & c & e \\ 0 & 0 & 2 & d & f \\ 0 & 0 & 0 & 3 & g \end{bmatrix},$$

$$\begin{bmatrix} 1 & a & b & d & f \\ 0 & 2 & c & e & g \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & a & b & c \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution. The first two systems are consistent by Theorem 1.2.10, because their last columns are not pivot columns. The last two systems are inconsistent by Theorem 1.2.10, because their last columns are pivot columns. By Theorem 1.2.12 the first system has exactly one solution, because each column except for the last one is a pivot column. Also, by Theorem 1.2.12 the second system has infinitely many solutions, because there is a nonpivot column (the second one) other than the last column.

1.2.6 Homogeneous linear systems

Let us now specialize to homogeneous linear systems.

Theorem 1.2.14 (Solutions of homogeneous systems). *For a homogeneous linear system, we have the following:*

1. *The system has either only the trivial solution or infinitely many solutions.*
2. *The system has infinitely many solutions if and only if it has free variables.*
3. *The system has infinitely many solutions if and only the coefficient matrix has at least one nonpivot column.*
4. *If the system has more unknowns than equations, then it has infinitely many solutions.*

Proof. A homogeneous linear system is always consistent because it has the trivial solution as a solution. Thus Parts 1, 2, and 3 follow from Theorem 1.2.12. For Part 4, we observe that the system corresponding to the reduced echelon form of the augmented matrix has more unknowns than equations, so there are free variables. Hence, the system has infinitely many solutions by Part 2. □

1.2.7 Numerical considerations

1. The introduction of leading 1s in Gauss elimination can be included in Step 3. This variant is preferred in floating point arithmetic. Furthermore, in Step 2 the row with the nonzero entry of largest absolute value is moved to the top. This method is called *partial pivoting*. Partial pivoting helps control round-off errors.
2. When a computer is used, there is no need for "physical interchanges" of rows, which can be computationally costly. Instead, a vector called a *permutation vector* is used to keep record of the performed interchanges.
3. Consider a linear system with n equations and n unknowns. For large n, it can be shown that Gauss elimination requires approximately $2n^3/3$ arithmetic operations and Gauss–Jordan elimination requires approximately n^3 operations.
4. Gauss elimination is not suitable for solving large systems because entries that are supposed to be zero are often nonzero due to round-off errors. These errors propagate with each step. In practice, a variety of numerical methods are used, such as the *iterative methods* studied in Section 1.4.

Exercises 1.2

Echelon Form

In Exercises 1–4, place each matrix into one of the following categories: (i) row echelon but *not* reduced row echelon form, (ii) reduced row echelon form, and (iii) not row echelon form.

1. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 17 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & 4 & 0 & -7 & 0 \\ 0 & 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

3. $\begin{bmatrix} 0 & 1 & 8 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & 3 & 0 & -7 & 4 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

5. What values of a, b, c, and d make the matrix in reduced row echelon form?

$$\begin{bmatrix} a & b & 0 & -7 & d \\ 0 & 0 & 1 & c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

6. Write the possible reduced row echelon forms of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Equivalent matrices; row reduction

7. Prove that each of the elementary row operations is *reversible*. This means that if an operation is used to produce matrix B from matrix A, then there is an elementary row operation that will reverse the effect of the first one and transform B back to A. This operation is called the *inverse* operation of the original one.

8. Use Exercise 7 to prove that

$$\text{If} \quad A \sim B, \quad \text{then} \quad B \sim A.$$

9. Prove that

$$\text{If} \quad A \sim B \quad \text{and} \quad B \sim C, \quad \text{then} \quad A \sim C.$$

10. Prove that

$$\begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

11. Prove that $A \sim C$. (*Hint:* Use Exercises 8 and 9 with $B = I_3$).

$$A = \begin{bmatrix} 7 & 2 & 3 \\ 0 & 4 & 1 \\ 5 & 6 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

12. *True or False?*

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

13. Prove that if $ad - bc \neq 0$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim I_2$$

14. Use Exercise 13 to prove that for any θ

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \sim I_2$$

15. *True or False?* If the $n \times n$ matrix A has n pivots, then $A \sim I_n$.

In Exercises 16–19, find the reduced row echelon forms of the matrices.

16. $\begin{bmatrix} -1 & -6 & 0 & 1 & 0 & -2 \\ 5 & 30 & 1 & -6 & 0 & 13 \\ 4 & 24 & 1 & -5 & 1 & 15 \end{bmatrix}$.

17. $\begin{bmatrix} -1 & -4 & 0 & -5 & 0 & -6 \\ 5 & 20 & 1 & 29 & 0 & 34 \\ 4 & 16 & 1 & 24 & 1 & 30 \end{bmatrix}$.

18. $\begin{bmatrix} 0 & -2 & -2 & 2 & 0 \\ 0 & -2 & -2 & 2 & 0 \\ -2 & 0 & 2 & 0 & 2 \\ 0 & 0 & -2 & 0 & 2 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$.

19. $\begin{bmatrix} 2 & 3 & 9 & 1 & 0 \\ 6 & 9 & 12 & -10 & 2 \\ 0 & 0 & 5 & 10 & 15 \\ 0 & 0 & 0 & -12 & 0 \end{bmatrix}$.

20. Find the reduced row echelon form of the complex matrix

$$\begin{bmatrix} i & -1+i & 0 \\ 0 & -2+i & i \\ -i & 0 & 2i \\ 1-2i & 4i & -2-i \end{bmatrix}.$$

21. Let R be the reduced row echelon form of A. Find the reduced row echelon form of:[2]

(a) $3A$ (all entries of A are multiplied by 3);

(b) $[A\ A]$ (two copies of A next to each other);

(c) $[I_n\ A]$ (I_n and A next to each other A has n rows);

(d) $\left[\begin{smallmatrix} A \\ I_n \end{smallmatrix}\right]$ (A above I_n, and A has n columns).

Linear Systems

In Exercises 22–25, solve the systems.

22.
$$\begin{aligned} x_1 - 8x_2 \qquad + 7x_4 \qquad &= 9, \\ -2x_1 + 16x_2 - x_3 - 20x_4 \qquad &= -24, \\ 2x_1 - 16x_2 + 6x_3 + 50x_4 + x_5 &= 51. \end{aligned}$$

23.
$$\begin{aligned} x + z + w &= -5, \\ x - z + w &= -1, \\ x + y + z + w &= -3, \\ 2x \quad + 2z \qquad &= -2. \end{aligned}$$

---

[2] This problem is adapted from Jeffrey L. Stuart's linear algebra review in Mathematical Association of America, Monthly, 2005, pp. 281–288.

24.

$$\begin{aligned} x + y + z + w - t &= 1, \\ y \qquad\qquad &= -1, \\ -2z - w + t &= -3, \\ w - 3t &= -1, \\ t &= 1. \end{aligned}$$

25.

$$\begin{aligned} x \qquad\quad - t &= -2, \\ y - z \quad + t &= 5, \\ -y + z \quad - t &= -5, \\ y - z \quad + t &= 5, \\ -y \quad - w \qquad &= -1. \end{aligned}$$

26. Solve the system with augmented matrix

$$\begin{bmatrix} -1 & 0 & 1 & 1 & -1 & : & -1 \\ 0 & 1 & 0 & -1 & -1 & : & 0 \\ 0 & 1 & -1 & -1 & 1 & : & -6 \\ 0 & 1 & 1 & -1 & 0 & : & 3 \\ 1 & -1 & -1 & 0 & 1 & : & 2 \end{bmatrix}$$

Existence and uniqueness of solutions

In Exercises 27–28, consider linear systems whose augmented matrices reduce to the following echelon forms. What can you say about the number of solutions?

27. (a) $\begin{bmatrix} 2 & a & b & d & : & f \\ 0 & 2 & c & e & : & g \\ 0 & 0 & 0 & 2 & : & h \end{bmatrix}$; (b) $\begin{bmatrix} 2 & a & b & : & d \\ 0 & 2 & c & : & e \\ 0 & 0 & 2 & : & f \\ 0 & 0 & 0 & : & 2 \end{bmatrix}$.

28. (a) $\begin{bmatrix} 1 & a & b & c & : & d \\ 0 & 0 & 0 & 2 & : & 0 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 2 & a & b & : & c \\ 0 & 0 & 0 & : & 2 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$.

29. Consider the homogeneous linear system whose coefficient matrix reduces to the following echelon form. What can you say about the number of solutions?

$$\text{(a)} \begin{bmatrix} 2 & a & b & d & f \\ 0 & 2 & c & e & g \\ 0 & 0 & 0 & 2 & h \end{bmatrix}; \quad \text{(b)} \begin{bmatrix} 2 & a & b & d \\ 0 & 2 & c & e \\ 0 & 0 & 2 & f \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

In Exercises 30–31, each row of the table gives the size and the number of pivots of the augmented matrix of some system. What can you say about the system?

30.

Size	# pivots
3 × 5	3
4 × 4	4
4 × 4	3
5 × 3	3

31.

Size	# pivots
6×4	4
5×5	5
5×5	4
4×6	4

32. Prove that if a matrix has size $m \times n$, then the number of pivot columns is at most m and at most n.

33. Let A be an $m \times n$ matrix with $m < n$. Prove that either the linear system with augmented matrix $[A : \mathbf{b}]$ is inconsistent, or it has infinitely many solutions.

34. Let A be an $n \times n$ matrix. Prove that if the linear system with augmented matrix $[A : \mathbf{c}]$ has exactly one solution, then the linear system with augmented matrix $[A : \mathbf{b}]$ has also exactly one solution.

In Exercises 35–36, find the values of a such that the system with augmented matrix proven has (i) exactly one solution, (ii) infinitely many solutions, and (iii) no solutions.

35. (a) $\left[\begin{array}{cccc} 2 & 3 & : & 4 \\ 4 & a & : & 8 \end{array}\right]$; (b) $\left[\begin{array}{cccc} 2 & 3 & : & 4 \\ 4 & 6 & : & a \end{array}\right]$.

36. (a) $\left[\begin{array}{cccc} 2 & 3 & : & 4 \\ a & 6 & : & 8 \end{array}\right]$; (b) $\left[\begin{array}{ccccc} 1 & 2 & 1 & : & 3 \\ 1 & 3 & -1 & : & 4 \\ 1 & 2 & a^2-8 & : & a \end{array}\right]$.

37. *True or false?* All linear systems with 15 unknowns and 2 equations are consistent.

38. *True or false?* All linear systems with 2 unknowns and 15 equations are inconsistent.

39. (*Sums of squares*) Derive a formula for the sum of squares

$$1^2 + 2^2 + \cdots + n^2$$

by assuming that the answer is a polynomial of degree 3 in n, say $f(n) = an^3 + bn^2 + cn + d$.

40. (*Plane through 3 points*) Find the equation of the plane through the points $P(1,1,2)$, $Q(1,2,0)$, and $R(2,1,5)$.

Magic squares

A *magic square* of size n is an $n \times n$ matrix whose entries consist of all integers between 1 and n^2 such that the sum of the entries of each column, row, or diagonal is the same (Figure 1.16). The sum of the entries of any row, column, or diagonal of a magic square of size n is $\frac{n(n^2+1)}{2}$. (To see this, use the identity $1+2+\cdots+k = \frac{k(k+1)}{2}$.)[3]

41. Prove that magic squares of size 2 do not exist.

42. Find the magic square of size 3, whose first row is 8, 1, 6. Set up and solve a linear system in unknowns a, b, c, d, e, f.

---

[3] Analogous to magic squares are *magic cubes*. The rows, columns, pillars, and four space diagonals each sum to a single number. There are no magic cubes of size 4. In November 2003, Walter Trump from Germany and Christian Boyer from France found a perfect magic cube of order 5.

$$\begin{bmatrix} 8 & 1 & 6 \\ a & b & c \\ d & e & f \end{bmatrix}.$$

This magic square was mentioned in the ancient Chinese book *Nine Chapters of the Mathematical Art*.[4]

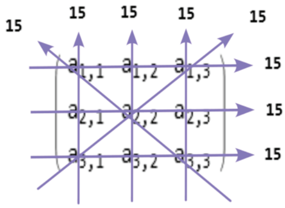


Figure 1.16: 3 × 3 magic square.

1.3 Applications: Economics, Chemistry, Physics, Engineering

We now discuss some of the many applications of linear systems. We only study toy problems involving small systems. Real-life applications may result in linear systems with thousands of equations. Solving such systems requires technology and some very sophisticated numerical methods.

1.3.1 Economics: The demand function, market equilibria

One of the functions economists use is the *demand function*, which expresses the number of items D of a certain commodity sold according to demand. The demand function D may depend on variables such as the price P of the commodity, the income I of the consumers, the price C of a competing commodity, etc. Furthermore, it may be a *linear* function of its variables, like $D = -15P + 0.05I + 2.5C$, or in general

$$D = aP + bI + cC.$$

Usually, the coefficients a, b, c of the variables are unknown, but they can be computed by solving a linear system.

Example 1.3.1. Sports Shoes Inc. plans to manufacture a new running shoe and researches the market for demand. It is found that if a pair of shoes costs \$60 in a \$60,000 average family income area and if their competitor, Athletic Shoes Inc., prices their competing shoes at \$60 a pair, then 1980 pairs will be sold. If on the other hand, the price remains the same and Athletic Shoes Inc. drops their price to \$30 a pair, then in a \$90,000 income area, 3390 pairs will be sold. Finally, if the shoes are priced at \$45 a pair while the competition remains at \$60 a pair, then in a \$75,000 income area, 3030 pairs will be sold.

---

[4] See Carl Boyer's "A History of Mathematics" and Shen's "Nine Chapters of the Mathematical Art" [15, 16].

(a) Compute the demand function by assuming that it depends linearly on its variables.
(b) How many pairs will be sold if the price is \$195 in a \$210,000 income area and if the competition prices their shoes at \$225?
(c) If the price of shoes P increases by one dollar while I and C remain constant, then how is D affected?

Solution.

(a) Let $D = aP + bI + cC$. We need a, b, c. According to the first research case, $60a + 60000b + 60c = 1980$. Similarly, we form the remaining equations to get the system

$$\begin{aligned}
60a + 60000b + 60c &= 1980,\\
60a + 90000b + 30c &= 3390,\\
45a + 75000b + 60c &= 3030,
\end{aligned}$$

with solution $a = -20$, $b = 0.05$, and $c = 3$. Hence, the demand function is

$$D = -20P + 0.05I + 3C.$$

(b) $D = -20(195) + 0.05(210000) + 3(225) = 7275$ pairs of shoes will be sold.
(c) If P increases by \$1, then the value of D will drop by 20 pairs of shoes.

Economists often study conditions for *market equilibria* of *related markets*. These are conditions under which the prices of various commodities are related. Several market conditions relate the prices of commodities in terms of certain linear equations. The *equilibrium prices* for a set of commodities in a market is a set of prices that satisfy all these equations.

Example 1.3.2. The dollar per pound equilibrium price conditions between three related markets, chicken, pork, and beef, are given by

$$\begin{aligned}
5P_c - P_p - 2P_b &= 1,\\
-2P_c + 6P_p - 3P_b &= 3,\\
-2P_c - P_p + 4P_b &= 10.
\end{aligned}$$

Compute the equilibrium price for each market.

Solution. We solve the system by Gauss elimination to find the equilibrium prices per pound. These are \$3 for chicken, \$4 for pork, and \$5 for beef.

1.3.2 Chemistry: Chemical solutions, balancing of reactions

There are many applications of linear systems to chemistry. Here we discuss chemical solutions and the balancing of chemical reactions.

Chemical solutions

In the following example, we discuss a typical application of linear systems to computing the volumes of reactants in chemical solutions.

Example 1.3.3. It takes three different ingredients A, B, and C, to produce a certain chemical substance. A, B, and C have to be dissolved in water, separately, before they interact to form the chemical. The solution containing A at concentration 1.5 grams per cubic centimeter ($\mathrm{g/cm^3}$) combined with the solution containing B at concentration $3.6\,\mathrm{g/cm^3}$, combined with the solution containing C at concentration $5.3\,\mathrm{g/cm^3}$, makes 25.07 g of the chemical. If the proportions for A, B, C in the above solutions are changed to 2.5, 4.3, $2.4\,\mathrm{g/cm^3}$, respectively (while the volumes remain the same), then 22.36 g of the chemical is produced. Finally, if the proportions are changed to 2.7, 5.5, $3.2\,\mathrm{g/cm^3}$, respectively, then 28.14 g of the chemical is produced. What are the volumes in $\mathrm{cm^3}$ of the solutions containing A, B, and C?

Solution. Let x_1, x_2, x_3 $\mathrm{cm^3}$ be the corresponding volumes of the solutions containing A, B, and C. Then $1.5x_1$ is the mass of A in the first case, $3.6x_2$ is the mass of B, and $5.3x_3$ is the mass of C. These masses add up to 25.07. Hence $1.5x_1 + 3.6x_2 + 5.3x_3 = 25.07$. Similarly, we form the remaining equations to get the system

$$\begin{aligned}
1.5x_1 + 3.6x_2 + 5.3x_3 &= 25.07, \\
2.5x_1 + 4.3x_2 + 2.4x_3 &= 22.36, \\
2.7x_1 + 5.5x_2 + 3.2x_3 &= 28.14
\end{aligned}$$

with solution $x_1 = 1.5$, $x_2 = 3.1$, and $x_3 = 2.2$. So, the requested volumes are, respectively, $1.5\,\mathrm{cm^3}$, $3.1\,\mathrm{cm^3}$, and $2.2\,\mathrm{cm^3}$.

Balancing of chemical reaction

Another application of systems to chemistry is the balancing of a chemical reaction. We need to insert *integer* coefficients in front of each one of the reactants so that the number of atoms of each element is the same on both sides of the equation.

Example 1.3.4. Consider the reaction of the burning of methane

$$x_1\ \mathrm{CH_4} + x_2\ \mathrm{O_2} \to x_3\ \mathrm{CO_2} + x_4\ \mathrm{H_2O}. \tag{1.8}$$

Balance the chemical reaction (1.8) by finding the coefficients x_i.

Solution. We have $x_1 = x_3$, because the number of carbon atoms should be the same on both sides. Likewise, we get the system

$$x_1 = x_3, \quad 4x_1 = 2x_4, \quad 2x_2 = 2x_3 + x_4$$

with solution $x_1 = r/2$, $x_2 = r$, $x_3 = r/2$, $x_4 = r$, $r \in \mathbf{R}$. Choosing $r = 2$ yields

$$\mathrm{CH_4} + 2\ \mathrm{O_2} \to \mathrm{CO_2} + 2\ \mathrm{H_2O}.$$

1.3.3 Physics and engineering: Circuits, heat conduction

Physics and engineering have their share of problems that reduce to solving a linear system. We discuss applications to electrical circuits, to heat conduction, and to weight balancing.

Electrical networks

Suppose we have the following electrical network (Figure 1.17) involving resistors R_i and a battery or generator of voltage E.

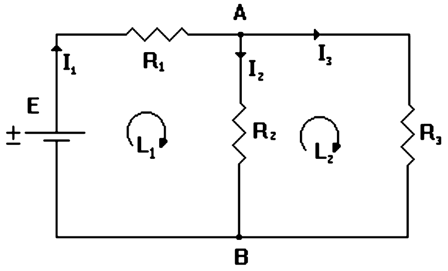


Figure 1.17: An electrical circuit with resistors.

The currents I and the voltage drops V satisfy *Kirchhoff's laws*, namely: (a) *the algebraic sum of all currents at any branch point is zero*, and (b) *the algebraic sum of all voltage changes around a simple loop is zero.*

The voltage drop V due to a resistor is related to the current I and resistance R by *Ohm's law*

$$V = IR.$$

The standard units are

V	Volt	(V)
I	Ampere	(A)
R	Ohm	(Ω)

A typical application is computing the currents, given the voltage of the electromotive force E (usually battery or generator) and the resistances of the resistors.

For each network element, a positive direction is chosen for the current through it. For the voltage source, the positive direction is the one from the negative pole to the positive. The voltage source adds voltage, and hence the voltage change is positive, whereas the voltage change through the resistors is negative due to the voltage drop.

Example 1.3.5. Find the currents I_1, I_2, I_3 in the above electrical circuit (Figure 1.17) if the voltage of the battery is $E = 6\,\mathrm{V}$ and the resistances are $R_1 = R_2 = 2\,\Omega$ and $R_3 = 1\,\Omega$.

Solution. By the current law we have $I_1 - I_2 - I_3 = 0$ from branch point A. Applying the voltage law to loop L_1 yields $6 - I_1R_1 - I_2R_2 = 0$, or $2I_1 + 2I_2 = 6$. Likewise, loop L_2 yields $I_3R_3 - I_2R_2 = 0$, or $2I_2 - I_3 = 0$. Hence

$$\begin{aligned} I_1 - I_2 - I_3 &= 0, \\ 2I_1 + 2I_2 &= 6, \\ 2I_2 - I_3 &= 0, \end{aligned}$$

and by elimination we get $I_1 = 2.25\,\mathrm{A}$, $I_2 = 0.75\,\mathrm{A}$, and $I_3 = 1.5\,\mathrm{A}$.

Heat conduction

Another typical application of linear systems is in heat transfer problems of physics and engineering. One such application requires the determination of the interior temperature of a *lamina* (thin metal plate), given the temperature of its boundary.

There are several ways of approaching this problem, some of which require more advanced mathematics. The approach here is the following approximation: The plate is overlaid by a *grid* or *mesh* (Figure 1.18). The intersections of the mesh lines are the *mesh points*. These are divided into *boundary points* and *interior points*. Given the temperature of the boundary points, we compute the temperature of the interior points according to the following principle.

(Mean value property for heat conduction) The temperature at any interior point is the average of the temperatures of its neighboring points.

It is clear that the finer the grid, the better the approximation of the temperature distribution of the plate.

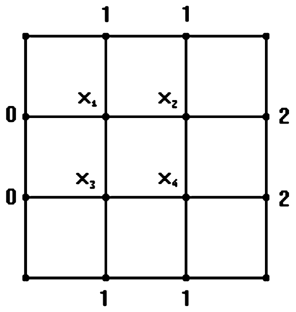


Figure 1.18: A simple heat conduction problem.

Example 1.3.6. Use the mean value property for heat conduction to compute the interior temperatures x_1, x_2, x_3, x_4 of the rectangular lamina of Figure 1.18 with left edge at 0 °C, the right edge at 2 °C, and the top and bottom edges at 1 °C.

Solution. According to the mean value property, we have

$$\begin{aligned} x_1 &= \frac{1}{4}(x_2 + x_3 + 1), \\ x_2 &= \frac{1}{4}(x_1 + x_4 + 3), \\ x_3 &= \frac{1}{4}(x_1 + x_4 + 1), \\ x_4 &= \frac{1}{4}(x_2 + x_3 + 3). \end{aligned}$$

The augmented matrix of the system reduces as

$$\begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & : & \frac{1}{4} \\ -\frac{1}{4} & 1 & 0 & -\frac{1}{4} & : & \frac{3}{4} \\ -\frac{1}{4} & 0 & 1 & -\frac{1}{4} & : & \frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & : & \frac{3}{4} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & : & \frac{3}{4} \\ 0 & 1 & 0 & 0 & : & \frac{5}{4} \\ 0 & 0 & 1 & 0 & : & \frac{3}{4} \\ 0 & 0 & 0 & 1 & : & \frac{5}{4} \end{bmatrix}.$$

Therefore $x_1 = 3/4$, $x_2 = 5/4$, $x_3 = 3/4$, $x_4 = 5/4$ degrees.

> One may also use the symmetry of the mesh to arrive at this answer quickly. (How?)

1.3.4 Traffic flow

Linear systems also arise in the studies of street traffic flow. A *network* consists of (a) a set of points called *nodes* and (b) a set of lines called *branches* (or *edges*) connecting the nodes. In the case of traffic flow, the branches are streets, and the nodes are street junctions.

In a typical network flow problem, one attaches numerical “flow” information to certain branches and seeks numerical flow information on other branches. For example, given the number of vehicles per hour along certain streets, one needs to calculate the number of vehicles per hour along some other streets. To ensure smooth flow it is assumed that:

1. The total flow into the network equals the total flow out;
2. The total flow into each node equals the total flow out.

? Are there any similarities between these conditions and *Kirchhoff’s laws* for electrical networks?

Example 1.3.7. The average rate of hourly traffic volumes for a downtown section consisting of one way streets is given in Figure 1.19. Find the missing amounts of hourly traffic rates x_i.

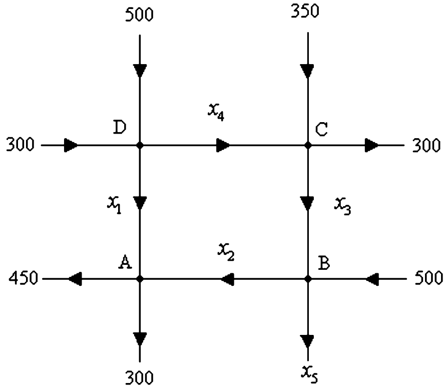


Figure 1.19: Traffic flow.

Solution. The total amount of traffic in $500+350+300+500 = 1650$ must equal the total amount of traffic out $300 + x_5 + 300 + 450 = 1050 + x_5$. Hence $x_5 = 600$. In addition, for each junction, we have

Junction	Traffic in		Traffic out
A	$x_1 + x_2$	=	$450 + 300$
B	$x_3 + 500$	=	$x_2 + x_5$
C	$x_4 + 350$	=	$300 + x_3$
D	$500 + 300$	=	$x_1 + x_4$

This information leads us to the linear system

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & & & & & & & = & 750, \\ & & x_2 & - & x_3 & & & + & x_5 & = & 500, \\ & & & & x_3 & - & x_4 & & & = & 50, \\ x_1 & & & & & + & x_4 & & & = & 800, \\ & & & & & & & & x_5 & = & 600, \end{array}$$

which we solve to get

$$\begin{aligned} x_1 &= 800 - r, \\ x_2 &= r - 50, \\ x_3 &= r + 50, \qquad r \in \mathbf{R}. \\ x_4 &= r, \\ x_5 &= 600, \end{aligned}$$

There are infinitely many solutions. For example, if $r = 400$, then $x_1 = 400$, $x_2 = 350$, $x_3 = 450$, $x_4 = 400$, $x_5 = 600$. There are also restrictions to ensure positive flow. A negative value would violate the one-way traffic assumption. In this case, we need $50 \leq r \leq 800$.

1.3.5 Statics and weight balancing

Let us now study a typical weight balancing lever problem in statics. We use *Archimedes' law of the lever*: *Two masses on a lever balance when their weights are inversely proportional to their distances from the fulcrum.*

Example 1.3.8. Find weights w_1, w_2, w_3, w_4 to balance the levers in Figure 1.20.

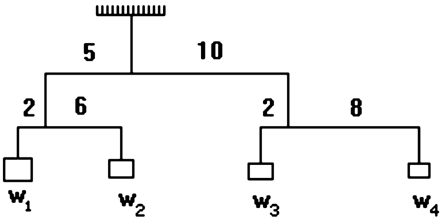


Figure 1.20: Weight balancing.

Solution. To balance the two small levers, according to Archimedes' law, we have $2w_1 = 6w_2$ for the lever on the left and $2w_3 = 8w_4$ for the lever on the right. To balance the main lever we need $5(w_1 + w_2) = 10(w_3 + w_4)$. Hence we have the following homogeneous system of three equations and four unknowns:

$$\begin{aligned} 5w_1 + 5w_2 - 10w_3 - 10w_4 &= 0, \\ 2w_1 - 6w_2 \phantom{{}- 10w_3 - 10w_4} &= 0, \\ 2w_3 - 8w_4 &= 0. \end{aligned}$$

The solution set is a one-parameter infinite solution set described by $w_1 = 7.5r$, $w_2 = 2.5r$, $w_3 = 4r$, and $w_4 = r$, $r \in \mathbf{R}$. Hence, infinitely many weights can balance this system, as expected by experience, provided that the weights are proportional to the numbers 7.5, 2.5, 4, and 1.

Exercises 1.3

1. Toys On Demand Inc. plans to manufacture a new toy train and researches the market for demand. It is found that if the train costs \$120 in a \$90,000 average family income area and if their competitor, Toys Supplies Inc., prices their competing toy train at \$90, then 3,480 trains will be sold. If on the other hand, the price remains the same and Toys Supplies raises their price to \$150 a train, then in a \$120,000 income area, 5,100 trains will be sold. Finally, if the train is priced at \$90 while the competition remains at \$120, then in a \$105,000 income area, 4,590 trains will be sold. Compute the demand function by assuming that it depends linearly on its variables.

2. Balance the reaction for the burning of propane

$$\mathrm{a\ C_3H_8 + b\ O_2 \rightarrow c\ CO_2 + d\ H_2O}.$$

3. It takes three different ingredients *A*, *B*, and *C* to produce a certain chemical substance. *A*, *B*, and *C* have to be dissolved in water, separately, before they interact to form the chemical. The solution containing *A* at 1.5 grams per cubic centimeter (g/cm^3) combined with the solution containing *B* at 1.8 g/cm^3, combined with the solution containing *C* at 3.2 g/cm^3 makes 15.06 g of the chemical. If the proportions for *A*, *B*, *C* in the above solutions are changed to 2.0, 2.5 2.8 g/cm^3, respectively (while the volumes remain the same), then 17.79 g of the chemical is produced. Finally, if the proportions are changed to 1.2, 1.5, 3.0 g/cm^3, respectively, then 13.05 g of the chemical is produced. What are the volumes in cm^3 of the solutions containing *A*, *B*, and *C*?

In Exercises 4–5, find the currents in the electrical circuits (Figures 1.21 and 1.22).

4. Figure 1.21: $E = 6\,\mathrm{V}$, $R_1 = 3$, $R_2 = 2$, $R_3 = 2$, $R_4 = 1$, and $R_5 = 2\,\Omega$.

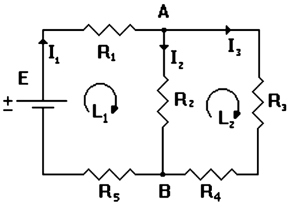


Figure 1.21: Two-loop circuit.

5. Figure 1.22: $E = 6$ V and all resistors are 1 Ω.

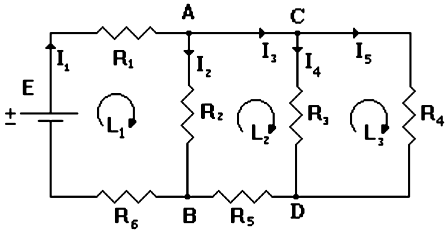

A
C
I_1
R_1
I_2
I_3
I_4
I_5
E
±
L_1
R_2
L_2
R_3
L_3
R_4
R_6
B
R_5
D

Figure 1.22: Three-loop circuit.

In Exercises 6–7, find the temperatures at x_i of the metal plate, given that the temperature of each interior point is the average of its four neighboring points (Figures 1.23 and 1.24).

6. Use Figure 1.23.

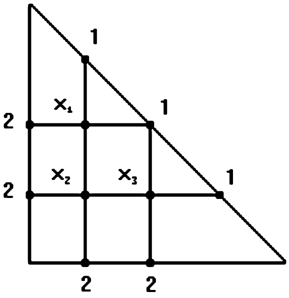

1
x_1
1
2
x_2
x_3
1
2
2
2

Figure 1.23: Heat conduction.

7. Use Figure 1.24.

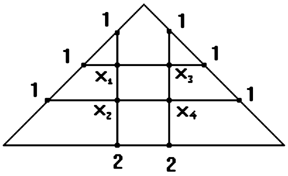

1
1
1
1
1
x_1
x_3
1
x_2
x_4
2
2

Figure 1.24: Heat conduction.

8. Balance the lever-weight system shown in Figure 1.25.

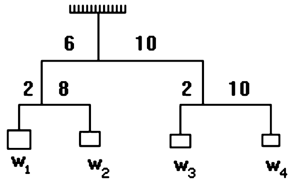


Figure 1.25: Lever-weight system.

9. The average rate of hourly traffic volumes for a downtown section consisting of one way streets is given in Figure 1.26. Find the missing amounts of hourly traffic rates x_i.

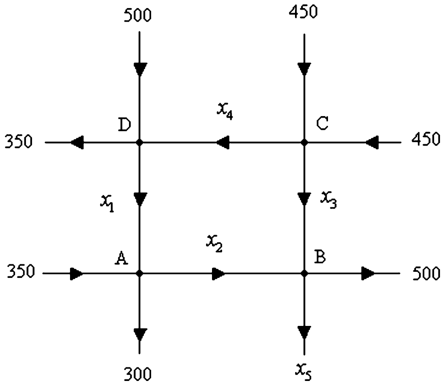


Figure 1.26: Traffic flow.

10. Find the equation of the cubic curve $y = ax^3 + bx^2 + cx + d$ in the xy-plane passing through the points $P(1,1)$, $Q(-1,5)$, $R(0,1)$, $S(-2,7)$.

11. Find an equation $ax + by + cz = d$ for the plane passing through the points $P(1,1,-1)$, $Q(2,1,2)$, $R(1,3,-5)$.

12. Use linear systems to compute constants A, B, C, and D in the following partial fractions decomposition:

$$\frac{1}{(x+1)(x-2)(x-3)(x-4)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-3} + \frac{D}{x-4}.$$

13. Suppose the numbers of bacteria of types A and B interdepend on each other according to the following experimental table. Is there a linear relation between A and B?

A	B
500	500
1,000	2,000
5,000	14,000
10,000	29,000

14. A business group needs, on average, fixed amounts of Japanese yen, French francs, and German marks during each of their business trips. They traveled three times this year. The first time they exchanged a total of \$2,400 at the following rates: the dollar was 100 yen, 1.5 francs, and 1.2 marks. The second time they exchanged a total of \$2,350 at the following rates: the dollar was 100 yen, 1.2 francs, and 1.5 marks. The third time they exchanged a total of \$2,390 at the following rates: the dollar was 125 yen, 1.2 francs, and 1.2 marks. What were the amounts of yen, francs, and marks that they bought each time?

15. *(The Fibonacci money pile problem).* Three men possess a single pile of money, their shares being 1/2, 1/3, and 1/6. Each man takes some money from the pile until nothing is left. The first man then returns 1/2 of what he took, the second man 1/3, and the third 1/6. When the total so returned is divided equally among the men, it is discovered that each man then possesses what he is entitled to. How much money was there in the original pile and how much did each man take from the pile? Use x, y, and z for each share and w for the pile.[5]

1.4 Numerical solutions of linear systems

Linear systems used in applications typically consist of hundreds or thousands of equations and unknowns. Solution methods require efficient computer programs. Although elimination is an important theoretical method of solving a linear system, in practice it is rarely used. We discuss two numerical methods that are widely used to find approximate solutions of linear systems.

First, we compare Gauss and Gauss–Jordan elimination discussed in Section 1.2. These are cases of *direct methods*, meaning that the solution is obtained in finitely many steps. In fact, the number of steps can be well estimated.

1.4.1 Computational efficiency of row reduction

In Section 1.2, we emphasized the use of Gauss elimination over Gauss–Jordan elimination. The reason is that Gauss–Jordan elimination, though seemingly more efficient (there is no backward pass), requires more arithmetic operations. In fact, for a system of n equations in n unknowns, it can be shown that for large n, Gauss elimination requires approximately $2n^3/3$ arithmetic operations and Gauss–Jordan elimination requires approximately n^3 operations. So for a medium-size system, say of 500 equations with 500 unknowns, Gauss–Jordan elimination requires approximately 125 million operations, whereas Gauss elimination requires only about 83 million operations. This is mainly why Gauss elimination is preferred. However, Gauss–Jordan elimination is favored in *parallel* computing, where it is slightly more efficient than Gauss elimination.

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[5] This problem was one of several in a mathematical competition set by Emperor Frederick II of Sicily. Several scholars were invited to solve these mathematical problems. One such scholar was Leonardo of Pisa (1175–1250), better known as *Fibonacci*. During his travels, Fibonacci learned the Arabic “new arithmetic”, which he later introduced to the West in his famous book *Liber abaci*. It is known that Fibonacci found the particular solution $w = 47$, $x = 33$, $y = 13$, and $z = 1$. Professor W. David Joyner introduced the author to this problem.

> In Section 1.5, it is explained why for large n, Gauss elimination requires approximately $2n^3/3$ arithmetic operations.

1.4.2 Iterative methods

In addition to the direct methods, we also have *iterative methods*, where we approximate the solution of a system by using iterations, starting with an initial guess. If the successive iterations approach the solution, then we say that the iteration *converges*. Otherwise, we say that it *diverges*. The procedure ends when two consecutive iterations yield the same answer within a desired accuracy. Unlike the direct methods, the number of steps needed is not known beforehand. We discuss two iterative methods, the *Jacobi iteration* and the *Gauss–Seidel iteration*.

1.4.3 Jacobi iteration

The Jacobi iteration applies to *square systems* as follows. We have a system with n equations in n unknowns $x_1, \dots, x_n$, such as

$$\begin{aligned} 5x + y - z &= 14, \\ x - 5y + 2z &= -9, \\ x - 2y + 10z &= -30. \end{aligned} \tag{1.9}$$

STEP 1. Solve the ith equation of the system for x_i:

$$\begin{aligned} x &= -0.2y + 0.2z + 2.8, \\ y &= 0.2x + 0.4z + 1.8, \\ z &= -0.1x + 0.2y - 3.0. \end{aligned} \tag{1.10}$$

STEP 2. Start with an initial guess $x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)}$ for the solution. In the absence of any information, we initialize all variables at zero: $x_1^{(0)} = 0$, $x_2^{(0)} = 0, \dots, x_n^{(0)} = 0$.

In the example, we let $x^{(0)} = 0$, $y^{(0)} = 0$, and $z^{(0)} = 0$.

STEP 3. Substitute the values $x_1^{(k-1)}, x_2^{(k-1)}, \dots, x_n^{(k-1)}$ obtained after the $(k-1)$th iteration into the right side of (1.10) to obtain the new values $x_1^{(k)}, x_2^{(k)}, \dots, x_n^{(k)}$.

In the example, the substitution $x = 0, y = 0, z = 0$ on the right side of (1.10) yields $x = 2.8, y = 1.8, z = -3$. Then substitution of these new values back into the right side of (1.10) again yields $x = 1.84, y = 1.16, z = -2.92$. We iterate again.

STEP 4. Stop the process when a desired accuracy has been achieved. Usually, one stops when two consecutive iterations yield the same values up to this accuracy.

In the example, we iterated using accuracy to four decimal places and stopped when two consecutive answers were the same.

Iteration	x	y	z
Initial guess	0.0000	0.0000	0.0000
1	2.8000	1.8000	−3.0000
2	1.8400	1.1600	−2.9200
3	1.9840	1.0000	−2.9520
4	2.0096	1.0160	−2.9984
5	1.9971	1.0026	−2.9978
6	1.9999	1.0003	−2.9992
7	2.0001	1.0003	−2.9999
8	2.0000	1.0001	−3.0000
9	2.0000	1.0000	−3.0000
10	2.0000	1.0000	−3.0000

The iterations suggest that $x = 2$, $y = 1$, and $z = -3$ is the solution of the system, correct at least to four decimal places. In fact, this is the exact solution in this case.

1.4.4 Gauss–Seidel iteration

The Gauss–Seidel iteration also applies to square systems. We have

STEP 1. The same as in the Jacobi iteration.

STEP 2. The same as in the Jacobi iteration.

STEP 3. Substitute the *most recently calculated* unknown into the right side of the equations obtained in Step 1 to get the new approximation $x_i^{(k)}$.

In the example, substitution of $y = 0, z = 0$ into the first equation yields $x = 2.8$. In the second equation, we substitute $z = 0$ and $x = 2.8$ (the most recent value of x) to get $y = 2.36$. In the third equation, we substitute $x = 2.8$ and $y = 2.36$ (the latest x and y) to get $z = -2.808$. We iterate again.

STEP 4. The same as in the Jacobi iteration.

In the example, we get

Iteration	x	y	z
Initial guess	0.0000	0.0000	0.0000
1	2.8000	2.3600	−2.8080
2	1.7664	1.0301	−2.9706
3	1.9999	1.0117	−2.9976
4	1.9981	1.0006	−2.9997
5	1.9999	1.0001	−3.0000
6	2.0000	1.0000	−3.0000
7	2.0000	1.0000	−3.0000

The essential difference between the two methods is that in the Jacobi iteration we update all variables simultaneously, whereas in Gauss-Seidel we update each unknown just when its new value becomes available (Figure 1.27).

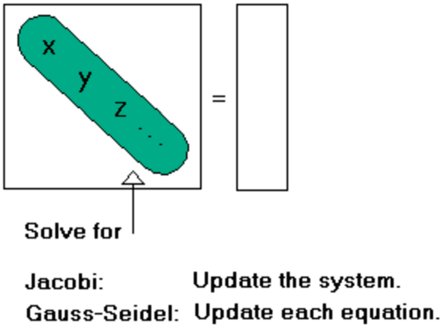
x
y
z
. . .
=
Solve for
Jacobi: Update the system.
Gauss-Seidel: Update each equation.

Figure 1.27: Jacobi compared with Gauss–Seidel.

Note that the Gauss–Seidel iteration required fewer iterations than the Jacobi iteration. This *appears* to be true in most cases, but it is not always true.

i It is not known in advance which of Jacobi or Gauss–Seidel is more efficient.

1.4.5 Convergence

A sufficient condition for the convergence of the Jacobi and Gauss–Seidel iterations is when the coefficient matrix of the system is *diagonally dominant*. This means that (a) the matrix is *square* and (b) each diagonal entry has absolute value larger than the sum of the absolute values of the other entries in the same row.

For example, system (1.9) has the coefficient matrix

$$\begin{bmatrix} 5 & 1 & -1 \\ 1 & -5 & 2 \\ 1 & -2 & 10 \end{bmatrix},$$

which is diagonally dominant because $|5| > |1| + |-1|$, $|-5| > |1| + |2|$, and $|10| > |1| + |-2|$. So we are guaranteed that both iterations will converge in this case.

The following matrix is not diagonally dominant:

$$\begin{bmatrix} 4 & 2 & -1 \\ 3 & -5 & 2 \\ 1 & -2 & 10 \end{bmatrix}.$$

The Jacobi and Gauss–Seidel iterations may converge, even if the coefficient matrix of the system is not diagonally dominant.

Sometimes, a rearrangement of equations may result in a diagonally dominant coefficient matrix. For example, the system

$$\begin{aligned} 2x + 4y - z &= 1, \\ 5x - y + 2z &= 2, \\ x - 2y + 10z &= 3 \end{aligned}$$

has the coefficient matrix that is not diagonally dominant. However, if we interchange the first and second equations, then the new coefficient matrix is diagonally dominant.

1.4.6 Comparison of elimination and Gauss–Seidel iteration

Let us see how our seemingly most efficient direct and iterative methods compare with each other. It can be shown that for large n, the Gauss–Seidel method requires approximately $2n^2$ arithmetic operations per iteration. If fewer than $n/3$ iterations are used, then the total amount of operations will be fewer than $2n^3/3$, and Gauss–Seidel iteration will be more efficient than Gauss elimination. For a square system of 500 equations, fewer than 166 iterations make Gauss–Seidel iteration a better choice.

Often in practice Gauss–Seidel iteration is preferred over Gauss elimination, even if more operations are used. Some of the reasons are:

1. During Gauss elimination the computer round-off errors accumulate and affect the final answer with each elementary row operation. In Gauss–Seidel iteration, there is only one round-off error, which is due to the last iteration. Indeed, the iteration before the last can be viewed as an excellent initial guess!
2. An additional virtue of Gauss–Seidel is that it is a *self-correcting* method. If at any stage there was a miscalculation, then the answer is still usable; it is simply considered as a new initial guess.
3. Both Jacobi and Gauss–Seidel iterations are excellent choices when the coefficient matrix is *sparse*, i. e., if it has many zero entries. This is because the same coefficients are used in each stage, so the zeros remain throughout the process.

1.4.7 Numerical considerations: Ill-conditioning and pivoting

Some systems exhibit behavior that requires a careful numerical analysis. Consider the almost identical systems

$$\begin{aligned} x + y &= 1, \\ 1.01x + y &= 2, \end{aligned} \quad \text{and} \quad \begin{aligned} x + y &= 1, \\ 1.005x + y &= 2. \end{aligned}$$

The exact solution of the first one is $x = 100, y = -99$, whereas the solution of the second is $x = 200, y = -199$. So a small change in the coefficients resulted into a dramatic change in the solution. Such a system is called *ill-conditioned*.

If, for example, we use floating-point arithmetic with accuracy of two decimal places and rounding up, then the second system becomes identical to the first one, so its our approximate solution yields an error of about 50 %. The reason for such a behavior is that the two lines defined by the first system are almost parallel. So a small change in the slope of one may move the intersection point quite some distance (Figure 1.28).

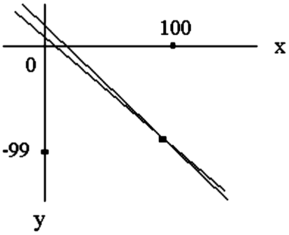

Figure 1.28: Almost parallel lines of an ill-conditioned system.

Another type of problem occurs when we use Gauss elimination with floating-point arithmetic and the entries of the augmented matrix of a system have vastly different sizes.

For example, consider the system

$$\begin{aligned} 10^{-3}x + y &= 2, \\ 2x - y &= 0. \end{aligned}$$

It is easy to see that the exact solution is $x = 2000/2001$ and $y = 4000/2001$. Suppose now we solve the system numerically, but we can only use floating-point arithmetic to three significant digits.

Solution 1.

$$\begin{bmatrix} 10^{-3} & 1 & 2 \\ 2 & -1 & 0 \end{bmatrix} \boxed{R_2 - 2 \cdot 10^3 R_1 \to R_2} \begin{bmatrix} 10^{-3} & 1 & 2 \\ 0 & \boxed{-2 \cdot 10^3} & -4 \cdot 10^3 \end{bmatrix}.$$

The actual (2, 2) entry of the last matrix was –2001, which was rounded to –2000, because we are working with 3 significant digits. The remaining of the reduction is as usual:

$$\sim \begin{bmatrix} 10^{-3} & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 10^{-3} & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

So we get $x = 0$ and $y = 2$. We see that the approximation for x is quite poor.

Solution 2. Suppose now we interchange equations 1 and 2 and scale the first row to get a leading 1:

$$\begin{bmatrix} 2 & -1 & 0 \\ 10^{-3} & 1 & 2 \end{bmatrix} \boxed{1/2R_1 \to R_1} \begin{bmatrix} 1 & -1/2 & 0 \\ 10^{-3} & 1 & 2 \end{bmatrix}.$$

Then

$$\boxed{R_2 - 10^{-3}R_1 \to R_2} \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & \boxed{1} & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The actual (2, 2) entry of the last matrix was $1 + (1/2) \cdot 10^{-3}$, which simplifies to 1 in our arithmetic. Hence $x = 1$ and $y = 2$, a much better approximation this time.

Let us explain what went wrong during the first solution. The small coefficient 10^{-3} at the first pivot position forced large coefficients in the second row, which resulted in a slight error for y due to rounding. This small error, however, caused a substantial error in estimating x in $10^{-3}x+y = 2$, because the coefficient of x was overpowered by that of y.

The second solution did not suffer from this problem, because the row with the larger size leading entry was moved to the pivot position. So elimination did not yield large coefficients, and although y suffered from the same small rounding error, the value of x was only slightly affected.

In practice, during Gauss elimination, we always move the row with the *largest absolute value* leading entry to the pivot position before we eliminate. This is called *partial pivoting*, and it helps us keep the round-off errors under control. There is also a variant where we pick the largest size entry in the entire matrix as pivot. This forces interchanging of *columns* in addition to rows, which means we have to change the variables as well. This method is called *full pivoting* and yields better numerical results, but it can be quite slow. Partial pivoting is the most popular modification of Gauss elimination.

Exercises 1.4

In Exercises 1–2, rewrite the system so that its coefficient matrix *is* diagonally dominant.

1. $x - 2y = -6,$
$5x + y = 14.$

2. $x + y + 5z = 15,$
$-x + 5y + z = -9,$
$5x + y - z = 5.$

3. Use Jacobi's method with 4 iterations and initial values $x = 0, y = 0$ to approximate the solution of the system

$$\begin{aligned} 5x + y &= 14, \\ x - 2y &= -6. \end{aligned}$$

Compare your answer with the exact solution.

4. Repeat Exercise 3 with Gauss–Seidel iteration.

In Exercises 5–7, find approximate solutions of the system using Jacobi's method with four iterations. Initialize all variables at 0.

5.
$$\begin{aligned} 7x - z &= 9, \\ -x + 4y &= 19, \\ y - 9z &= 23. \end{aligned}$$

6.
$$\begin{bmatrix} 6 & 1 & 1 & 1 \\ 1 & 6 & 1 & 1 \\ 1 & 1 & 6 & 1 \\ 1 & 1 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 15 \\ 30 \\ 20 \\ 25 \end{bmatrix}.$$

7.
$$\begin{aligned} 5x + y - z &= 5, \\ -x + 5y + z &= -9, \\ x + y + 5z &= 15. \end{aligned}$$

In Exercises 8–9, find approximate solutions of the system using the Gauss–Seidel method with four iterations. Initialize all variables at 0.

8. The system of Exercise 5.

9. The system of Exercise 7.

10. The system of Exercise 6.

11. The coefficient matrices of the systems below are not diagonally dominant. Apply Gauss–Seidel iteration initializing $x = 0, y = 0$ and 5 iterations. Prove that (a) the iteration for the first system *diverges* and (b) the iteration for the second system *converges* to 2 decimal places i. e., that the difference between the last two iterates of each variable is < 0.005).

$$\text{(a)} \quad \begin{aligned} x - y &= 2, \\ x + y &= 0; \end{aligned} \qquad \text{(b)} \quad \begin{aligned} 4x - y &= -3, \\ x + y &= 0. \end{aligned}$$

In Exercises 12–14, use partial pivoting in Gauss elimination to solve the system. Use four significant digit arithmetic.

12.
$$\begin{aligned} x - 3y &= -11, \\ 10x + 5y &= 30. \end{aligned}$$

13.
$$\begin{aligned} 1.2x - 4.5y &= -1.23, \\ -5.5x + y &= -15.95. \end{aligned}$$

14.
$$\begin{aligned} x + 2y + 2z &= 6, \\ 2x + 4y + z &= 9, \\ 8x + 2y + z &= 19. \end{aligned}$$

15. (*Scaling*) In the following system, all the coefficients of x are of different order of magnitude than the rest. In such cases the calculations are simplified if we scale the variable. In this case, let $x' = 0.001x$. Write the system in the variables x', y, z and solve it using Gauss elimination. Then compute x.

$$
\begin{aligned}
0.004x + y - z &= 15.8,\\
0.001x + 5y + z &= 14.2,\\
0.001x + y + 5z &= -29.8.
\end{aligned}
$$

1.5 Miniprojects

1.5.1 Psychology: Animal intelligence

A set of experiments in psychology deal with the study of teaching tasks to various animals such as pigs, rabbits, rats, etc. One such experiment involves the search for food. An animal is placed somewhere in a square mesh of corridors that may lead to food (points labeled 1) or to a dead-end (points labeled 0) (Figure 1.29).

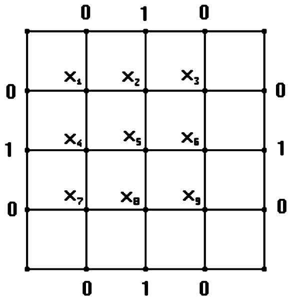

Figure 1.29: Animal intelligence experiments.

It is assumed that the probability for an animal to occupy position x_i is the *average* of the probabilities of occupying the neighboring positions, directly above, below, to the left, and to the right of it. If a neighboring position is one with food, then this being success, its probability is 100 % = 1. If a neighboring position is dead-end, then this being failure, its probability is 0 % = 0. For example, for Figure 1.29, we have

$$
\begin{aligned}
x_1 &= \frac{1}{4}(0 + 0 + x_4 + x_2),\\
x_2 &= \frac{1}{4}(1 + x_1 + x_5 + x_3),\\
x_3 &= \frac{1}{4}(0 + x_2 + x_6 + 0), \quad \ldots .
\end{aligned}
$$

(a) Compute the probabilities x_i, $i = 1, \dots, 9$, for Figure 1.29.

(b) Compare this type of problem with heat conduction problems studied in Section 1.3. Write a few sentences on this comparison.

! Exploiting any *symmetries* of the data may help avoid lengthy calculations.

1.5.2 Counting operations in Gauss elimination

The following pseudocode performs the forward pass of Gauss elimination for a linear system $A\mathbf{x} = \mathbf{b}$ where A is $n \times n$ with $a_{ii} \neq 0$, i. e., where row interchanges are not needed. This code allows for the counting of the exact number of numerical operations in the forward pass.

```
The Forward Pass of Gauss Elimination
Input: nxn matrix A=[aij] and n-vector b.
for i = 1 to n - 1
  for j = i+1 to n
  m := aji/aii
  aji := 0
  bj := bj - m*bi
    for k = i + 1 to n
      ajk := ajk - m*aik
    end for
  end for
end for
Output: An echelon form of matrix [A:b]
```

Prove that the total number of

(a) multiplications is $\frac{1}{3}n(n^2 - 1)$,

(b) subtractions is $\frac{1}{3}n(n^2 - 1)$,

(c) divisions is $\frac{1}{2}n(n - 1)$,

(d) multiplications or divisions is $\frac{1}{6}n(n - 1)(2n + 5)$.

For large n, the terms with n^3 are dominant, and only these count. So there are approximately $\frac{2n^3}{3}$ operations in the forward pass. The same estimate is used for a complete Gauss elimination because the back-substitution stage only contributes terms with n^2 or fewer powers of n for the number of operations.

1.5.3 Archimedes' cattle problem

This is a famous problem sent by the ancient Greek mathematician Archimedes of Syracuse to Eratosthenes in Alexandria. Its original form was a collection of epigrams in ancient Greek (Figure 1.30).[6]

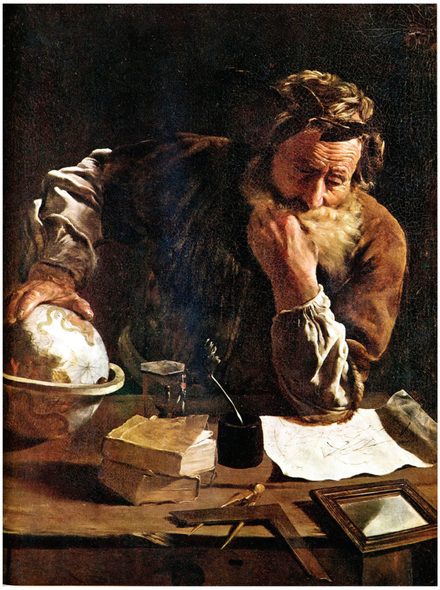

Figure 1.30: Archimedes of Syracuse.
Painting by Domenico Fetti 1640, Public Domain https://commons.wikimedia.org/w/index.php?curid=146592.
Archimedes (c. 287–212 BCE) is considered as one of the greatest mathematicians of all time. He lived in Syracuse, a Greek settlement in Sicily. He studied mathematics in Alexandria, Egypt. Among his many accomplishments were the computations of the volume and the surface area of the sphere. His work covers geometry, hydrostatics, physics, and engineering. He was killed by a Roman soldier during the capture of Syracuse by the Roman general Marcellus. Archimedes had helped defend the city by employing military machines that he had invented.

We outline the part of the problem that is relevant to the current project. For an authoritative translation, see Sir Thomas L. Heath's *The Works of Archimedes* (Dover Edition, 1953), pp. 319–326. The epigrams start as follows:

Compute, O stranger, the number of the oxen of the Sun which once grazed upon the fields of the Sicilian isle of Thrinacia and which were divided, according to color, into four herds, one white, one black, one yellow, and one dappled…

Then the manuscript goes on to describe the relations between the cows and the bulls of the four herds. Let W, B, D, Y be the numbers of the bulls in the white, black, dappled, and yellow herds, respectively. Likewise, let w, b, d, y be the numbers of the cows in the same order. Then $W + w, B + b, D + d, Y + y$ are the numbers of the oxen in the white, black, dappled, and yellow herds, respectively. The manuscript gives us the following relationships for the bulls and cows:

---

[6] It is generally believed that Archimedes worked on this problem, but it is not known whether he is the author of it.

$$\begin{aligned} W &= \left(\frac{1}{2}+\frac{1}{3}\right)B + Y, & w &= \left(\frac{1}{3}+\frac{1}{4}\right)(B+b),\\ B &= \left(\frac{1}{4}+\frac{1}{5}\right)D + Y, & b &= \left(\frac{1}{4}+\frac{1}{5}\right)(D+d),\\ D &= \left(\frac{1}{6}+\frac{1}{7}\right)W + Y, & d &= \left(\frac{1}{5}+\frac{1}{6}\right)(Y+y),\\ & & y &= \left(\frac{1}{6}+\frac{1}{7}\right)(W+w). \end{aligned}$$

Solve this system of seven equations with eight unknowns. The system is homogeneous, with more unknowns than equations, so there are infinitely many solutions. Prove that the smallest integer solution is given by

	Bulls	**Cows**
White	10,366,482	7,206,360
Black	7,460,514	4,893,246
Yellow	4,149,387	5,439,213
Dappled	7,358,060	3,515,820
Total		50,389,082

Can the reader imagine performing these computations “by hand”?

i

The cattle problem has a much more advanced part that involves number theory. The ancient manuscript continues giving formation conditions of the herds that result in eight integers, each having 206,545 digits. After efforts that started in 1880, a complete solution, generated by a Cray 1 supercomputer in 1965, was published by Williams, German, and Zarnke. In 1998, Vardi developed explicit formulas to generate solutions using the *Wolfram Mathematica* software.

1.6 Technology-aided problems and answers

1. Solve the system numerically. Display both in default and in higher accuracy. If your program supports rational arithmetic, then find the exact answer. Finally, verify your answer.

$$\begin{aligned} \frac{1}{5}x + \frac{1}{6}y + \frac{1}{7}z &= \frac{241}{1260},\\ \frac{1}{6}x + \frac{1}{7}y + \frac{1}{8}z &= \frac{109}{672},\\ \frac{1}{7}x + \frac{1}{8}y + \frac{1}{9}z &= \frac{71}{504}. \end{aligned} \tag{1.11}$$

2. Solve the following system for x and y:

$$\begin{aligned} a_1x + b_1y &= c_1,\\ a_2x + b_2y &= c_2. \end{aligned}$$

3. Enter the augmented matrix of system (1.11) and find (a) a row echelon (if available) and (b) the reduced echelon form. What is the solution of the system?

4. Let A be the coefficient matrix of system (1.11). Is A row equivalent to

$$B = \begin{bmatrix} 1 & 5 & 3 \\ 6 & 2 & 4 \\ 2 & 1 & 7 \end{bmatrix}?$$

5. Consider the following system. Use your program to prove that if $c = -250/3$, then the system has infinitely many solutions. If, on the other hand, $c \neq -250/3$, then the system has no solutions.

$$\begin{aligned} \frac{1}{5}x - \frac{1}{6}y &= 100, \\ -\frac{1}{6}x - \frac{5}{36}y &= c. \end{aligned}$$

6. For matrix B of Exercise 4, use your software to display the matrix consisting of

1. The first column;
2. The second row;
3. The first two columns;
4. The last two rows;
5. $\begin{bmatrix} 1 & 5 \\ 6 & 2 \end{bmatrix}$.

7. If your program supports random numbers, then generate and solve a random system of three equations and three unknowns. If you repeat this several times, then do you mostly get consistent or inconsistent systems?

8. Use your program to sketch the lines defined by a system of two equations and two unknowns on the same graph.

9. Use your program to sketch the planes defined by a system of three equations and three unknowns on the same graph.

10. Find the temperatures of the nine interior points of a square plate that has been subdivided by three equally spaced parallel vertical lines and three equally spaced parallel horizontal lines. Assume that the two vertical sides of the square are kept at 85 degrees while the two horizontal sides are at 110 degrees. Use the mean value property for heat conduction.

11. Write a program that solves a square linear system by Jacobi's iteration. Use your program to find approximate solutions of the linear system. Use with six iterations. Initialize all variables at 0. Make a guess on what the exact solution is.

$$\begin{bmatrix} 8 & 1 & 1 & 1 \\ 1 & 8 & 1 & 1 \\ 1 & 1 & 8 & 1 \\ 1 & 1 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -16 \\ 19 \\ -30 \end{bmatrix}$$

12. Repeat Exercise 11 by using Gauss–Seidel.

1.6.1 Selected solutions with Mathematica

```
(* EXERCISE 1 - Partial *)
sys={1/5 x+1/6 y+1/7 z == 241/1260, 1/6 x+1/7 y+1/8 z == 109/672,
      1/7 x+1/8 y+1/9 z == 71/504}    (* Assigning a name to the system. *)
Solve[sys, {x,y,z}]                   (* Rational number arithmetic.     *)
N[%]                (* Evaluation of the last output to default accuracy. *)
N[%%,15]                              (* Evaluation to higher accuracy.   *)
NSolve[sys, {x,y,z}]                  (* A one-step alternative.          *)
Solve[{1./5 x+1/6 y+1/7 z==241/1260,  (* Also, forcing floating point     *)
1/6 x+1/7 y+1/8 z == 109/672,         (* arithmetic with 1./5 .           *)
1/7 x+1/8 y+1/9 z == 71/504},{x,y,z}]
LinearSolve[{{1/5,1/6,1/7},{1/6,1/7,1/8},  (* LinearSolve with the coeff. *)
{1/7,1/8,1/9}},{241/1260,109/672,71/504}]  (* matrix and constant vector. *)
(* EXERCISE 2  *)
Solve[{a1 x + b1 y == c1, a2 x + b2 y == c2},{x,y}]
Simplify[%]                            (* Answer needs simplification.    *)
(* EXERCISE 3  *)
m={{1/5,1/6,1/7,241/1260},{1/6,1/7,1/8,109/672},{1/7,1/8,1/9,71/504}}
RowReduce[m]  (* The reduced row echelon form. The soln is the last coln.*)
(* EXERCISE 4  *)
A={{1/5,1/6,1/7},{1/6,1/7,1/8},{1/7,1/8,1/9}}
B = {{1,5,3},{6,2,4},{2,1,7}}
RowReduce[A]                       (* The 2 reduced echelon forms are     *)
RowReduce[B]                       (* the same, so A and B are equivalent.*)
(* EXERCISE 6  *)
B[[All, 1]] // MatrixForm                                     (* Column 1*)
B[[2]]                                                           (* Row 2*)
B[[All,1;;2]] // MatrixForm                          (* First two columns *)
B[[2;;3,All]] // MatrixForm                              (* Last two rows *)
B[[1;;2,1;;2]] // MatrixForm                                (* submatrix. *)
(* EXERCISE 7 - Hint  *)
Random[]                                        (* A random real in [0,1]. *)
(* EXERCISE 8 - Hint  *)
Plot[2*x-1, {x,0,4}]              (* Plots 2x-1 as x varies from 0 to 4.  *)
Plot[{2*x-1,x+2}, {x,0,4}]        (* Plots 2x-1 and x+2 on the same graph.*)
(* EXERCISE 9 - Hint  *)
p1=Plot3D[x-y, {x,-3,3},{y,-2,2}]     (* 3D plot of x-y on [-3,3]x[-2,2]. *)
p2=Plot3D[x+y, {x,-3,3},{y,-2,2}]       (* A second plot.                 *)
Show[{p1,p2}]                           (* Displayed together.            *)
(* Exercise 11. *)
JacobiIteration[A_, b_, x0_] := Module[{n, r, i, j, xnew, xcurrent},
n = Length[b]; xcurrent = N[x0];
r = 0;
While[r <= 6, xnew = b;
 For[i = 1, i <= n, i++,
  For[j = 1, j <= n, j++,
   If[i != j, xnew[[i]] = xnew[[i]] - A[[i, j]]*xcurrent[[j]]]];
     xnew[[i]] = xnew[[i]]/A[[i, i]]] ;
 xcurrent = xnew;
 Print[xcurrent];
r++];
xcurrent];
(* Then type *)
A = {{8, 1, 1, 1}, {1, 8, 1, 1}, {1, 1, 8, 1}, {1, 1, 1, 8}}
b = {5, -16, 19, -30}
x0 = {0, 0, 0, 0}
JacobiIteration[A, b, x0]   (* Guess for exact solution: 1,-2,3,-4. *)
```

1.6.2 Selected solutions with MATLAB

```
% EXERCISE 1 - Partial
A = [1/5 1/6 1/7; 1/6 1/7 1/8; 1/7 1/8 1/9]  % To solve a square system,
b = [241/1260; 109/672; 71/504]   % form the coefficient matrix A, then
A\b                               % the constant vector b and type A\b.
format long                       % For higher displayed accuracy, use
ans                               % long format and call the last output.
format short                      % Back to short format.
linsolve(A,b)                     % We may also use linsolve (ST).
% EXERCISE 3
m=[1/5 1/6 1/7 241/1260;1/6 1/7 1/8 109/672; 1/7 1/8 1/9 71/504]
rref(m)                 % The RREF. The last column is the solution.
% EXERCISE 4
B=[1 5 3; 6 2 4; 2 1 7]        % Matrix A was entered in Exer. 1.
rref(A)                        % The 2 reduced echelon forms are
rref(B)                        % the same,  so A and B are equivalent.
% EXERCISE 6
B(:,1)                         % Column 1.
B(2,:)                         % Row 2.
B(:,1:2)                       % Columns 1 and 2.
B(2:3,:)                       % Rows 2 and 3.
B(1:2,1:2)                     % Upper left 2-block.
% EXERCISE 7 - Hint
rand                           % A random real in [0 1].
% Also related: randn .
% EXERCISE 8 - Hint
x = 0:.1:4;                  % Define an x-vector.
y1 = 2*x-1; y2 = x+2;        % then apply the functions to get the y-vectors
plot(x,y1,x,y2)              % and plot.
% EXERCISE 9 - Hint
x = -3:1/4:3;        % To plot x-y and x+y on [-3,3]x[-2,2] on the same graph:
y = -2:1/6:2;      % Create vectors for the x- and y-coordinates of the points.
[X,Y]=meshgrid(x,y); % Builds an array for x and y suitable for 3-d plotting.
Z=[X-Y,X+Y];                 % Define Z in terms of the two functions in
mesh(Z);                     % X and Y and use mesh to plot.
% Related:  Explore the command linspace!
% Exercise 11.
function [B] = JacobiIteration(A,b,x0)
% system A x = b, starting at the vector x0 .
[n,m] = size(A); xcurrent=x0; r=0;
while r <= 6
xneu = b;
 for i = 1:n
  for j = 1:n
   if i~=j
     xneu(i) = xneu(i)-A(i,j)*xcurrent(j);
    else
   end;
  end;
 xneu(i) = xneu(i)/A(i,i);
 end
xcurrent = xneu
r = r + 1;
end
B=xcurrent;
% Then type
A = [8 1 1 1; 1 8 1 1; 1 1 8 1; 1 1 1 8]
b=[5 -16 19 -30]
x0=[0 0 0 0]
JacobiIteration(A, b, x0) % Guess for exact solution: 1,-2,3,-4.
```

1.6.3 Selected solutions with Maple

```
# EXERCISE 1 - Partial
sys:={1/5*x+1/6*y+1/7*z =241/1260, 1/6*x+1/7*y+1/8*z = 109/672,
1/7*x+1/8*y+1/9*z = 71/504};      # Assigning a name to the system.
solve(sys, {x,y,z});                  # Rational number arithmetic.
evalf(%);        # Evaluation of the last output to default accuracy.
evalf(%%,15);                         # Evaluation to higher accuracy.
fsolve(sys, {x,y,z});                 # A one-step alternative.
solve({1./5*x+1/6*y+1/7*z=241/1260,     # Also, forcing floating point
1/6*x+1/7*y+1/8*z = 109/672,          # arithmetic with 1./5 .
1/7*x+1/8*y+1/9*z = 71/504},{x,y,z});
# Another way of solving a linear system is using linsolve. First load
with(LinearAlgebra);                  # the linear algebra package
A:=<<1/5|1/6|1/7>,<1/6|1/7|1/8>,<1/7|1/8|1/9>>; # Then use the
b:=<241/1260,109/672,71/504>;                # coeff. matrix
LinearSolve(<A|b>);                          # and constant vector.
# EXERCISE 2
solve({a1*x + b1*y = c1, a2*x + b2*y = c2},{x,y});
# EXERCISE 3
m:=<<1/5|1/6|1/7|241/1260>,<1/6|1/7|1/8|109/672>,<1/7|1/8|1/9|71/504>>;
GaussianElimination(m);              # Gauss Elimination; A row echelon form.
```

```
ReducedRowEchelonForm(m);              # The reduced row echelon form.
# EXERCISE 4
A:=Matrix([[1/5,1/6,1/7],[1/6,1/7,1/8],[1/7,1/8,1/9]]); # Another syntax
B:=Matrix([[1,5,3],[6,2,4],[2,1,7]]);                   # for matrix.
ReducedRowEchelonForm(A);            # The two reduced echelon forms are
ReducedRowEchelonForm(B);            # the same, so A and B are equivalent.
# EXERCISE 6
Column(B,1); Row(B,2); SubMatrix(B, [1..3], [1,2]);
SubMatrix(B, [2,3], [1..3]); SubMatrix(B, [1,2],[1,2]);
# EXERCISE 7 - Hint
rin := rand(1..1000):    # generates a random integer between 1 and 1000.
a1 := evalf(rin()/1000); # Division by 1000 generates a random real in [0,1].
# Also related: RandomMatrix and RandomVector.
# EXERCISE 8 - Hint
plot(2*x-1, x=0..4);              # Plots 2x-1 as x varies from 0 to 4.
plot({2*x-1,x+2}, x=0..4);        # Plots 2x-1 and x+2 on the same graph.
# EXERCISE 9 - Hint
plot3d(x-y, x=-3..3,y=-2..2);           # 3D plot of x-y on [-3,3]x[-2,2]
plot3d({x-y,x+y}, x=-3..3,y=-2..2);     # 3D plots in one graph.
# Exercise 11.
JacobiIteration := proc(A, b, x0)
  local n, r, i, j, xnew, xcurrent;
  n := nops(b);
  xcurrent := evalf(x0);
  r := 0;
  while r <= 6 do
    xnew := b;
     for i to n do
      for j to n do
        if i <> j then xnew[i] := -xcurrent[j]*A[i, j] + xnew[i];
        end if;
      end do;
     xnew[i] := xnew[i]/A[i, i];
     end do;
    xcurrent := xnew;
   print(xcurrent);
   r := r + 1;
   end do;
xcurrent;
end proc
# Then type
A := [[8, 1, 1, 1], [1, 8, 1, 1], [1, 1, 8, 1], [1, 1, 1, 8]];
b := [5, -16, 19, -30];
x0 := [0, 0, 0, 0];
JacobiIteration(A, b, x0); # Guess for exact solution: 1,-2,3,-4.
```

2 Vectors

> It is impossible to be a mathematician without being a poet in soul.
>
> Sofia Kovalevskaya, Russian mathematician (Figure 2.1).

Figure 2.1: Sofia Kovalevskaya.
By Unknown author, after 1880, Mittag-Leffler Institute, Public Domain, https://commons.wikimedia.org/w/index.php?curid=4581849.
Sofia Kovalevskaya (1850–1891) Russian born mathematician who made remarkable contributions to the theory of differential equations. In 1888, she won the Prix Bordin of the French Academy of Science, for her work "On the Problem of the Rotation of a Solid Body about a Fixed Point". She was one of the first women in history to obtain a doctorate in mathematics and to hold a university professorship.

Introduction

In this chapter, we introduce the remaining two fundamental objects of linear algebra, matrices and vectors. We discuss addition and scalar multiplication and their basic properties. Then we introduce the matrix–vector multiplication, which opens the door to a variety of applications such as geometric transformations, computer graphics, discrete dynamical systems, population models, and fractals.

The notion of vector is old. Physicists use vectors for the study of forces and velocities. According to A. P. Knott, "Aristotle knew that forces can be represented by directed line segments", "Simon Stevin used the parallelogram law to solve problems in statics", and "This law was explicitly stated by Galileo".[1]

Plane and space vectors have a dual existence, algebraic and geometric. This duality makes it possible to study geometry by algebraic means.

The modern development of vectors starts with the geometric treatment of complex numbers by Argand and Wessel, the discovery of quaternions by W. R. Hamilton, and the hypercomplex numbers by H. Grassmann. In 1881 and 1884, American mathematical physicist J. Willard Gibbs published a modern theory of vectors titled *Vector Analysis* (Figure 2.2).

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[1] "The History of Vectors and Matrices" by A. P. Knott in Mathematics in School, Vol. 7, No. 5 (November 1978), pp. 32–34. See [23].

https://doi.org/10.1515/9783111331850-002

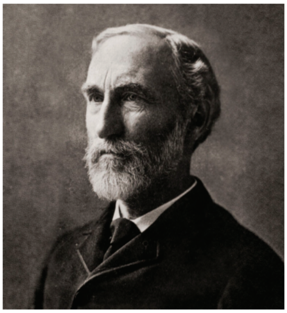

Figure 2.2: Josiah Willard Gibbs.
By Unknown. Frontispiece of The Scientific Papers of J. Willard Gibbs, eds. H. A. Bumstead and R. G. Van Name, Public Domain. https://commons.wikimedia.org/w/index.php?curid=7919387.
Josiah Willard Gibbs (1839–1903) was an American theoretical physicist and chemist. He was one of the founders of modern physical chemistry and statistical mechanics. He made groundbreaking contributions to thermodynamics.

The notion of matrix is old as well. Leibnitz had already been working with arrays of numbers, and in 1693, he introduced determinants to solve linear systems. J. J. Sylvester first used the term “matrix” in 1848. It was, however, A. Cayley who in 1855 defined matrix operations as we know them today.

2.1 Matrices and vectors

In Section 1.1, we introduced the notion of matrix and used it to represent linear systems in the form of the augmented matrix, the coefficient matrix, and the vector of constants. We also defined a vector as a one-column matrix. In this section, we study the basic properties of matrices and vectors.

Definition 2.1.1. A general matrix A of size $m \times n$ with (i, j) entry a_{ij} is denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

This is abbreviated by

$$A = [a_{ij}],$$

where i and j are indices such that $1 \le i \le m$ and $1 \le j \le n$. The set of all $m \times n$ matrices with real entries is denoted by $\mathbf{M}_{mn}$. In the special case where $m = n$, the matrix is called a *square matrix* of size n. If $n = 1$, then A is called a *column matrix*, or an *m-vector*, or a *vector*. The set of all n-vectors with real components is denoted by $\mathbf{R}^n$. If $m = 1$, then A is called a *row matrix*, or an *n-row vector*, or a *row vector*. The entries of vectors are also called *components*.

Example 2.1.2. The following are matrices of respective sizes 4×2, 2×3, 3×3, 5×1, and 1×2:

$$\begin{bmatrix} 1 & -2 \\ -3 & 5 \\ 0 & 6 \\ 2 & -8 \end{bmatrix}, \quad \begin{bmatrix} 7 & 21 & -1 \\ 9 & \sqrt{5} & 4 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{bmatrix}, \quad \begin{bmatrix} 7.1 \\ 3.2 \\ -1.5 \\ 4.9 \\ 6.9 \end{bmatrix}, \quad \begin{bmatrix} a & b \end{bmatrix}.$$

The $(3, 2)$ entry of the first matrix is 6. The third matrix is a square matrix of size 3. The fourth matrix is a 5-vector. The last matrix is a row matrix, or a row vector.

A *zero* matrix, denoted by $\mathbf{0}$, is a matrix with zero entries:

$$\mathbf{0} = [0], \quad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We say that two matrices A and B are *equal* and we write $A = B$, if A and B have the same size and their corresponding entries are equal.

Matrices with entries complex numbers are also useful. The set of all $m \times n$ matrices with complex entries is denoted by $\mathbf{M}_{mn}(\mathbf{C})$.

Example 2.1.3. The following are respectively, matrices from $\mathbf{M}_{3,2}(\mathbf{C})$ and $\mathbf{M}_{2,3}(\mathbf{C})$:

$$\begin{bmatrix} 1+i & -2+3i \\ -3 & 5i \\ 0 & 6-2i \end{bmatrix}, \quad \begin{bmatrix} i & 0.2+i & 0 \\ 9 & 1-2i & 4.5i \end{bmatrix}.$$

Notational Convention: On occasion, to save space, we use the notation $(x_1, x_2, \ldots, x_n)$ to denote the vector with components $x_1, x_2, \ldots, x_n$. So, for example, we may use $(1, 2, 3, 4)$ for $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

2.1.1 Addition and scalar multiplication

We *add* two matrices of the same size A and B by adding the corresponding entries. The resulting matrix is the *sum* of the two matrices and is denoted by $A + B$. So, if $A = [a_{ij}]$ and $B = [b_{ij}]$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, then

$$A + B = [a_{ij} + b_{ij}].$$

We may also multiply a real number c times a matrix A by multiplying all entries of A by c. The resulting matrix is denoted by cA. We have

$$cA = [ca_{ij}].$$

This operation is called *scalar multiplication*. The multiplier c is often called a *scalar*, because it *scales* A.

Example 2.1.4. We have

$$\begin{bmatrix} 1 & -3 & 0 \\ 2 & -4 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 5 \\ -1 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 5 \end{bmatrix},$$

$$(-2)\begin{bmatrix} 1 & 0 \\ -3 & 4 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 6 & -8 \\ -10 & 2 \end{bmatrix}.$$

The matrix $(-1)A$ is called the *opposite* of A and is denoted by $-A$. The matrix $A + (-1)B$ is denoted by $A - B$ and is called the *difference* of A and B. This is the *subtraction* operation:

$$A - B = A + (-1)B.$$

The operations of addition and scalar multiplication satisfy some basic properties described in the following theorem.

Theorem 2.1.5 (Properties of addition and scalar multiplication). *Let A, B, and C be any $m \times n$ matrices, and let a, b, c be any scalars. Then*

1. $(A + B) + C = A + (B + C)$, *(Associativity law)*
2. $A + B = B + A$, *(Commutativity law)*
3. $A + \mathbf{0} = \mathbf{0} + A = A$,
4. $A + (-A) = (-A) + A = \mathbf{0}$,
5. $c(A + B) = cA + cB$, *(Distributivity law)*
6. $(a + b)C = aC + bC$, *(Distributivity law)*
7. $(ab)C = a(bC) = b(aC)$,
8. $1A = A$,
9. $0A = \mathbf{0}$.

Proof of 1 and 6.

1. $(A + B) + C$ and $A + (B + C)$ have the same size. Moreover,

$$\begin{aligned}(A + B) + C &= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}] \\ &= [a_{ij} + (b_{ij} + c_{ij})] = [a_{ij}] + [b_{ij} + c_{ij}] \\ &= A + (B + C)\,.\end{aligned}$$

6. $(a + b)C$ and $aC + bC$ have the same size. In addition,

$$(a + b)\,C = (a + b)\,[c_{ij}] = [(a + b)\,c_{ij}]$$

$$\begin{aligned} &= [ac_{ij} + bc_{ij}] = [ac_{ij}] + [bc_{ij}] \\ &= a\,[c_{ij}] + b\,[c_{ij}] = aC + bC. \end{aligned}$$ ☐

By the associativity and commutativity laws in Theorem 2.1.5 there is no need to use parentheses for writing sums of scaled matrices of the same size. So for any of the equal expressions

$$(-3A_1 + 8A_2) + A_3 = -3A_1 + (8A_2 + A_3) = (A_3 - 3A_1) + 8A_2,$$

we simply write $-3A_1 + 8A_2 + A_3$.

Definition 2.1.6. Let $A_1, A_2, \dots, A_k$ be given $m \times n$ matrices, and let $x_1, x_2, \dots, x_k$ be any scalars. The $m \times n$ matrix

$$A = x_1A_1 + x_2A_2 + \cdots + x_kA_k$$

is called a *linear combination* of $A_1, \dots, A_k$. The scalars $x_1, \dots, x_k$ are called the *coefficients* of the linear combination. If not all c_i are zero, then we have a *nontrivial linear combination*. If all x_i are zero, then we have the *trivial linear combination*. The trivial linear combination represents the $m \times n$ zero matrix.

Example 2.1.7. Find the linear combination $3A_1 - 2A_2 + A_3$, where

$$A_1 = \begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -7 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 9 & -2 \\ -4 & 0 \end{bmatrix}.$$

Solution. We have

$$3\begin{bmatrix} 1 & -3 \\ 2 & -4 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ -7 & 3 \end{bmatrix} + \begin{bmatrix} 9 & -2 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 12 & -13 \\ 16 & -18 \end{bmatrix}.$$

2.1.2 Transpose, symmetric and Hermitian matrices

Let A be an $m \times n$ matrix. The *transpose* of A, denoted by A^T, is the $n \times m$ matrix obtained from A by switching all columns of A to rows and maintaining the same order.

Example 2.1.8. We have

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad \begin{bmatrix} a & b & c \end{bmatrix}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \begin{bmatrix} \pi \\ e \end{bmatrix}^T = \begin{bmatrix} \pi & e \end{bmatrix}.$$

In general, if $A = [a_{ij}]$, $1 \le i \le m$, $1 \le j \le n$, then

$$A^T = [a_{ji}]\,.$$

Theorem 2.1.9 (Properties of transposition). *Let A and B be $m \times n$ matrices, and let c be any scalar. Then*

1. $(A+B)^T = A^T + B^T$;
2. $(cA)^T = cA^T$;
3. $(A^T)^T = A$.

Proof of 1. The matrices $(A+B)^T$ and $A^T + B^T$ have the same size, and

$$\begin{aligned}(A+B)^T &= [a_{ij}+b_{ij}]^T \\ &= [a_{ji}+b_{ji}] \\ &= [a_{ji}] + [b_{ji}] \\ &= A^T + B^T.\end{aligned}$$ ☐

A matrix A such that $A^T = A$ is called *symmetric*. A symmetric matrix is necessarily square (why?). The following matrices are symmetric:

$$\begin{bmatrix} 5 & -7 \\ -7 & 6 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 3 \\ -1 & 4 & 9 \\ 3 & 9 & 6 \end{bmatrix}, \quad \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}.$$

Note the mirror symmetry of a symmetric matrix with respect to the *main diagonal*, i. e., to the upper-left to lower-right diagonal line.

A matrix A such that $A^T = -A$ is called *skew-symmetric*. A skew-symmetric matrix has to be square (why?). The following matrices are skew-symmetric:

$$\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & -9 \\ -3 & 9 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -b & c & -d \\ b & 0 & f & -g \\ -c & -f & 0 & i \\ d & g & -i & 0 \end{bmatrix}.$$

Note that the main diagonal is zero and there is opposite mirror symmetry with respect to it.

For matrices with complex entries, we have the corresponding notions of Hermitian and skew-Hermitian matrices. Recall that the complex conjugate of $z = a + ib$ is the complex number $\overline{z} = a - ib$. If $A = [a_{ij}]$ is a complex matrix, then the complex conjugate of A is the matrix $\overline{A}$ with entries the complex conjugates of the entries of A, i. e., $\overline{A} = [\overline{a_{ij}}]$. For a complex matrix A, we consider the matrix A^H that is the transpose of its conjugate matrix:

$$A^H = \overline{A}^T = [\overline{a_{ji}}].$$

A complex matrix A is called *Hermitian* if

$$A^H = A.$$

A Hermitian matrix is necessarily square. A Hermitian matrix with real entries is just symmetric.

Example 2.1.10. Prove that the matrix

$$A = \begin{bmatrix} -1 & 4+2i & -3i \\ 4-2i & -2 & 1-i \\ 3i & 1+i & -3 \end{bmatrix}$$

is Hermitian.

Solution. We have

$$A^H = (\overline{A})^T = \begin{bmatrix} -1 & 4-2i & 3i \\ 4+2i & -2 & 1+i \\ -3i & 1-i & -3 \end{bmatrix}^T = \begin{bmatrix} -1 & 4+2i & -3i \\ 4-2i & -2 & 1-i \\ 3i & 1+i & -3 \end{bmatrix} = A.$$

Hermitian matrices are used in engineering and physics, especially in atomic physics.

A complex square matrix A is called *skew-Hermitian* if

$$A^H = -A.$$

A skew-Hermitian matrix is necessarily square. A skew-Hermitian matrix with real entries is just skew-symmetric.

The matrix

$$A = \begin{bmatrix} -i & 3+2i \\ -3+2i & 0 \end{bmatrix}$$

is skew-Hermitian, because

$$A^H = (\overline{A})^T = \begin{bmatrix} i & 3-2i \\ -3-2i & 0 \end{bmatrix}^T = \begin{bmatrix} i & -3-2i \\ 3-2i & 0 \end{bmatrix} = -A.$$

2.1.3 Special square matrices, trace

Let A be a square matrix of size n. As mentioned before, the *main diagonal* is the upper left to lower right diagonal line. Its entries are a_{ii}, $1 \leq i \leq n$. A matrix A is called *upper triangular* if all entries below the main diagonal are zero, i. e., if $a_{ij} = 0$ for $j < i$. A matrix

A is called *lower triangular* if the entries above the main diagonal are all zero, so $a_{ij} = 0$ for $i < j$. If the main diagonal is also zero, then we talk about *strictly upper triangular* and *strictly lower triangular* matrices.

Consider the matrices below. A, D, E are upper triangular, B, C, D, E are lower triangular, and C is strictly lower triangular.

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \quad B = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad E = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}.$$

If the nondiagonal entries of a square matrix are zero, then the matrix is called *diagonal*. If all entries of a diagonal matrix are equal, then we have a *scalar* matrix. The matrices D and E are diagonal. The matrix E is a scalar matrix.

Note that a scalar matrix of size n with common diagonal entry 1 is just the identity matrix.

Let A be a square matrix. The *trace*, $\mathrm{tr}(A)$, of A is the sum of the main diagonal elements. For example, if

$$A = \begin{bmatrix} -7 & 5 \\ 6 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{23} & b_{33} \end{bmatrix},$$

then

$$\mathrm{tr}\,(A) = -7 + 4 = -3, \quad \mathrm{tr}\,(B) = b_{11} + b_{22} + b_{33}.$$

2.1.4 Geometric Interpretation of vectors

Although n-vectors were introduced as column matrices, their true origins lie in geometry and physics. Vectors with two or three components are used to represent physical quantities determined by both direction and magnitude, such as *force*, *velocity*, and *displacement*, as follows: The 2-vector $[\, x_1 \; x_2 \,]^T$ can be graphically represented in a Cartesian coordinate plane either by the *arrow* starting at the origin $(0, 0)$ with tip the point with coordinates (x_1, x_2) or by the *point* with coordinates (x_1, x_2). Such an arrow provides a direction, and its length may represent the magnitude of the quantity. In this way, we can identify a Cartesian plane with a chosen origin and the set of 2-vectors with $\mathbf{R}^2$. Similarly, we can use 3-vectors and identify a Cartesian space with a chosen origin and $\mathbf{R}^3$ (Figure 2.3). It is customary to denote vectors by boldface letters such as $\mathbf{a}$, $\mathbf{b}$, $\mathbf{v}$, and $\mathbf{u}$.

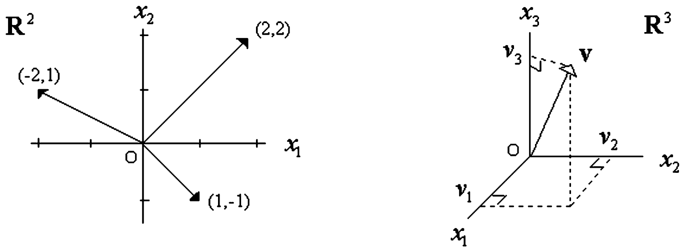

Figure 2.3: Plane and space vectors viewed as arrows starting at the origin.

Vectors as special cases of matrices can be added, if they have the same size. They can also be multiplied by scalars. In fact, all operation properties described in Theorem 2.1.5 apply to n-vectors.

Vector addition, subtraction, and scalar multiplication for 2-vectors and 3-vectors have familiar geometric meanings. We add two vectors geometrically by using the *parallelogram law of addition*. We multiply a vector by a number geometrically by appropriately scaling the vector (Figure 2.4).

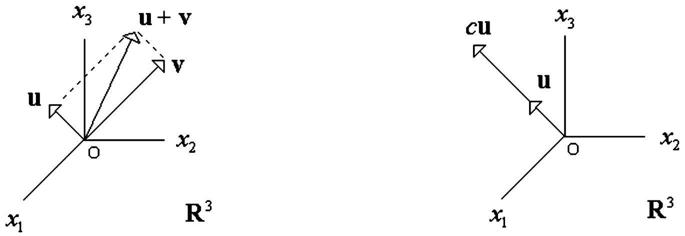

Figure 2.4: The parallelogram law for vector addition and scalar product.

The notion of linear combination of matrices also applies to the special case of n-vectors. If $n = 2$ or 3, then we can depict linear combinations geometrically.

Example 2.1.11. Compute and sketch the linear combination $\frac{1}{2}\mathbf{v}_1 - 3\mathbf{v}_2$, where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Solution. We have $\frac{1}{2}\left[\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}\right] - 3\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] + \left[\begin{smallmatrix} 3 \\ -3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 4 \\ -1 \end{smallmatrix}\right]$ (Figure 2.5).

2.1.5 Application of linear combinations

Example 2.1.12. A sports company owns two factories, each making aluminum and titanium mountain bikes. The first factory makes 150 aluminum and 15 titanium bikes a

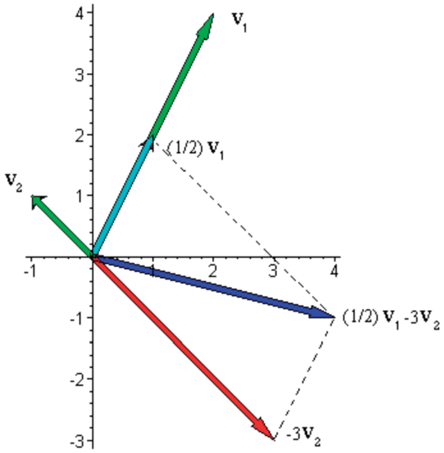

Figure 2.5: The linear combination $\frac{1}{2}\mathbf{v}_1 - 3\mathbf{v}_2$.

day. For the second factory, the numbers are 220 and 20, respectively. If $\mathbf{v}_1 = \begin{bmatrix} 150 \\ 15 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 220 \\ 20 \end{bmatrix}$, then compute and discuss the meaning of (a)–(d):

(a) $\mathbf{v}_1 + \mathbf{v}_2$;
(b) $\mathbf{v}_2 - \mathbf{v}_1$;
(c) $10\mathbf{v}_1$;
(d) $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$ for $x_1, x_2 > 0$.

Solution. (a) $\mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} 370 \\ 35 \end{bmatrix}$ represents the total number of aluminum (370) and titanium (35) bikes produced by the two factories in one day.

(b) $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 70 \\ 5 \end{bmatrix}$ represents how many more bikes the second factory makes over the first one in one day.

(c) $10\mathbf{v}_1 = \begin{bmatrix} 1500 \\ 150 \end{bmatrix}$ represents how many bikes the first factory makes in 10 days.

(d) $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \begin{bmatrix} 150x_1+220x_2 \\ 15x_1+20x_2 \end{bmatrix}$ represents the total number of bikes produced, if the first factory operates for x_1 days and the second for x_2 days.

2.1.6 Digital signals

Various signals, such as a sound wave, that occur in nature or in a laboratory are usually continuous, or *analog*. It is often desirable to *filter* such a signal. For example, our voice is filtered and converted into an electrical pulse when we talk on the phone. Many of the filters are discrete, or *digital*. These filters take a continuous signal and sample it at a discrete sequence of values. When we digitize a signal, we use vectors to save, transform, or otherwise manipulate the discrete sample. In Figure 2.6, we used 20 equally spaced points from 0 to 2π to sample the sine function. This was done by evaluating the 21-vector $\mathbf{v} = [\frac{\pi k}{10},\ k = 0, \dots, 20]^T$ with the sine function to get the 21-vector

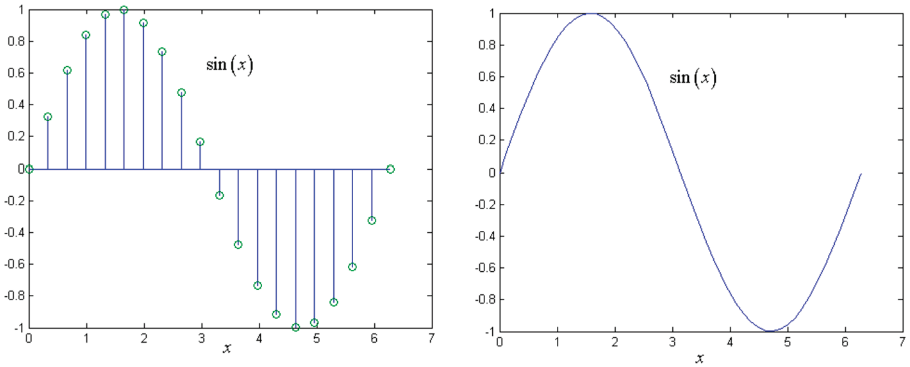

Figure 2.6: Sampling the sine function at a vector of x values.

$$\sin \mathbf{v} = \left[\sin\left(\frac{\pi k}{10}\right),\ \ k = 0, \dots, 20\right]^T.$$

This vector represents a digitized sample of the sine function.

Exercises 2.1

Matrix Operations

1. Identify the sizes and the $(2,2)$ entries of matrices A and B. Find the $(3,1)$ entry of A and the $(2,3)$ entry of B.

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 2 & 1 \end{bmatrix}.$$

2. Find the values of x, y, and z such that the following matrices are equal:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} x+z & 0 & 1 \\ -y & 2 & -z \end{bmatrix}.$$

3. Compute the following, if possible. If the operations cannot be performed, explain why not.

(a) $\begin{bmatrix} -2 & 3 \\ 4 & -5 \\ -6 & 7 \end{bmatrix} - \begin{bmatrix} 7 & -6 \\ -5 & 4 \\ 3 & -2 \end{bmatrix}.$

(b) $-\begin{bmatrix} 1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$

(c) $3\begin{bmatrix} 0 & 2 \\ -4 & -6 \end{bmatrix} - 4\begin{bmatrix} 3 & -5 \\ 7 & 0 \end{bmatrix}.$

4. Compute the expressions

(a) $-2A + B - 4I_3$,

(b) $-3(A + I_3) + 5B$,

where

$$A = \begin{bmatrix} -2 & 3 & 8 \\ 4 & -5 & 0 \\ -6 & 7 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & -4 \\ 3 & 1 & 0 \\ -6 & 5 & 2 \end{bmatrix}.$$

5. Find the matrix X that satisfies the equations
(a) $2X + B = -3A$,
(b) $\frac{1}{2}A - \frac{3}{2}X = B$,

where

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 5 & -2 \end{bmatrix}.$$

6. Prove Parts 2–5 of Theorem 2.1.5.

7. Prove Parts 7–9 of Theorem 2.1.5.

Transpose; special matrices

8. Find A^T, where $A = \begin{bmatrix} a & 1 & \alpha \\ b & 2 & \beta \end{bmatrix}$.

9. Verify that $(A + B)^T = A^T + B^T$ if

$$A = \begin{bmatrix} -2 & 3 \\ 4 & -5 \\ -6 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -6 \\ -5 & 4 \\ 3 & -2 \end{bmatrix}.$$

10. Prove Parts 2 and 3 of Theorem 2.1.9.

11. Let A and B be $n \times n$ matrices, and let c be a scalar. Prove that
(a) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$;
(b) $\operatorname{tr}(cA) = c \operatorname{tr}(A)$;
(c) $\operatorname{tr}(A) = \operatorname{tr}(A^T)$.

12. Which of the following matrices are symmetric? Explain why.

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 2 & 0 \\ 2 & 5 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

13. Let A and B be $n \times n$ matrices, and let c be a scalar. Prove that
(a) If A, B are upper triangular, then $A + B$ and cA are upper triangular;
(b) If A, B are scalar matrices, then $A + B$ and cA are scalar matrices.

14. Let A and B be symmetric, and let c be a scalar. Prove that
(a) $A + B$ is symmetric;
(b) cA is symmetric.

15. Which of the following matrices are skew-symmetric? Explain why.

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

16. Answer each question.
(a) Give an example of a symmetric matrix and an example of a skew-symmetric matrix.
(b) Prove that each diagonal entry of a skew-symmetric matrix is 0.
(c) If B is a square matrix, then prove that $B + B^T$ is symmetric.
(d) If B is a square matrix, then prove that $B - B^T$ is skew-symmetric.
(e) By using Parts (c) and (d) find a formula that writes any square matrix as a sum of a symmetric matrix and a skew-symmetric matrix.
(f) Test your formula from Part (e) on a square matrix of your choice.

17. Let A and B be skew-symmetric matrices, and let c be a scalar. Prove that
(a) $A + B$ is skew-symmetric;
(b) cA is skew-symmetric.

18. Which of the following matrices are Hermitian? Explain why.

$$A = \begin{bmatrix} 8 & 2-8i \\ 2+8i & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2+8i & 0 \\ 2-8i & 3 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2+8i \\ -2-8i & i \end{bmatrix}.$$

19. Prove that the diagonal entries of a Hermitian matrix are real numbers.

20. *True or False?* If A Hermitian, then
(a) $-2A$ is Hermitian;
(b) iA is Hermitian;
(c) iA is skew-Hermitian.

21. Let A and B be Hermitian $n \times n$ matrices, and let c be a real scalar. Prove that $A + B$ and cA are Hermitian. Is cA Hermitian, if c is a complex number that is not real? Explain.

22. Which of the following matrices are skew-Hermitian?

$$A = \begin{bmatrix} i & 2+4i \\ -2+4i & 0 \end{bmatrix}, \quad B = \begin{bmatrix} i & 2+4i & 0 \\ -2-4i & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} i & 2+4i \\ -2+4i & 1 \end{bmatrix}.$$

23. *True or False?* If A skew-Hermitian, then
(a) $-2A$ is skew-Hermitian;
(b) iA is skew-Hermitian;
(c) iA is Hermitian.

24. Please answer each question.
(a) Give an example of a Hermitian matrix and an example of a skew-Hermitian matrix.
(b) Prove that each diagonal entry of a skew-Hermitian matrix is either 0 or pure imaginary.
(c) If B is a complex square matrix, then prove that $B + B^H$ is Hermitian.
(d) If B is a complex square matrix, then prove that $B - B^H$ is skew-Hermitian.
(e) By using Parts (c) and (d) find a formula that writes any complex square matrix as a sum of a Hermitian matrix and a skew-Hermitian matrix.
(f) Test your formula from Part (e) on a complex square matrix of your choice.

25. Let A and B be skew-Hermitian $n \times n$ matrices, and let c be a real scalar. Prove that $A + B$ and cA are skew-Hermitian. Is cA skew-Hermitian if c is a complex number that is not real? Explain.

26. *True or False?*

(a) A is skew-Hermitian if and only if iA is Hermitian.

(b) If A is skew-Hermitian and k is an even integer, then A^k is Hermitian.

(c) If A is skew-Hermitian and k is an odd integer, then A^k is skew-Hermitian.

Applications

27. Prove that the following *Pauli spin matrices*, used in particle physics, are Hermitian:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

28. An airline buys food supplies for three of its planes. The average dollar cost per trip is given by the following matrix A with columns $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$:

Class	Plane 1	Plane 2	Plane 3
First	350	300	450
Business	500	600	700
Economy	800	700	900

Compute and explain the meaning of

(a) $10\mathbf{a}_3$,

(b) $\mathbf{a}_3 - \mathbf{a}_2$,

(c) $7\mathbf{a}_1 + 8\mathbf{a}_2 + 9\mathbf{a}_3$.

29. (*Centroid*) The centroid of the n-vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the vector $\mathbf{v}$

$$\mathbf{v} = \frac{1}{k}(\mathbf{v}_1 + \cdots + \mathbf{v}_k).$$

Figure 2.7 shows the centroid of three vectors. Find the centroid of the triangle PQR, where $P(1,2)$, $Q(2,-4)$, and $R(-1,7)$, i. e., find the centroid of the vectors with tips at P, Q, R.

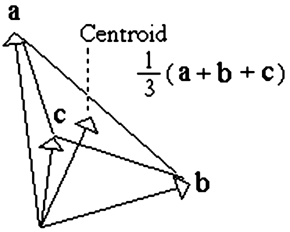

Figure 2.7: The centroid of three vectors.

30. (*Center of mass*) Let $m_1, \dots, m_n$ be n masses located at the tips of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, and let $M = m_1 + \cdots + m_n$ be the total mass. The center of mass of these systems is defined by

$$\frac{1}{M}(m_1\mathbf{v}_1 + \cdots + m_n\mathbf{v}_n).$$

Find the center of mass of the system with masses 1, 4, 5, 2 kg located respectively at $P_1(-1,2,0)$, $P_2(0,5,-1)$, $P_3(1,1,-3)$, and $P_4(-6,1,-3)$.

2.2 Matrix transformations

In this section, we define a useful operation between a matrix A and a vector $\mathbf{x}$, the product $A\mathbf{x}$. As $\mathbf{x}$ varies, so does $A\mathbf{x}$, defining a transformation, called a matrix transformation. There are numerous applications of matrix transformations, such as in differential equations, computer graphics, and generation of fractal images.

2.2.1 The matrix–vector product

Definition 2.2.1. Let A be an $m \times n$ matrix, and let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ be an n-vector. Let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ be the columns of A viewed as m-vectors. We define the *product* $A\mathbf{x}$ as the m-vector

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n. \tag{2.1}$$

The product $A\mathbf{x}$ is the *linear combination* of the columns of A with coefficients the components of $\mathbf{x}$. The particular case where $\mathbf{x} = \mathbf{0}$ yields

$$A\mathbf{0} = \mathbf{0}.$$

Example 2.2.2. Compute the product

$$\begin{bmatrix} -1 & 5 & -3 & 7 \\ 6 & 0 & 2 & 8 \\ 5 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}.$$

Solution. This product equals

$$1\begin{bmatrix} -1 \\ 6 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix} + 3\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ 4 \end{bmatrix}.$$

The product $A\mathbf{x}$ is only defined when the number of columns of the matrix equals the number of the components of the vector. This is the reason why the following "products" are undefined:

$$\begin{bmatrix} -1 & 5 \\ 6 & 0 \\ 9 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Theorem 2.2.3. *Let A be an $m \times n$ matrix, let $\mathbf{x}$ and $\mathbf{y}$ be n-vectors, and let c be a scalar. Then*

1. $A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y}$;
2. $A(c\mathbf{x}) = cA\mathbf{x}$.

Proof of 1. If A has columns $\mathbf{a}_i$ and if $\mathbf{x} = [x_i]$, $\mathbf{y} = [y_i]$, then by the definition of the matrix–vector product and Theorem 2.1.5 in Section 2.1 we have

$$\begin{aligned} A(\mathbf{x}+\mathbf{y}) &= (x_1+y_1)\,\mathbf{a}_1 + \cdots + (x_n+y_n)\,\mathbf{a}_n \\ &= (x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n) + (y_1\mathbf{a}_1 + \cdots + y_n\mathbf{a}_n) \\ &= A\mathbf{x} + A\mathbf{y}. \end{aligned}$$

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Definition 2.2.4. The columns of the identity matrix I_n viewed as n-vectors are called the *standard basis vectors* in $\mathbf{R}^n$ and are denoted by $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. Hence we have

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Theorem 2.2.5. *Let A be an $m \times n$ matrix with columns $\mathbf{a}_i$. Let $\mathbf{x} = [\, x_1 \cdots x_n \,]^T$ be an n-vector, and let $\mathbf{e}_i$ be the ith standard basis n-vector. Then*

1. $I_n\mathbf{x} = \mathbf{x}$;
2. $A\mathbf{e}_i = \mathbf{a}_i$.

Proof. For Part 1, we have

$$I_n\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n = \mathbf{x},$$

and for Part 2, we have

$$\begin{aligned} A\mathbf{e}_i &= [\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_i\ \ldots\ \mathbf{a}_n]\,\mathbf{e}_i \\ &= 0\mathbf{a}_1 + 0\mathbf{a}_2 + \cdots + 1\mathbf{a}_i + \cdots + 0\mathbf{a}_n \\ &= \mathbf{a}_i. \end{aligned}$$

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Sometimes we use the *physics* (or *engineering*) notation **i**, **j** for $\mathbf{e}_1$, $\mathbf{e}_2$ in $\mathbf{R}^2$ and **i**, **j**, **k** for $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ in $\mathbf{R}^3$ (Figure 2.8):

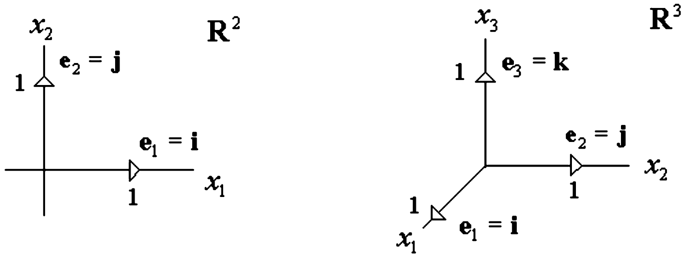
x_2
$\mathbf{R}^2$
1
$\mathbf{e}_2 = \mathbf{j}$
$\mathbf{e}_1 = \mathbf{i}$
1
x_1
x_3
$\mathbf{R}^3$
1
$\mathbf{e}_3 = \mathbf{k}$
$\mathbf{e}_2 = \mathbf{j}$
x_2
1
1
x_1
$\mathbf{e}_1 = \mathbf{i}$

Figure 2.8: Standard basis vectors in $\mathbf{R}^2$ and in $\mathbf{R}^3$.

!

Equation (2.1) shows that the *i*th component of the product $A\mathbf{x}$ is obtained by multiplying the entries of the *i*th row of the matrix by the corresponding vector components and then adding the products. In other words, the product $A\mathbf{x}$ is the *m*-vector with entries c_i given by

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ \hline \multicolumn{1}{|c}{a_{i1}} & a_{i2} & \cdots & \multicolumn{1}{c|}{a_{in}} \\ \hline \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right], \quad \mathbf{x} = \left[\boxed{\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array}}\right],$$

$$c_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n. \tag{2.2}$$

2.2.2 Matrix transformations

A *transformation*, or *map* T from a set A to a set B, denoted by $T : A \to B$, is a correspondence that with each element a of A, associates a unique element b of B. This unique element b is denoted by $T(a)$ and is called the *image* of a under T. We say that a is *mapped* to b. The set A is called the *domain* of T, and the set B is the *codomain*. The subset of B that consists of all possible images is called the *range* of T. The range of T is denoted by $\mathrm{Range}(T)$ (Figure 2.9).

In a transformation, each element $a \in A$ has only one image $T(a)$. However, there are transformations where two or more elements have the same image. For example, for $T : \mathbf{R} \to \mathbf{R}$, $T(x) = x^2$, the elements 2 and -2 both have the image 4. This is not true for $T : \mathbf{R} \to \mathbf{R}$, $T(x) = x - 5$, where different elements always have different images. Now we return to the product $A\mathbf{x}$. If $\mathbf{x}$ varies in $\mathbf{R}^n$, then $A\mathbf{x}$ varies in $\mathbf{R}^m$, defining a special transformation, called a matrix transformation.

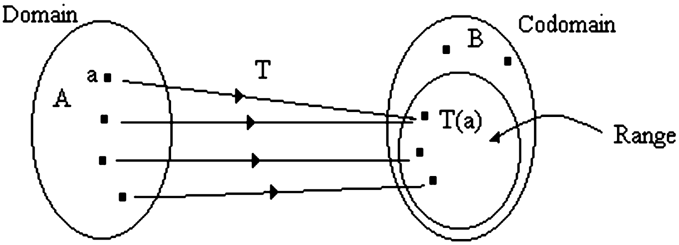

Figure 2.9: A transformation T from a set A to a set B.

Definition 2.2.6. A *matrix transformation* $T : \mathbf{R}^n \to \mathbf{R}^m$ is a transformation for which there is an $m \times n$ matrix A such that for all $\mathbf{x}$ in $\mathbf{R}^n$,

$$T(\mathbf{x}) = A\mathbf{x}.$$

(Figure 2.10.)

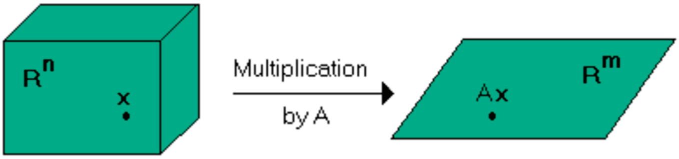

Figure 2.10: Multiplication by A defines a transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$.

Example 2.2.7. Consider the matrix transformation $T(\mathbf{x}) = \left[\begin{smallmatrix} 3 & -7 & 8 \\ 2 & 1 & -4 \end{smallmatrix}\right] \mathbf{x}$.

(a) Determine the domain and codomain.

(b) Find the image of $\mathbf{x} = [\, x_1 \; x_2 \; x_3 \,]^T$.

(c) Prove that $\mathbf{u} = [\, 21 \; 29 \; 18 \,]^T$ and $\mathbf{v} = [\, 1 \; 1 \; 1 \,]^T$ have the same image under T.

Solution. (a) The matrix is of size 2×3. Thus the domain is $\mathbf{R}^3$, and the codomain is $\mathbf{R}^2$.

(b) We have

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -7 & 8 \\ 2 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 7x_2 + 8x_3 \\ 2x_1 + x_2 - 4x_3 \end{bmatrix}.$$

(c) By Part (b) the choice $x_1 = 21$, $x_2 = 29$, $x_3 = 18$ yields $T(\mathbf{u}) = \left[\begin{smallmatrix} 4 \\ -1 \end{smallmatrix}\right]$, and the choice $x_1 = 1$, $x_2 = 1$, $x_3 = 1$ yields $T(\mathbf{v}) = \left[\begin{smallmatrix} 4 \\ -1 \end{smallmatrix}\right]$.

Example 2.2.8. Is the following transformation a matrix transformation?

$$T : \mathbf{R}^2 \to \mathbf{R}^2, \quad T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}.$$

Solution. Yes, because

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Hence T is represented by the matrix $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

A matrix transformation satisfies the following important identities: For all $\mathbf{x}_1$, $\mathbf{x}_2$ in $\mathbf{R}^n$ and c in $\mathbf{R}$,

$$T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2) \tag{2.3}$$

and

$$T(c\mathbf{x}_1) = cT(\mathbf{x}_1). \tag{2.4}$$

This is because if $T : \mathbf{R}^n \to \mathbf{R}^m$ is defined by $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A, then by Theorem 2.2.3 we have

$$T(\mathbf{x}_1 + \mathbf{x}_2) = A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = T(\mathbf{x}_1) + T(\mathbf{x}_2)$$

and

$$T(c\mathbf{x}_1) = A(c\mathbf{x}_1) = cA\mathbf{x}_1 = cT(\mathbf{x}_1).$$

In general, if a transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ satisfies identities (2.3) and (2.4) for all $\mathbf{x}_1$, $\mathbf{x}_2$ in $\mathbf{R}^n$ and c in $\mathbf{R}$, then it is called a *linear transformation* from $\mathbf{R}^n$ to $\mathbf{R}^m$.

Identities (2.3) and (2.4) can be repeated and combined into the following identity, whose proof is left as an exercise. For all scalars c_i, we have

$$T(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k) = c_1T(\mathbf{x}_1) + c_2T(\mathbf{x}_2) + \cdots + c_kT(\mathbf{x}_k). \tag{2.5}$$

The fact that a matrix transformation satisfies (2.3) and (2.4) makes it a linear transformation. What is surprising is the converse: any linear transformation is a matrix transformation. Hence the notions of linear and matrix transformations are identical when the domain is $\mathbf{R}^n$ and the codomain is $\mathbf{R}^m$.

Theorem 2.2.9. *Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a transformation.*

1. *If T is a matrix transformation, then T is a linear transformation.*
2. *If T is a linear transformation, then T is a matrix transformation.*

Proof. Part 1 is already proved. For Part 2, let A be the matrix with columns $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$. We claim that $T(\mathbf{x}) = A\mathbf{x}$, which shows that T is a matrix transformation. If $\mathbf{x} = [\, x_1 \cdots x_n \,]^T$, then $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$. Therefore by linearity expressed in equation (2.5) we have

$$\begin{aligned} T(\mathbf{x}) &= T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) \\ &= x_1 T(\mathbf{e}_1) + \cdots + x_n T(\mathbf{e}_n) \\ &= [T(\mathbf{e}_1) \ \cdots \ T(\mathbf{e}_n)]\mathbf{x} \\ &= A\mathbf{x}. \end{aligned}$$

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Notation. The matrix A with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is denoted by

$$A = [\mathbf{a}_1 \ \mathbf{a}_2 \ldots \ \mathbf{a}_n].$$

Definition 2.2.10. The matrix A in the proof of Part 2 of Theorem 2.2.9 is called *the standard matrix* or simply *the matrix* of T. Hence, the standard matrix of any linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is the matrix A with columns $T(\mathbf{e}_i)$:

$$A = [T(\mathbf{e}_1) \ \cdots \ T(\mathbf{e}_n)].$$

Example 2.2.11. Find the standard matrix A of the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 7x_3 \\ -3x_1 + 8x_3 \end{bmatrix}.$$

Solution. We compute the images $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$, $T(\mathbf{e}_3)$:

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

Thus $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = [T(\mathbf{e}_1) \ T(\mathbf{e}_2) \ T(\mathbf{e}_3)] = \begin{bmatrix} 1 & -2 & 7 \\ -3 & 0 & 8 \end{bmatrix}.$$

2.2.3 Matrix form of linear systems

Linear systems are intimately connected with matrix–vector products and matrix transformations.

The linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{2.6}$$

can be written as equality of two vectors by using the matrix–vector product as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

This is abbreviated by

$$A\mathbf{x} = \mathbf{b}. \tag{2.7}$$

System (2.6) may also be written in linear combination form as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

and can be abbreviated by

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}, \tag{2.8}$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the columns of A. Therefore we have the important equivalence of expressions

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \Leftrightarrow A\mathbf{x} = \mathbf{b}. \tag{2.9}$$

The associated homogeneous system of (2.7) can now be written as the vector equation

$$A\mathbf{x} = \mathbf{0} \tag{2.10}$$

or as the linear combination

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}. \tag{2.11}$$

Example 2.2.12. Write the linear system

$$\begin{aligned} 7x_1 + 4x_2 + 5x_3 &= 1, \\ 2x_1 - 3x_2 + 9x_3 &= -8 \end{aligned}$$

both in matrix–vector product form and in linear combination form.

Solution. We have

$$\begin{bmatrix} 7 & 4 & 5 \\ 2 & -3 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$$

and

$$x_1\begin{bmatrix} 7 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ -3 \end{bmatrix} + x_3\begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}.$$

The matrix notation of a linear system shows that *finding a solution* $\mathbf{x}$ *of the linear system* (2.6) *is identical to finding a vector* $\mathbf{x}$ *with image* $\mathbf{b}$ *under the matrix transformation* $T(\mathbf{x}) = A\mathbf{x}$.

Example 2.2.13. Find all vectors $\mathbf{x}$ with image $[\, 4\ -1\,]^T$ under the transformation

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 4 & 5 \\ -1 & -3 & 9 \end{bmatrix}\mathbf{x}.$$

What geometric object do these vectors form?

Solution. We need all $\mathbf{x}$ such that

$$T(x) = \begin{bmatrix} 1 & 4 & 5 \\ -1 & -3 & 9 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

This is equivalent to solving the linear system with augmented matrix

$$\begin{bmatrix} 1 & 4 & 5 & : & 4 \\ -1 & -3 & 9 & : & -1 \end{bmatrix}.$$

By row reduction we find that all such $\mathbf{x}$ are of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 51r - 8 \\ -14r + 3 \\ r \end{bmatrix}, \quad r \in \mathbf{R}.$$

This set of vectors is the space straight line through the point $(-8, 3, 0)$ in the direction of the vector $[\, 51\ -14\ 1\,]^T$.

2.2.4 Relation between the solutions of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

Next, we find a relation between the solutions of a linear system

$$A\mathbf{x} = \mathbf{b} \tag{2.12}$$

and its associated homogeneous system

$$A\mathbf{x} = \mathbf{0}. \tag{2.13}$$

Definition 2.2.14. If $\mathbf{v}$ is an n-vector and if S is a set of n-vectors, then we denote by $\mathbf{v}+S$ the set of all vectors that are sums of the form $\mathbf{v}+\mathbf{w}$, where $\mathbf{w}$ is in S:

$$\mathbf{v} + S = \{\mathbf{v} + \mathbf{w}, \;\; \mathbf{w} \in S\}\,.$$

We say that the set $\mathbf{v}+S$ is a *translation* of set S by the vector $\mathbf{v}$. We simply add $\mathbf{v}$ to each vector of S (Figure 2.11).

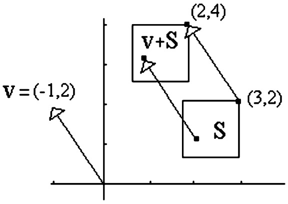

Figure 2.11: $\mathbf{v}+S$ is the translation of S by $\mathbf{v} = [\,-1\;\;2\,]^T$.

Let $\mathbf{p}$ be a solution of (2.12), and let $\mathbf{h}$ be a solution of (2.13). Then $\mathbf{p}+\mathbf{h}$ is a solution of (2.12), because by Theorem 2.2.3

$$A\,(\mathbf{p} + \mathbf{h}) = A\mathbf{p} + A\mathbf{h} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

If $\mathbf{p}_1$ is another solution of (2.12), then the difference $\mathbf{p}-\mathbf{p}_1$ is a solution of (2.13), because, again by Theorem 2.2.3,

$$A\,(\mathbf{p} - \mathbf{p}_1) = A\mathbf{p} - A\mathbf{p}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

We conclude that the general solution of the nonhomogeneous system can be obtained by adding to a particular solution the general solution of the associated homogeneous system. This relation between solutions can be nicely expressed in terms of solution sets. If S_N is the solution set of (2.12), S_H is the solution set of (2.13), and if $\mathbf{p}$ is a particular solution of (2.12), then

$$S_N = \mathbf{p} + S_H.$$

In other words S_N is a *translation* of S_H (Figure 2.12). Let us collect our observations into the following theorem.

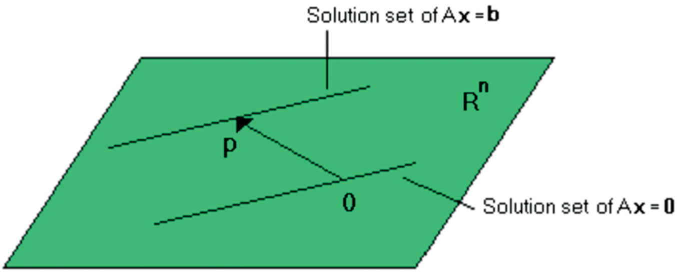

Figure 2.12: The solution set of $A\mathbf{x} = \mathbf{b}$ is a translation of the solution set of $A\mathbf{x} = \mathbf{0}$.

Theorem 2.2.15. *Let* $\mathbf{p}$ *be a particular solution of a consistent linear system* $A\mathbf{x} = \mathbf{b}$, *and let* S_N *be the solution set. Let* $\mathbf{h}$ *be the general solution of* $A\mathbf{x} = \mathbf{0}$, *and let* S_H *be the solution set. Then*

1. $\mathbf{p} + \mathbf{h}$ *is the general solution of* $A\mathbf{x} = \mathbf{b}$;
2. $S_N = \mathbf{p} + S_H$.

Exercises 2.2

Matrix–vector product

In Exercises 1–4, use

$$A = \begin{bmatrix} -3 & -2 \\ -1 & 0 \\ 5 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 7 & -3 & -2 \\ 4 & 6 & -1 & 0 \end{bmatrix}$$

and

$$\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

to compute the indicated product, if possible. If it is not possible, then explain why.

1. $A\mathbf{u}$, $A\mathbf{v}$, $A\mathbf{w}$.

2. $B\mathbf{u}$, $B\mathbf{v}$, $B\mathbf{w}$.

3. $\mathbf{u}\mathbf{u}$, $\mathbf{u}\mathbf{v}$, $\mathbf{u}^T\mathbf{u}$.

4. $B^T\mathbf{u}$, $B^T\mathbf{w}$, $\mathbf{A}^T\mathbf{v}$.

5. Let $A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Find, if possible:

(a) $A\mathbf{v}$;
(b) $A(A\mathbf{v})$.

6. Write the product

$$\begin{bmatrix} 2 & 7 & -2 \\ 9 & 6 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$$

as a linear combination.

7. Write the linear combination

$$(-3)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 9 \end{bmatrix} + \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

as a matrix–vector product.

8. *True or False?* If A is an $m \times n$ matrix and $\mathbf{v}$ is an n-vector, then $A(A\mathbf{v})$ is always defined.

9. Let A be an $m \times n$ matrix, and let $\mathbf{v}$ be an n-vector. If A has equal columns, then describe in words the product $A\mathbf{v}$.

10. Let A be an $m \times n$ matrix, and let $\mathbf{v}$ be an n-vector. If $\mathbf{v}$ has equal components, then describe in words the product $A\mathbf{v}$.

11. Find all 2-vectors of the form

$$\begin{bmatrix} -2 & 1 & -4 \\ 2 & -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

12. Prove Part 2 of Theorem 2.2.3.

13. Given the apparent equation

$$3\begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix} - 7\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} - \begin{bmatrix} -9 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

find scalars c_1 and c_2 such that

$$\begin{bmatrix} 4 & 3 \\ 0 & -1 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -9 \\ 7 \\ 4 \end{bmatrix}.$$

14. Solve the linear system

$$\begin{bmatrix} -2 & 3 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}.$$

15. For $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, find all vectors $\mathbf{x}$ such that $A\mathbf{x} = 4\mathbf{x}$.

16. Find a vector $\mathbf{x}$ such that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$, where

$$A = \begin{bmatrix} a & 2a & 6a \\ 3a & 3a & 3a \\ 2a & 2a & 5a \end{bmatrix}.$$

17. Compute the product

$$\begin{bmatrix} 2+i & i & 7 \\ 1 & 4i & 1-i \end{bmatrix}\begin{bmatrix} i \\ 1+i \\ 3-i \end{bmatrix}.$$

18. A sports company sells bicycles of types 1, 2, 3, 4 at three outlets. The outlets are supplied as follows:

Type	Out. 1	Out. 2	Out. 2
Bike 1	25	15	35
Bike 2	20	25	25
Bike 3	15	35	20
Bike 4	20	30	10

If M is the matrix defined by the above table, then compute and interpret the products

$$M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad M\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad M\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Matrix transformations

In Exercises 19–20, for the given matrix transformation $T(\mathbf{x}) = A\mathbf{x}$, find the image $T(\mathbf{v})$ for the given A and $\mathbf{v}$.

19. $A = \begin{bmatrix} 5 & 6 & 1 & 4 \\ 7 & -3 & -2 & 0 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -6 \\ 3 \end{bmatrix}.$

20. $A = \begin{bmatrix} -4 & 9 \\ 1 & 6 \\ 7 & -3 \\ 0 & 2 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$

In Exercises 21–26, consider the matrix transformation $T(\mathbf{x}) = A\mathbf{x}$. Find, if possible, all vectors $\mathbf{v}$ whose image is $\mathbf{b}$, i. e., such that $T(\mathbf{v}) = \mathbf{b}$.

21. $A = \begin{bmatrix} 1 & 2 \\ 3 & -6 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$

22. $A = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$

23. $A = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -4 \\ 12 \end{bmatrix}.$

24. $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 0 & 1 \\ 0 & -1 & 4 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}.$

25. $A = \begin{bmatrix} 1 & 2 & 7 & -5 \\ 0 & 1 & 6 & 2 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$

26. $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 9 & 1 \\ 0 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

In Exercises 27–31, for the given transformations $T : \mathbf{R}^n \to \mathbf{R}^m$, find
(a) n and m;
(b) the domain and codomain of T;
(c) all vectors of the domain whose image is the zero m-vector.

27. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 0 \end{bmatrix}$.

28. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y \\ x - z \end{bmatrix}$.

29. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \\ y \end{bmatrix}$.

30. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - z \\ -x + z \\ x - z \end{bmatrix}$.

31. $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 \end{bmatrix}\mathbf{x}$.

32. If $T : \mathbf{R}^{10} \to \mathbf{R}^{25}$ defines a matrix transformation with matrix A, then what is the size of A?

33. If A is a 4×7 matrix, find m and n for the matrix transformation $T : \mathbf{R}^n \to \mathbf{R}^m$, $T(\mathbf{x}) = A\mathbf{x}$.

34. Find the matrix of the linear transformation

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 2x_1 + 3x_2 \end{bmatrix}.$$

35. Find the matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}.$$

36. Find the matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ by using the graphical information in Figure 2.13. In each graph the dotted line segments are of equal length.

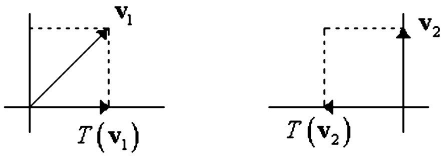

Figure 2.13: Find the transformation.

37. Find the image of $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

38. Find a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ whose range is the xy-plane (Figure 2.14).

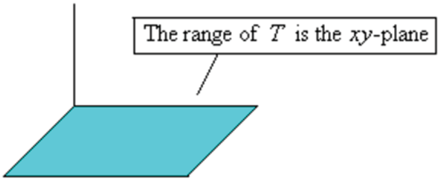

Figure 2.14: Find a transformation.

39. Explain why each transformation is nonlinear.

(a) $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ |x_2| \end{bmatrix}$;

(b) $T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In Exercises 40–44, determine whether or not the range and codomain of the linear transformation are equal.

40. $T(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\mathbf{x}$.

41. $T(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ -3 & -6 \end{bmatrix}\mathbf{x}$.

42. $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & -8 \\ -2 & 0 & 16 \end{bmatrix}\mathbf{x}$.

43. $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 0 & 1 \\ 0 & -1 & 4 \end{bmatrix}\mathbf{x}$.

44. $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x}$.

45. Prove that if each row of a matrix A has a pivot, then the codomain and range of $T(\mathbf{x}) = A\mathbf{x}$ are equal.

46. Give an example of a matrix A that has fewer pivots than the number of rows. For this matrix, find a vector that is in the codomain of $T(\mathbf{x}) = A\mathbf{x}$ but not in the range.

Applications to physics

47. (*Galilean transformation*) Let $\mathbf{x} = (x, y, z, t)$ and $\mathbf{x}' = (x', y', z', t')$ be the space-time coordinates of two frames F and F' with parallel coordinate axes. Let us assume that the frame F' is moving away from the frame F at a constant relative velocity v in a direction along the x- and x'-axes (Figure 2.15). Prove that $\mathbf{x}'$ is a matrix transformation of $\mathbf{x}$ and find the standard matrix.

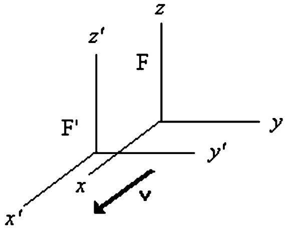

Figure 2.15: Frames moving at a constant relative speed.

48. (*Lorentz transformation*) Let $\mathbf{x} = (x, y, z, t)$ and $\mathbf{x}' = (x', y', z', t')$ be the space-time coordinates of two frames F and F' with parallel coordinate axes, and let the frame F' move away from the frame F at a constant relative velocity v in a direction along the x- and x'-axes (Figure 2.15). In Einstein's theory of *special relativity* the frames F and F' are related by the *Lorentz transformation*

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}},$$

where c is the speed of light. Prove that a Lorentz transformation defines a matrix transformation $L : \mathbf{R}^4 \to \mathbf{R}^4$. Find the matrix of this transformation.

2.3 The span

In this section, we introduce the basic notion of the span of a set of n-vectors. This, once more, is related to linear systems and plays an important role in the sequel.

Definition 2.3.1. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be fixed m-vectors. The set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the *span* of these vectors and is denoted by

$$\mathrm{Span}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}.$$

The span consists of all possible linear combinations of these vectors, i. e., all vectors of the form

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_k\mathbf{v}_k,$$

where the coefficients x_i may take on any real values. The span of one vector $\mathbf{v}$ consists of all scalar products of $\mathbf{v}$:

$$\mathrm{Span}\,\{\mathbf{v}\} = \{x\mathbf{v},\ \ x \in \mathbf{R}\}\,.$$

As an example, the span of $\mathbf{v} = [\,1\;1\;1\,]^T$ in $\mathbf{R}^3$ is the space line through the origin and the tip of $\mathbf{v}$ (Figure 2.16).

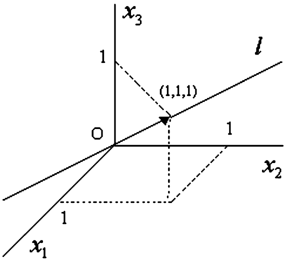
x_3
1
l
(1,1,1)
O
1
x_2
1
x_1

Figure 2.16: The span of $[\,1\;\;1\;\;1\,]^T$ is the line l.

The span is an *infinite set*, unless all $\mathbf{v}_i$ are $\mathbf{0}$.

Geometrically, the span of two 2-vectors or 3-vectors that are not multiples of each other is the unique plane through the origin containing these vectors. For example, $\mathrm{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\} = \mathbf{R}^2$ (Figure 2.17).

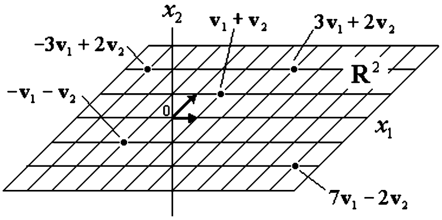
x_2
$\mathbf{v}_1 + \mathbf{v}_2$
$3\mathbf{v}_1 + 2\mathbf{v}_2$
$-3\mathbf{v}_1 + 2\mathbf{v}_2$
$\mathbf{R}^2$
$-\mathbf{v}_1 - \mathbf{v}_2$
0
x_1
$7\mathbf{v}_1 - 2\mathbf{v}_2$

Figure 2.17: The span of $[\,1\;\;0\,]^T$ and $[\,1\;\;1\,]^T$ is $\mathbf{R}^2$.

An m-vector $\mathbf{b}$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ if the coefficients x_i can be found such that $\mathbf{b} = x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k$. By the definition of the matrix–vector product (Equation (2.9), Section 2.2) we have

$$\mathbf{b} = x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k \Leftrightarrow A\mathbf{x} = \mathbf{b}, \tag{2.14}$$

where A is the matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_k$, and $\mathbf{x}$ is the vector with components

$x_1, \dots, x_k$. Therefore saying that $\mathbf{b}$ is in the span of the $\mathbf{v}_i$ is equivalent to saying that the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. We have proved the following theorem.

Theorem 2.3.2. *Let* $\mathbf{b}$ *and* $\mathbf{v}_1, \dots, \mathbf{v}_k$ *be in* $\mathbf{R}^m$ *and let* A *be the matrix with columns* $\mathbf{v}_1, \dots, \mathbf{v}_k$. *Then the following statements are equivalent.*

1. $\mathbf{b}$ *is in* $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$.
2. *The linear system* $A\mathbf{x} = \mathbf{b}$ *is consistent.*

Example 2.3.3. Determine whether each of $\mathbf{u}$ and $\mathbf{v}$ is in $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{u} = \begin{bmatrix} -50 \\ 20 \\ 10 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 10 \\ -4 \\ -2 \end{bmatrix}.$$

Solution. By Theorem 2.3.2 it suffices to check whether the two systems with the following augmented matrices are consistent:

$$\begin{bmatrix} 1 & -5 & 10 & : & -50 \\ 0 & 2 & -4 & : & 20 \\ -2 & 1 & -2 & : & 10 \end{bmatrix}, \quad \begin{bmatrix} 1 & -5 & 10 & : & 0 \\ 0 & 2 & -4 & : & 1 \\ -2 & 1 & -2 & : & -3 \end{bmatrix}.$$

Row reduction yields, respectively,

$$\begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & -2 & : & 10 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & : & 0 \\ 0 & 1 & -2 & : & 0 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}.$$

The first system has solutions, and thus $\mathbf{u}$ is in the span. The second system has no solutions, and hence $\mathbf{v}$ is not in the span. Geometrically, the vectors $\mathbf{u}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are on the same plane through the origin, and the vector $\mathbf{v}$ is not on this plane.

Definition 2.3.4. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a finite set of m-vectors, and let V be a subset of $\mathbf{R}^m$. If $V = \operatorname{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, then we say that S *spans* V or that S is a *spanning set* or *generating set* of V. We also say that the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ *generate* V.

Example 2.3.5. Find a spanning set for the solution set S_H of the homogeneous system

$$\begin{aligned} x_1 - 2x_2 - x_3 + x_4 &= 0, \\ x_1 - 3x_2 + x_3 - 2x_4 &= 0. \end{aligned}$$

Solution. Row reduction yields the two-parameter general solution

$$x_1 = 5r - 7s, \quad x_2 = 2r - 3s, \quad x_3 = r, \quad x_4 = s, \quad r, s \in \mathbf{R}.$$

In vector form the solution can be written as a linear combination of two vectors:

$$\mathbf{x} = \begin{bmatrix} 5r-7s \\ 2r-3s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \quad r, s \in \mathbf{R}.$$

Thus, a spanning set for S_H is

$$\left\{ \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Example 2.3.6. Write the solution set S_N of the following nonhomogeneous system as a translation of a spanning set:

$$\begin{aligned} x_1 - 2x_2 - x_3 + x_4 &= 2, \\ x_1 - 3x_2 + x_3 - 2x_4 &= -8. \end{aligned}$$

Solution. Row reduction yields the two-parameter general solution

$$\mathbf{x} = \begin{bmatrix} 22+5r-7s \\ 10+2r-3s \\ r \\ s \end{bmatrix} = r\begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 22 \\ 10 \\ 0 \\ 0 \end{bmatrix}, \quad r, s \in \mathbf{R}.$$

Hence

$$S_N = \begin{bmatrix} 22 \\ 10 \\ 0 \\ 0 \end{bmatrix} + \mathrm{Span}\left\{ \begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The last two examples verify the claim of Theorem 2.2.15 in Section 2.2 that the general solution of a nonhomogeneous system is a translation by a particular solution of the general solution of the associated homogeneous system.

The next theorem provides pivot conditions such that the span of the m-vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is the entire $\mathbf{R}^m$.

Theorem 2.3.7 (Condition for generating $\mathbf{R}^m$). *Let A be an $m \times k$ matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_k$. Then the following statements are equivalent.*

1. $\mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \mathbf{R}^m$.
2. $A\mathbf{x} = \mathbf{b}$ *is consistent for all* $\mathbf{b} \in \mathbf{R}^m$.
3. *Every m-vector* $\mathbf{b}$ *is a linear combination in the columns of A.*
4. *Each row of A has a pivot position.*

Proof. Statements 1 and 3 are identical by the definition of span. Statements 1 and 2 are equivalent by Theorem 2.3.2. It suffices to prove the equivalence of Statements 2 and 4.
2 ⇒ 4: Suppose that the system is consistent for all $\mathbf{b} \in \mathbf{R}^m$. If one row of A is not a pivot row, then any echelon of $[A : \mathbf{b}]$ will have a row of the form $[0\ 0\ \cdots\ 0\ :\ b]$. Since $\mathbf{b}$ is *any* m-vector, we may choose components for $\mathbf{b}$ so that $b \neq 0$. But then $[A : \mathbf{b}]$ will be inconsistent for the particular $\mathbf{b}$. This contradicts our assumption that $[A : \mathbf{b}]$ is consistent for all m-vectors $\mathbf{b}$. Therefore each row of A must have a pivot position.
4 ⇒ 2: Conversely, suppose that each row of A has a pivot position. Then the last row of any echelon form of A is nonzero. Therefore the last column of the reduced augmented matrix is never a pivot column, no matter what $\mathbf{b}$ is. Hence by Theorem 1.2.10 in Section 1.2 the system $[A : \mathbf{b}]$ is consistent for all m-vectors $\mathbf{b}$. □

Example 2.3.8. Which of the systems $[A : \mathbf{b}]$, and $[B : \mathbf{b}]$ is consistent for all $\mathbf{b} \in \mathbf{R}^3$, where

$$A = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 0 & -1 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 0 & -1 & 1 & -2 \end{bmatrix}?$$

Solution. The row reductions

$$A \sim \begin{bmatrix} -1 & 3 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad B \sim \begin{bmatrix} -1 & 3 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

show that A has one pivot position in each row, and hence $[A : \mathbf{b}]$ is solvable for all $\mathbf{b} \in \mathbf{R}^3$ by Theorem 2.3.7. The third row of B has no pivot, so the system $[B : \mathbf{b}]$ is not solvable for all $\mathbf{b} \in \mathbf{R}^3$.

Example 2.3.9. Can fifteen 16-vectors span $\mathbf{R}^{16}$?

Answer. No. The matrix with columns these vectors can only have a maximum of 15 pivots. However, by Theorem 2.3.7, to span $\mathbf{R}^{16}$, 16 pivots are required.

Theorem 2.3.10 (Reduction of spanning set). *If one of the m-vectors* $\mathbf{v}_1, \dots, \mathbf{v}_k$ *is a linear combination of the remaining ones, then the span remains the same if we remove this vector.*

Proof. By renaming, if necessary, we may assume that $\mathbf{v}_k$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{k-1}$. Let

$$\mathbf{v}_k = c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1}. \tag{2.15}$$

Let $V = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and $V' = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}\}$. We prove that

$$V = V'.$$

Any linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, as we see from $r_1\mathbf{v}_1 + \cdots + r_{k-1}\mathbf{v}_{k-1} = r_1\mathbf{v}_1 + \cdots + r_{k-1}\mathbf{v}_{k-1} + 0\mathbf{v}_k$. Hence $V' \subseteq V$.

Now let $\mathbf{u} \in V$. Then $\mathbf{u} = d_1\mathbf{v}_1 + \cdots + d_k\,\mathbf{v}_k$ for some scalars d_i. By equation (2.15) we have

$$\begin{aligned} \mathbf{u} &= d_1\mathbf{v}_1 + \cdots + d_{k-1}\mathbf{v}_{k-1} + d_k(c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1}) \\ &= (d_1 + d_k c_1)\mathbf{v}_1 + \cdots + (d_{k-1} + d_k c_{k-1})\mathbf{v}_{k-1}. \end{aligned}$$

Thus $\mathbf{u}$ is a linear combination in $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$. Therefore $\mathbf{u} \in V'$. Hence $V \subseteq V'$. We conclude that $V = V'$. □

Exercises 2.3

1. Let $\mathbf{v} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, and let $S = \{\mathbf{w}\}$. *True or False?*

(a) **v** is in S.
(b) **w** is in S.
(c) **v** is in Span(S).
(d) **w** is in Span(S).

2. Let S be a set of n-vectors that contains at least one nonzero vector. Explain why Span(S) has *infinitely* many vectors.

In Exercises 3–4, let

$$\mathbf{a} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}.$$

3. Answer the following questions. Explain your answers.

(a) Is **c** in Span{**a**, **b**}?
(b) Is **b** in Span{**a**, **c**}?
(c) Is **a** in Span{**b**, **c**}?
(d) Is **d** in Span{**a**}?
(e) Is **a** in Span{**d**}?
(f) Is **d** in Span{**c**}?
(g) Is **d** in Span{**b**, **c**}?

4. Answer the following questions. Explain your answers. Is it true that

(a) Span{**a**, **b**} = $\mathbf{R}^2$?
(b) Span{**a**, **d**} = $\mathbf{R}^2$?
(c) Span{**a**, **c**} = Span{**b**, **c**}?
(d) Span{**a**} = Span{**d**}?

In Exercises 5–6, let

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}.$$

5. Answer the following questions and explain your answers.
(a) Is $\mathbf{v}_4$ in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?
(b) Is $\mathbf{v}_4$ in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$?
(c) Is $\mathbf{v}_4$ in $\text{Span}\{\mathbf{v}_2, \mathbf{v}_3\}$?

6. Prove the following statements:
(a) $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbf{R}^3$;
(b) $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\} = \mathbf{R}^3$;
(c) $\text{Span}\{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\} = \text{Span}\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

In Exercises 7–10, determine whether or not the columns of the given $m \times n$ matrix span $\mathbf{R}^m$.

7. (a) $\begin{bmatrix} a & 2a \\ b & 2b \end{bmatrix}$; (b) $\begin{bmatrix} a & b \\ 2a & 2b \end{bmatrix}$.

8. (a) $\begin{bmatrix} a & b & 0 \\ a & b & 0 \end{bmatrix}$; (b) $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$.

9. (a) $\begin{bmatrix} 2 & 0 & 5 \\ 0 & -2 & 0 \\ -1 & 1 & 2 \end{bmatrix}$; (b) $\begin{bmatrix} a & a & 0 \\ a & a & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & 2 & -3 & 2 & -5 & 5 \\ 0 & 0 & 0 & 2 & -5 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

11. Under what restriction(s) on a and b will the columns of $\begin{bmatrix} a & b & 0 \\ a & 0 & 0 \\ a & 0 & 1 \end{bmatrix}$ span $\mathbf{R}^3$?

12. *True or False?*
(a) $\mathbf{R}^{10}$ can be spanned by exactly nine 10-vectors.
(b) $\mathbf{R}^{10}$ cannot be spanned by nine 10-vectors.
(c) $\mathbf{R}^{10}$ can be spanned by ten 9-vectors.
(d) $\mathbf{R}^{10}$ can be spanned by ten 10-vectors.
(e) $\mathbf{R}^{10}$ can be spanned by eleven 10-vectors.
(f) Any twenty 10-vectors span $\mathbf{R}^{10}$.
(g) Twenty 10-vectors can span $\mathbf{R}^{10}$.

13. Prove that

$$\text{Span}\{\mathbf{u}, \mathbf{v}\} = \text{Span}\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}.$$

14. Prove that

$$\text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \text{Span}\{\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w}\}.$$

15. Prove that $S_1 = S_2$, where

$$S_1 = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}, \quad S_2 = \text{Span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix}\right\}.$$

16. Let A be a matrix whose columns span $\mathbf{R}^{10}$. What can you say about

(a) the size of A?

(b) the linear system $A\mathbf{x} = \mathbf{b}$?

17. Let A be a 10×9 matrix.

(a) Can the columns of A span $\mathbf{R}^{10}$? Explain.

(b) Is it true that there is a 10-vector $\mathbf{b}$ such that the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent? Explain.

18. Find a finite spanning set for

$$V = \left\{ \begin{bmatrix} 3a - b \\ 4b \\ -a \end{bmatrix}, \ a, b \in \mathbf{R} \right\}.$$

19. Determine the number of pivots of the matrix A if the system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbf{R}^{35}$.

20. Draw the span of the columns of $\begin{bmatrix} 3 & 5 \\ -3 & 5 \end{bmatrix}$.

21. Find all values of x such that

$$\mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix} \right\} = \mathbf{R}^3.$$

22. Draw the following sets:

$$\text{(a)} \quad \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}; \quad \text{(b)} \quad \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

23. Describe geometrically the set of linear combinations of the form $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where c_1 and c_2 are any *positive* real numbers.

2.4 Linear independence

We now introduce the fundamental concept of linear independence of a sequence of given vectors. This basic concept is related to homogenous linear systems and their solutions.

Definition 2.4.1. The m-vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are called *linearly dependent* if there are scalars $x_1, \dots, x_k$ *not all zero* such that

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0}. \tag{2.16}$$

Thus, there is a nontrivial linear combination of the $\mathbf{v}_i$ representing the zero vector. Equation (2.16) with not all x_i zero is called a *linear dependence relation*. If the vectors are not linearly dependent, then they are called *linearly independent*.

Saying that the vectors are linearly independent means that *there is no linear dependence relation* among them. Therefore *all nontrivial linear combinations of the* $\mathbf{v}_i$ *s yield nonzero vectors*. Equivalently, $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent if

$$\text{from} \quad x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0} \quad \text{it follows that} \quad x_1 = 0, \dots, x_k = 0.$$

Example 2.4.2. The vectors $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 4 \\ 14 \end{smallmatrix}\right]$ are linearly dependent, because if we choose $x_1 = 2$, $x_2 = -6$, and $x_3 = 1$ for the left-hand side of (2.16), then

$$2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + (-6)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 14 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In Example 2.4.2, checking for linear independence involved some guessing of the coefficients. This actually is not necessary as we see in the next two theorems.

Theorem 2.4.3. *Let A be an $m \times k$ matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_k$, and let $\mathbf{x}$ be the k-vector with components $x_1, \dots, x_k$. Then the following statements are equivalent:*

1. $\mathbf{v}_1, \dots, \mathbf{v}_k$ *are linearly dependent.*
2. $A\mathbf{x} = \mathbf{0}$ *has nontrivial solutions.*
3. *A has nonpivot columns.*

Proof. Statements 2 and 3 are equivalent by Theorem 1.2.14 in Section 1.2. The equivalence of Statements 1 and 2 follows from

$$x_1\mathbf{v}_1 + \cdots + x_k\mathbf{v}_k = \mathbf{0} \Leftrightarrow A\mathbf{x} = \mathbf{0}.$$

Indeed, if $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent, then there is a relation $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ with c_i not all zero. This implies that $A\mathbf{c} = \mathbf{0}$ where $\mathbf{c}$ is nonzero and has components c_i. Thus $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions. Thus 1 implies 2. The proof of the converse consists of the reversion of the order of these statements. ☐

Theorem 2.4.3 may also be expressed in terms of linear independence as follows.

Theorem 2.4.4. *In the notation of Theorem* 2.4.3 *the following are equivalent:*

1. $\mathbf{v}_1, \dots, \mathbf{v}_k$ *are linearly independent.*
2. $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*
3. *Each column of A is a pivot column.*

Example 2.4.5. Let

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 14 \\ 9 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ -6 \\ 4 \end{bmatrix}.$$

Check for linear independence the vectors (a) $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and (b) $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$.

Solution. By either Theorem 2.4.3 or Theorem 2.4.4 we need to find the number of pivots of the matrix that has columns the given vectors.

(a) Row reduction yields

$$\begin{bmatrix} 0 & 1 & 3 \\ -2 & 2 & 14 \\ 3 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Column 3 is a nonpivot column, and thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

(b) By row reduction we have

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 14 & -6 \\ 7 & 9 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

All columns are pivot columns, and hence $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent.

1. If one of the m-vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is zero, then the vectors are *linearly dependent*. This is because if, say, $\mathbf{v}_1 = \mathbf{0}$, then

   $$1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \cdots + 0\mathbf{v}_k = \mathbf{0}$$

   is a linear dependence relation.
2. Two vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent if and only if one is a scalar multiple of the other. Indeed, if $\mathbf{v}_1 = k\mathbf{v}_2$, then

   $$1\mathbf{v}_1 + (-k)\,\mathbf{v}_2 = \mathbf{0}$$

   is a linear dependence relation. Conversely, if the vectors are linearly dependent, then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ for some c_1, c_2 not both zero. If $c_1 \neq 0$, then $\mathbf{v}_1 = (-c_2/c_1)\mathbf{v}_2$. So $\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_2$.

The next theorem gives us a useful criterion for linear dependence.

Theorem 2.4.6. *Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be m-vectors with $k \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$. Then these vectors are linearly dependent if and only if at least one vector, say, $\mathbf{v}_i$ $(i \geq 2)$ is a linear combination of the vectors that precede it, i. e., $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}$.*

Proof. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be linearly dependent. Then there are scalars $c_1, \dots, c_k$ not all zero such that

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Let c_i be the last nonzero scalar in this equation. Then

$$c_1\mathbf{v}_1 + \cdots + c_i\mathbf{v}_i = \mathbf{0}$$

with $c_i \neq 0$. Now $i \geq 2$, because $\mathbf{v}_1 \neq \mathbf{0}$. Therefore we can solve for $\mathbf{v}_i$ to get

$$\mathbf{v}_i = \left(-\frac{c_1}{c_i}\right)\mathbf{v}_1 + \cdots + \left(-\frac{c_{i-1}}{c_i}\right)\mathbf{v}_{i-1}.$$

Hence $\mathbf{v}_i$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}$.

Conversely, if $\mathbf{v}_i$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}$, then there are scalars $c_1, \ldots, c_{i-1}$ such that

$$\mathbf{v}_i = c_1\mathbf{v}_1 + \cdots + c_{i-1}\mathbf{v}_{i-1}.$$

Therefore we get the linear dependence relation

$$c_1\mathbf{v}_1 + \cdots + c_{i-1}\mathbf{v}_{i-1} + (-1)\mathbf{v}_i + 0\mathbf{v}_{i+1} + \cdots + 0\mathbf{v}_k = \mathbf{0}.$$

Hence the vectors are linearly dependent. □

1. From Theorem 2.4.6 we conclude that the set $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is linearly dependent if and only if at least one of the vectors is in the span of the remaining vectors (Figure 2.18). Hence the set is linearly independent if and only if none of the vectors is in the span of the others (Figure 2.19).

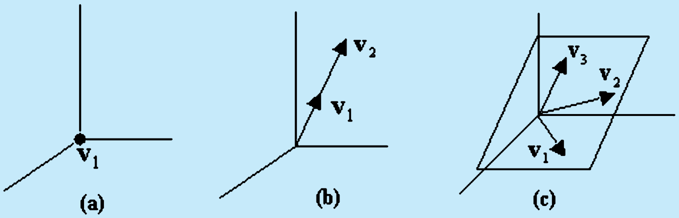

Figure 2.18: Linear dependence: (a) one vector; (b) two vectors; (c) three vectors.

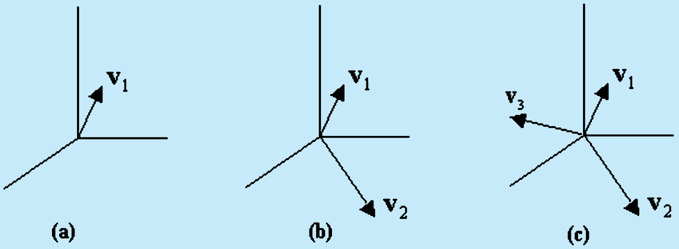

Figure 2.19: Linear independence: (a) one vector; (b) two vectors; (c) three vectors.

2. Theorem 2.4.6 does *not* say that *every* vector is a linear combination of the remaining (or the preceding) vectors. For example, $\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\}$ is linearly dependent, but the last vector is not a linear combination of the first two.

One useful consequence of Theorem 2.4.4 is the following theorem.

Theorem 2.4.7. *If the m-vectors* $\mathbf{v}_1, \dots, \mathbf{v}_k$ *are linearly independent, then* $k \le m$.

Proof. Due to linear independence, the matrix $[\mathbf{v}_1 \cdots \mathbf{v}_k]$ has k pivot columns by Theorem 2.4.4. The number of pivots cannot exceed either the number of columns or the number of rows of a matrix. Therefore $k \le m$, as stated. □

Theorem 2.4.7 really says the following:

If there are more vectors than components, then the vectors are linearly dependent.

Finally, in the following theorem, we give two more useful properties of linearly independent sets.

Theorem 2.4.8. *In* $\mathbf{R}^m$, *let* $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *be linearly independent, and let* $\mathbf{v}$ *be a vector.*

1. *If* $\mathbf{v}$ *is in* Span(S), *then* $\mathbf{v}$ *is a linear combination of vectors in S with unique coefficients.*
2. *If* $\mathbf{v}$ *is not in* Span(S), *then the set* $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}\}$ *is linearly independent.*

Proof. 1. By assumption, $\mathbf{v}$ is a linear combination of the $\mathbf{v}_i$s. If $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ and $\mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_k\mathbf{v}_k$, then

$$(c_1 - d_1)\mathbf{v}_1 + \cdots + (c_k - d_k)\mathbf{v}_k = \mathbf{0}.$$

Hence $c_1 - d_1 = \cdots = c_k - d_k = 0$, because S is linearly independent. Therefore

$$c_1 = d_1, \dots, c_k = d_k,$$

and we have the uniqueness of the coefficients.
2. Suppose, on the contrary, that $S' = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}\}$ is linearly dependent. Then there is a linear dependence relation

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c\mathbf{v} = \mathbf{0}.$$

If $c \ne 0$, then we may solve the equation for $\mathbf{v}$, but then $\mathbf{v}$ would be in the span of S, which contradicts our assumption. If on the other hand, if $c = 0$, then the equation reduces to

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

with at least one $c_i \ne 0$. This contradicts the assumption that S is linearly independent. We conclude that S' has to be linearly independent. □

Exercises 2.4

In Exercises 1–4, determine whether the vectors are linearly independent.

1. $\begin{bmatrix} 6 \\ -2 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}.$

2. $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -9 \\ -5 \end{bmatrix}.$

3. $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix}.$

4. $\begin{bmatrix} a \\ a \\ 1 \end{bmatrix}, \begin{bmatrix} b \\ b \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for $a \neq b$.

5. Prove that the following vectors are linearly dependent and find a linear dependence relation:

$$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

In Exercises 6–7, determine whether the columns of the matrix are linearly independent.

6. $\begin{bmatrix} -3 & 1 & 1 & 0 \\ 2 & 1 & 0 & -1 \\ 4 & 1 & -1 & 1 \end{bmatrix}.$

7. $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

In Exercises 8–11, use inspection to determine whether or not the vectors are linearly independent.

8. $\begin{bmatrix} 555 \\ 123 \end{bmatrix}, \begin{bmatrix} 55500 \\ 12300 \end{bmatrix}.$

9. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$

10. $\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 20 \\ 10 \\ 30 \\ -10 \end{bmatrix}.$

11. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ a \\ 0 \end{bmatrix}.$

12. Determine in Figure 2.20 which of (a), (b), and (c) show linearly independent vectors in $\mathbf{R}^3$.

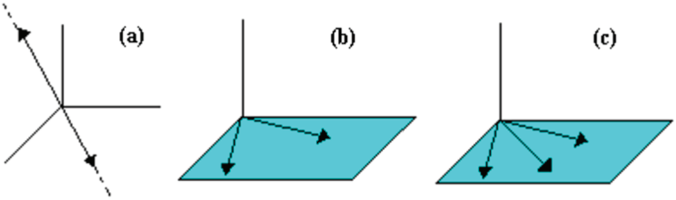

Figure 2.20: Check for linear independence.

13. Are there real values a for which the given set is linearly dependent? If true, find these values.

(a) $\left\{\begin{bmatrix} a \\ 1 \end{bmatrix}, \begin{bmatrix} a+2 \\ a \end{bmatrix}\right\}$;

(b) $\left\{\begin{bmatrix} a \\ 2 \end{bmatrix}, \begin{bmatrix} a-2 \\ a \end{bmatrix}\right\}$.

14. What condition on a and b will make the vectors $\begin{bmatrix} a \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ b \end{bmatrix}$ linearly dependent?

15. By noting that the first column of the matrix A equals the difference of the third and second columns, find a nontrivial solution of the system $A\mathbf{x} = \mathbf{0}$ without actually solving the system.

$$A = \begin{bmatrix} 9 & -6 & 3 \\ 9 & 0 & 9 \\ -1 & 5 & 4 \end{bmatrix}.$$

16. If the 3×3 matrix A has linearly independent columns, then what are possible reduced row echelon forms of A?

17. If the 3×2 matrix A has linearly independent columns, then what are possible reduced row echelon forms of A?

18. If the 2×2 matrix A has linearly dependent columns, then what are possible reduced row echelon forms of A?

19. *True or False?*

(a) Any 2 distinct n-vectors are linearly independent.

(b) Any n linearly independent n-vectors span $\mathbf{R}^n$.

(c) $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2\}$ is linearly independent.

(d) 5001 vectors in $\mathbf{R}^{5000}$ are always linearly dependent.

20. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be linearly independent. Prove that each of the following is also linearly independent:

(a) $\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_1\}$;

(b) $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$;

(c) $\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$.

21. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be linearly independent. Find c_1, c_2, and c_3 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = (2c_2 - c_1)\mathbf{v}_1 + (c_3 - c_2)\mathbf{v}_2 + (c_2 - 1)\mathbf{v}_3.$$

22. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be linearly dependent. Prove that for any scalar c, the set $\{c\mathbf{v}_1, \dots, c\mathbf{v}_k\}$ is also linearly dependent.

23. Let S be linearly independent. Prove that any nonempty subset of S is also linearly independent.

24. Let S be a set that contains a linearly dependent subset. Prove that S is also linearly dependent.

25. Suppose the columns of the $m \times n$ matrix A are linearly independent. Prove that for any m-vector $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has at most one solution.

26. Give an example of three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ such that $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent, but $\{\mathbf{v}_1, \mathbf{v}_3\}$ is linearly dependent.

27. Give an example of three vectors in $\mathbf{R}^3$ that are linearly dependent but all possible pairs are linearly independent.

28. Let A be an $n \times n$ matrix with linearly independent columns. Prove that for any n-vector $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

29. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation, and let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be linearly dependent n-vectors. Prove that $T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)$ are linearly dependent m-vectors.

30. Suppose that $S_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ are linearly independent subsets of $\mathbf{R}^3$. What geometric object is the intersection $\mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$?

31. Prove that the pivot columns of any reduced row echelon form matrix are linearly independent.

32. Write the solution set of the single linear equation

$$x_1 + \cdots + x_n = 0$$

as the span of $n-1$ linearly independent n-vectors.

2.5 Dot product, lines, hyperplanes

In this section, we focus on a useful special case of matrix–vector product, called the *dot product*. Using this product, we study geometric objects in $\mathbf{R}^n$ such as lines and hyperplanes. The solution sets of linear systems are, in general, intersections of hyperplanes.

2.5.1 Dot product

Definition 2.5.1. The *dot product* $\mathbf{u} \cdot \mathbf{v}$ of two n-vectors $\mathbf{u} = [u_1 \cdots u_n]^T$ and $\mathbf{v} = [v_1 \cdots v_n]^T$ is the matrix–vector product $\mathbf{u}^T\mathbf{v}$:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} = [u_1 \cdots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + \cdots + u_nv_n. \tag{2.17}$$

For simplicity, we identified the 1×1 matrix $[u_1v_1 + \cdots + u_nv_n]$ with its single entry $u_1v_1 + \cdots + u_nv_n$. Thus we view the dot product as a *number*. If the dot product of two vectors is zero, then we call these vectors *orthogonal*.

Example 2.5.2. For $\mathbf{u} = [\,-3\ 2\ 1\,]^T$, $\mathbf{v} = [\,4\ -1\ 5\,]^T$, and $\mathbf{w} = [\,-2\ 1\ -8\,]^T$, we have

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = (-3)\,4 + 2\,(-1) + (1)\,(5) = -9.$$

Likewise, we see that $\mathbf{u} \cdot \mathbf{w} = 0$. Hence $\mathbf{u}$ and $\mathbf{w}$ are orthogonal.

In the following definition, we generalize the notion of length for plane or space vectors.

Definition 2.5.3. The *norm*, or *length*, or *magnitude* of an n-vector $\mathbf{u}$ is the positive square root

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \left(u_1^2 + \cdots + u_n^2\right)^{\frac{1}{2}}.$$

The (*Euclidean*) *distance* between two n-vectors $\mathbf{u}$ and $\mathbf{v}$ is

$$\|\mathbf{u} - \mathbf{v}\|.$$

An n-vector is a *unit* vector, if its norm is 1. Note that for any scalar c, we have

$$\|c\mathbf{u}\| = |c|\,\|\mathbf{u}\|.$$

Example 2.5.4. Let $\mathbf{v} = [\,1\ 2\ -3\ 1\,]^T$ and $\mathbf{u} = [\,\frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\,]^T$. The length of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \left(1^2 + 2^2 + (-3)^2 + 1^2\right)^{\frac{1}{2}} = \sqrt{15}.$$

The distance between $\mathbf{v}$ and $\mathbf{u}$ is

$$\|\mathbf{v} - \mathbf{u}\| = \left\|\left(\frac{1}{2}, \frac{5}{2}, -\frac{7}{2}, \frac{3}{2}\right)\right\| = \sqrt{21}.$$

Furthermore, $\mathbf{u}$ is a unit vector.

The definition of length agrees with our geometric intuition of length for 2-vectors or 3-vectors. This can be seen by the use of the Pythagorean theorem (Figure 2.21).

The dot product for plane and space vectors is related to the length and angle between the vectors by the formula

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\ \|\mathbf{v}\| \cos\theta. \tag{2.18}$$

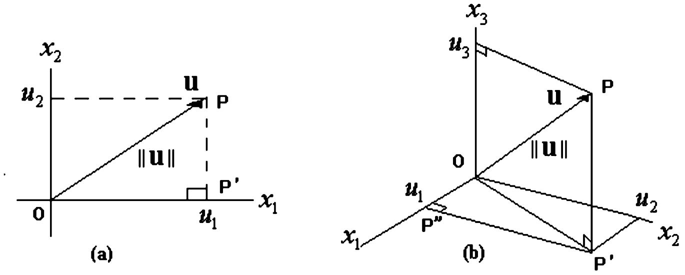

Figure 2.21: The norm for plane and space vectors.

This is proved by using the *law of cosines* on the space triangle OPQ, where θ is the angle between OP and OQ:

$$\begin{aligned}\|\mathbf{u}\|\ \|\mathbf{v}\|\ \cos\theta &= \frac{1}{2}\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|PQ\|^2\right)\\ &= \frac{1}{2}\left(\sum_{i=1}^{3} u_i^2 + \sum_{i=1}^{3} v_i^2 - \sum_{i=1}^{3}(v_i - u_i)^2\right)\\ &= \sum_{i=1}^{3} u_i v_i = \mathbf{u}\cdot\mathbf{v}.\end{aligned}$$

Note that if $\theta = \frac{\pi}{2}$, i. e., if $\mathbf{u}$ and $\mathbf{v}$ are at right angles, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Theorem 2.5.5 (Properties of dot product). *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be any n-vectors and let c be any scalar. Then*

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$; *(Symmetry)*
2. $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$; *(Additivity)*
3. $c\,(\mathbf{u}\cdot\mathbf{v}) = (c\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(c\mathbf{v})$; *(Homogeneity)*
4. $\mathbf{u}\cdot\mathbf{u} \geq 0$. *Also,* $\mathbf{u}\cdot\mathbf{u} = 0$ *if and only if* $\mathbf{u} = \mathbf{0}$. *(Positive definiteness)*

Proof of 4. If $\mathbf{u} = [\,u_1 \cdots u_n\,]^T$, then $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2 = u_1^2 + \cdots + u_n^2 \geq 0$, and

$$\begin{aligned}\mathbf{u}\cdot\mathbf{u} = 0 &\Leftrightarrow u_1^2 + \cdots + u_n^2 = 0\\ &\Leftrightarrow u_1 = \cdots = u_n = 0\\ &\Leftrightarrow \mathbf{u} = \mathbf{0}.\end{aligned}$$

The proof of the remaining properties is left as an exercise. □

Theorem 2.5.5 can be used to expand the square of the length of a sum of vectors as follows:

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v} \tag{2.19}$$

This can be seen from

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\cdot\mathbf{u} + +\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}\\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}\\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}.\end{aligned}$$

We now discuss the following useful inequality.

Theorem 2.5.6 (Cauchy–Bunyakovsky–Schwarz inequality (CBSI)). *For any n-vectors* $\mathbf{u}$ *and* $\mathbf{v}$*, we have*

$$|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\;\|\mathbf{v}\|\,. \tag{2.20}$$

Furthermore, equality holds if and only if $\mathbf{u}$ *and* $\mathbf{v}$ *are scalar multiples of each other.*

Proof. Let x be any scalar. We use identity (2.19) and Part 4 of Theorem 2.5.5 to get

$$0 \le \|x\,\mathbf{u}+\mathbf{v}\|^2 = x^2\,\|\mathbf{u}\|^2 + x\,(2\,\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|^2\,.$$

If we let $p(x) = ax^2 + bx + c$ with $a = \|\mathbf{u}\|^2$, $b = 2\,\mathbf{u}\cdot\mathbf{v}$, and $c = \|\mathbf{v}\|^2$, we see that $a \ge 0$ and $p(x) \ge 0$. So the graph of $p(x)$ is a parabola in the upper half-plane that opens upward. Hence the parabola is either above the x-axis or tangent to it. This means that $b^2 - 4ac \le 0$. Therefore

$$(2\,\mathbf{u}\cdot\mathbf{v})^2 - 4\,\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2 \le 0,$$

which implies the stated inequality. The proof of the statement about the case of equality is left as an exercise. □

The Cauchy–Bunyakovsky–Schwarz inequality implies that

$$\frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|\;\|\mathbf{v}\|} \le 1 \quad \text{or} \quad -1 \le \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\;\|\mathbf{v}\|} \le 1.$$

Since any number between –1 and 1 can be written as $\cos\theta$ for a unique $0 \le \theta \le \pi$, the last inequality allows us to define the angle between two n-vectors.

The *angle* between two nonzero n-vectors $\mathbf{u}$ and $\mathbf{v}$ is the unique number θ such that

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\;\|\mathbf{v}\|}\,, \qquad 0 \le \theta \le \pi. \tag{2.21}$$

We can also write the dot product in terms of the angle

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\;\|\mathbf{v}\|\cos\theta. \tag{2.22}$$

2.5.2 Orthogonal projections

The dot product can be used to write any n-vector as a sum of orthogonal vectors. A typical application occurs in physics where a *force* vector is broken up into a sum of orthogonal components along desirable directions.

Let $\mathbf{u}$ and $\mathbf{v}$ be given nonzero vectors. We want to write $\mathbf{u}$ as

$$\mathbf{u} = \mathbf{u}_{\mathrm{pr}} + \mathbf{u}_{\mathrm{c}},$$

where $\mathbf{u}_{\mathrm{pr}}$ is a scalar multiple of $\mathbf{v}$, and $\mathbf{u}_{\mathrm{c}}$ is orthogonal to $\mathbf{u}_{\mathrm{pr}}$ (Figure 2.22). Such a decomposition of $\mathbf{u}$ is always possible and it is *unique*. The vector $\mathbf{u}_{\mathrm{pr}}$ is called the *orthogonal projection of* $\mathbf{u}$ *on* $\mathbf{v}$. The vector $\mathbf{u}_{\mathrm{c}}$ is called the *vector component of* $\mathbf{u}$ *orthogonal to* $\mathbf{v}$.

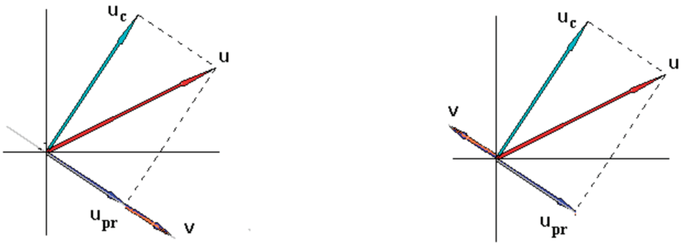

Figure 2.22: The orthogonal projection of **u** on **v**.

We compute $\mathbf{u}_{\mathrm{pr}}$ and $\mathbf{u}_{\mathrm{c}}$ in terms of $\mathbf{u}$ and $\mathbf{v}$. Since $\mathbf{u}_{\mathrm{pr}}$ and $\mathbf{v}$ have the same direction, we have $\mathbf{u}_{\mathrm{pr}} = c\,\mathbf{v}$ for some scalar c. In addition, since $\mathbf{u}_{\mathrm{c}}$ and $\mathbf{v}$ are orthogonal, we have that $\mathbf{u}_{\mathrm{c}} \cdot \mathbf{v} = 0$. Hence,

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= (\mathbf{u}_{\mathrm{pr}} + \mathbf{u}_{\mathrm{c}})\cdot\mathbf{v} = \mathbf{u}_{\mathrm{pr}}\cdot\mathbf{v} + \mathbf{u}_{\mathrm{c}}\cdot\mathbf{v} \\ &= (c\mathbf{v})\cdot\mathbf{v} + 0 = c\,(\mathbf{v}\cdot\mathbf{v}) \\ &\Rightarrow c = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}. \end{aligned}$$

Therefore,

$$\mathbf{u}_{\mathrm{pr}} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v} \quad \text{the orthogonal projection of } \mathbf{u} \text{ on } \mathbf{v} \tag{2.23}$$

and

$$\mathbf{u}_{\mathrm{c}} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\,\mathbf{v} \quad \text{the vector component of } \mathbf{u} \text{ orthogonal to } \mathbf{v}. \tag{2.24}$$

Example 2.5.7. Let $\mathbf{u} = [\,1\;1\;1\,]^T$ and $\mathbf{v} = [\,2\;2\;0\,]^T$. Find the orthogonal projection $\mathbf{u}_{\mathrm{pr}}$ of $\mathbf{u}$ on $\mathbf{v}$ and the vector component $\mathbf{u}_{\mathrm{c}}$ of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

Solution. We have

$$\mathbf{u}_{\mathrm{pr}} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{4}{8}\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

$$\mathbf{u}_{\mathrm{c}} = \mathbf{u} - \mathbf{u}_{\mathrm{pr}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The answer is geometrically obvious as seen from Figure 2.23.

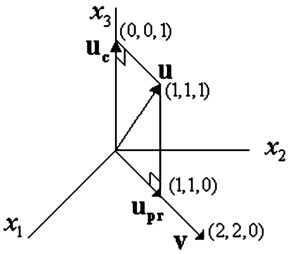

Figure 2.23: Projecting (1, 1, 1) on (2, 2, 0).

2.5.3 Lines, planes, and hyperplanes

The dot product can be used to deduce the equations of geometric objects such as planes in $\mathbf{R}^3$ and, more generally, hyperplanes in $\mathbf{R}^n$. First, we discuss the equations for lines.

Lines in $\mathbf{R}^n$ Let l be the space line passing through a given point $P(x_0, y_0, z_0)$ and parallel to a given *nonzero* vector $\mathbf{n} = [\, a \; b \; c \,]^T$ (Figure 2.24). Let $X(x, y, z)$ be any point in l. By $\mathbf{p}$ we denote all scalar multiples $t\,\mathbf{n}$ $(-\infty < t < \infty)$. These represent all possible vectors parallel to $\mathbf{n}$. Since $\mathbf{x} - \mathbf{p}$ is parallel to $\mathbf{n}$, we must have $\mathbf{x} - \mathbf{p} = t\,\mathbf{n}$ for some scalar t.

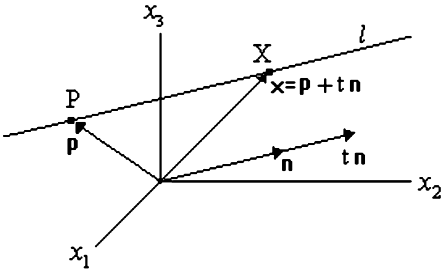

Figure 2.24: The line through **p** in the direction of **n**.

Therefore, $\mathbf{x}$ represents points that cover l as t runs through $\mathbf{R}$. We have

$$\mathbf{x} = \mathbf{p} + t\,\mathbf{n}, \quad -\infty < t < \infty. \tag{2.25}$$

This vector equation is called a *parametric equation* of the line with *parameter* t. Equation (2.25) is equivalent to three equations in the components, called the *parametric equations* of the line:

$$x = x_0 + t\,a, \quad y = y_0 + t\,b, \quad z = z_0 + t\,c. \tag{2.26}$$

Equation (2.25) is also equivalent to the set equality

$$l = \mathbf{p} + \operatorname{Span}\{\mathbf{n}\},$$

which expresses the geometric fact that line l is a translation by $\mathbf{p}$ of the line through the origin and the tip of $\mathbf{n}$.

Example 2.5.8. The vector parametric equation of the line through $P(1,-1,2)$ in the direction of $\mathbf{n} = [\,1\;1\;1\,]^T$ is given by

$$\mathbf{x} = \mathbf{p} + t\,\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Equation (2.25) is also valid for plane lines. In fact, it is used for n-vectors to define lines in $\mathbf{R}^n$. Let $\mathbf{n} = [\,a_1 \cdots a_n\,]^T$ be a nonzero n-vector. The *line* in $\mathbf{R}^n$ through the point $\mathbf{p}$ (viewed as a vector) in the direction of $\mathbf{n}$ is the set of all points $\mathbf{x}$ (vectors) of the form

$$\mathbf{x} = \mathbf{p} + t\,\mathbf{n}, \quad -\infty < t < \infty. \tag{2.27}$$

Example 2.5.9. The parametric equation of the line in $\mathbf{R}^5$ passing through the point $(1,2,3,4,5)$ in the direction $[\,1\;1\;1\;1\;1\,]^T$ is

$$\begin{aligned} [\,x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5\,]^T &= [\,1 \quad 2 \quad 3 \quad 4 \quad 5\,]^T \\ &\quad + t\,[\,1 \quad 1 \quad 1 \quad 1 \quad 1\,]^T \end{aligned}$$

Equation (2.27) is also equivalent to the set equality

$$l = \mathbf{p} + \operatorname{Span}\{\mathbf{n}\}. \tag{2.28}$$

Planes in $\mathbf{R}^3$ Next, we define planes in $\mathbf{R}^3$. A nonzero vector $\mathbf{n} = [\,a\;b\;c\,]^T$ is called a *normal* to a plane $\mathcal{P}$ if it is perpendicular to $\mathcal{P}$. Let $P(x_0, y_0, z_0)$ be a given point in $\mathcal{P}$, and let $X(x, y, z)$ be any other point. If $\mathbf{p}$ and $\mathbf{x}$ are the corresponding vectors, then $\mathbf{x} - \mathbf{p}$ is parallel to $\mathcal{P}$ and hence orthogonal to the normal $\mathbf{n}$ (Figure 2.25). Therefore, we have

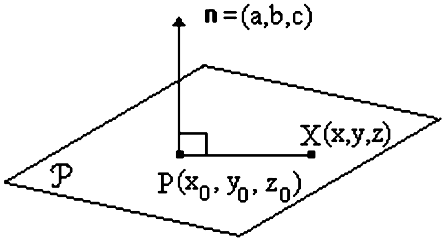

Figure 2.25: The plane through P with normal $\mathbf{n}$.

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0. \tag{2.29}$$

In terms of components, this equation can be rewritten in the form

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0. \tag{2.30}$$

Equation (2.30) characterizes all the points $\mathbf{x}$ of $\mathcal{P}$ in terms of a normal vector $\mathbf{n}$ and a point $\mathbf{p}$ of $\mathcal{P}$. It is called a *point-normal form* of the equation of the plane. Equation (2.30) can be rewritten in the form

$$ax + by + cz = d \tag{2.31}$$

with $d = ax_0 + by_0 + cz_0$. This is the *(general) equation* of the plane.

Example 2.5.10. The equation of the plane passing through $(-1, 2, 3)$ and perpendicular to $\begin{bmatrix} -2 & 1 & 4 \end{bmatrix}^T$ is

$$-2 \cdot (x + 1) + 1 \cdot (y - 2) + 4 \cdot (z - 3) = 0$$

or in the general equation form (2.31)

$$-2x + y + 4z = 16.$$

Hyperplanes in $\mathbf{R}^n$ Equations (2.29), (2.30), and (2.31) can be used with n-vectors. More precisely, if $\mathbf{n} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix}^T \neq \mathbf{0}$ and $\mathbf{p} = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix}^T$, then the set $\mathcal{H}$ of points (vectors) $\mathbf{x} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$ in $\mathbf{R}^n$ that satisfy the equation

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0 \tag{2.32}$$

is called the *hyperplane* in $\mathbf{R}^n$ passing through $\mathbf{p}$ with *normal* $\mathbf{n}$. In terms of components, we have the *point-normal equation* of the hyperplane

$$a_1(x_1 - p_1) + \cdots + a_n(x_n - p_n) = 0. \tag{2.33}$$

Equation (2.33) can also be written in the form

$$a_1x_1 + \cdots + a_nx_n = d. \tag{2.34}$$

This is the *general equation of a hyperplane*, and it is a *linear equation*.

Example 2.5.11. Find an equation of the hyperplane $\mathcal{H}$ in $\mathbf{R}^4$ passing through the point $(1, 2, 3, 4)$ and normal to the direction $[\,-1\ 2\ -2\ 1\,]^T$.

Solution. Equation (2.33) with $\mathbf{n} = [\,-1\ 2\ -2\ 1\,]^T$ and $\mathbf{p} = [\,1\ 2\ 3\ 4\,]^T$ yields

$$-1 \cdot (x_1 - 1) + 2 \cdot (x_2 - 2) - 2 \cdot (x_3 - 3) + 1 \cdot (x_4 - 4) = 0$$

or in general equation form

$$x_1 - 2x_2 + 2x_3 - x_4 = -1.$$

2.5.4 Hyperplanes and solutions of linear systems

Hyperplanes are intimately related to solution sets of linear systems. The general equation (2.34) of the hyperplane is a consistent linear equation with $[\,a_1 \cdots a_n\,]^T \neq \mathbf{0}$. Hence, a hyperplane is the solution set of a linear equation. We have the following theorem.

Theorem 2.5.12. *Let S be a set of n-vectors. The following are equivalent:*
1. *S is a hyperplane.*
2. *S is the solution set of a consistent linear equation with nonzero coefficient matrix.*

As an immediate consequence, we have the important fact that, in general, the solution sets of linear systems are intersections of hyperplanes.

Theorem 2.5.13. *The solution set of a linear system $A\mathbf{x} = \mathbf{b}$ is the intersection of hyperplanes, provided that each linear equation is consistent with nonzero coefficient matrix.*

Theorem 2.5.13 does not assume that the linear system is consistent.

Exercises 2.5

Dot Product

In Exercises 1–5, use

$$\mathbf{d} = \begin{bmatrix} -1 & -2 & 1 & \sqrt{3} \end{bmatrix}^T, \quad \mathbf{u} = \begin{bmatrix} -1 & 2 & -2 \end{bmatrix}^T, \quad \mathbf{v} = \begin{bmatrix} 4 & -3 & 5 \end{bmatrix}^T.$$

1. Compute the expressions

(a) $\mathbf{u} \cdot \mathbf{v}$, (b) $\mathbf{u} \cdot \mathbf{u} - \|\mathbf{u}\|^2$, (c) $(\mathbf{d} \cdot \mathbf{d})\,\mathbf{d}$,

(d) $\|\mathbf{v}\|\,\mathbf{v} + \|\mathbf{u}\|\,\mathbf{u}$, (e) $\|\mathbf{u}\| - \|\mathbf{v}\|$, (f) $(1/\|\mathbf{d}\|)\,\mathbf{d}$.

2. Which expressions are *undefined* and why? Compute the expressions that *are* defined.

(a) $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{v})$, (b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{v}$, (c) $(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{v})$,
(d) $\mathbf{u}\cdot(3\mathbf{d})$, (e) $(\mathbf{d}\cdot\mathbf{d})^3$, (f) $\mathbf{d}\cdot\mathbf{d}+2$.

3. Find the unit vector in the direction of **u**.

4. Find a vector of length 2 in the opposite direction of that of **u**.

5. Find a vector in the direction of **v** with length 9 times the length of **v**.

6. Find the distance between the points $P(4,-3,2,0)$ and $Q(0,2,-6,4)$ in $\mathbf{R}^4$.

7. Which value of x makes the two vectors

$$\begin{bmatrix} 1 & 2 & 4 & -3 \end{bmatrix}^T, \quad \begin{bmatrix} x & 7 & 3 & x \end{bmatrix}^T$$

orthogonal?

8. Find two vectors orthogonal to the vector $\begin{bmatrix} 0 & -5 & 2 & 7 \end{bmatrix}^T$.

9. Find the orthogonal projection of **u** onto **v**.

$$\begin{aligned} &\text{(a)}\quad \mathbf{u}=\begin{bmatrix} 0 & -1 & 6 \end{bmatrix}^T, \quad \mathbf{v}=\begin{bmatrix} -1 & -3 & 5 \end{bmatrix}^T; \\ &\text{(b)}\quad \mathbf{u}=\begin{bmatrix} -2 & -1 & 0 & 1 \end{bmatrix}^T, \quad \mathbf{v}=\begin{bmatrix} 0 & 0 & -1 & 3 \end{bmatrix}^T. \end{aligned}$$

10. Referring to Exercise 9, find the vector component of **u** orthogonal to **v**.

11. Verify CBSI for the pairs **u** and **v** of Exercise 9.

12. Prove that equality holds in CBSI if and only if **u** and **v** are scalar multiples of each other.

13. Prove Parts 1–3 of Theorem 2.5.5.

14. *True of false?* Explain. If the claim is false, then give an example that shows this.
(a) If $\mathbf{u}\cdot\mathbf{v}=0$, then either $\mathbf{u}=\mathbf{0}$, or $\mathbf{v}=\mathbf{0}$.
(b) If $\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{u}\neq\mathbf{0}$, then $\mathbf{v}=\mathbf{w}$.
(c) The sum of two unit vectors is a unit vector.

15. *(Pythagorean theorem)* Prove the Pythagorean theorem, which states that two n-vectors **u** and **v** are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2=\|\mathbf{u}\|^2+\|\mathbf{v}\|^2 .$$

16. *(Triangle inequality)* Prove the triangle inequality, which states that for any n-vectors **u** and **v**, we have

$$\|\mathbf{u}+\mathbf{v}\|\le\|\mathbf{u}\|+\|\mathbf{v}\| .$$

Geometrically, the triangle inequality expresses the fact that the sum of two sides of a triangle is less than the third side (Figure 2.26).

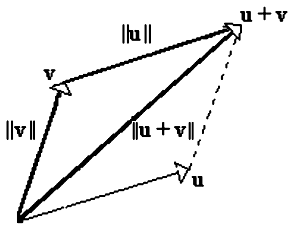

Figure 2.26: The triangle inequality.

17. (*Parallelogram law*) Prove the following identity valid for n-vectors (Figure 2.27):

$$\|\mathbf{u}+\mathbf{v}\|^2+\|\mathbf{u}-\mathbf{v}\|^2=2\|\mathbf{u}\|^2+2\|\mathbf{v}\|^2.$$

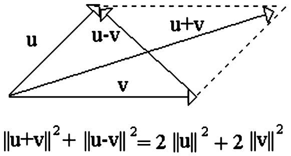

Figure 2.27: The parallelogram law.

18. (*Polarization identity*) Prove the following identity that expresses the dot product in terms of the norm:

$$\mathbf{u}\cdot\mathbf{v}=\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2-\frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2.$$

19. Let $\|\mathbf{u}+\mathbf{v}\|=4$ and $\|\mathbf{u}-\mathbf{v}\|=2$. Find $\mathbf{u}\cdot\mathbf{v}$. (*Hint:* Use Exercise 18.)

20. Prove that **u** and **v** are orthogonal if and only if $\|\mathbf{u}+\mathbf{v}\|=\|\mathbf{u}-\mathbf{v}\|$.

21. Prove that if **u** is orthogonal to **v** and to **w**, then it is orthogonal to any linear combination $c\mathbf{v}+d\mathbf{w}$.

22. Let **u** and $\mathbf{v}_1,\dots,\mathbf{v}_n$ be n-vectors such that **u** is orthogonal to each $\mathbf{v}_i$. Prove that **u** is orthogonal to each vector in the span of $\mathbf{v}_1,\dots,\mathbf{v}_n$.

23. Describe the geometrical shape of the set of 3-vectors **x** with the property

$$\mathbf{x}\cdot\begin{bmatrix}1 & 1 & 1\end{bmatrix}^T=0.$$

24. Describe geometrically the set of all 3-vectors **v** such that $\|\mathbf{v}\|=1$.

25. Let $T:\mathbf{R}^n\to\mathbf{R}$ be a linear transformation. Prove that there exists an n-vector **u** such that $T(\mathbf{v})=\mathbf{u}\cdot\mathbf{v}$ for all $\mathbf{v}\in\mathbf{R}^n$.

Lines, planes, hyperplanes

In Exercises 26–36, we let l_1, l_2, l_3, and l_4 be the lines with respective parametric equations

$$x=-3t+5,\quad y=2t-4,\quad z=-t+2;$$

$$x=6t+1,\quad y=-4t+6,\quad z=2t+8;$$

$$x = -s + 5, \quad y = -s - 7, \quad z = s + 11;$$
$$x = s + 14, \quad y = 2s - 2, \quad z = 3s + 13.$$

Also, let $P(5, -4, 2)$, $Q(2, -2, 1)$, and $R(1, -2, 2)$ be points in $\mathbf{R}^3$.

26. Which of P, Q, and R are in l_1?

27. Find the intersection of l_3 with the coordinate planes.

28. Find all pairs of parallel lines from l_1, l_2, l_3, l_4.

29. Find all pairs of perpendicular lines from l_1, l_2, l_3, l_4.

30. Prove that l_1 and l_4 intersect. Find their point of intersection.

31. Prove that l_1 and l_3 are *skew lines* (i. e., they are not parallel and they do not intersect).

32. Find the parametric equations of the line through P and parallel to $\begin{bmatrix} 4 & -3 & 1 \end{bmatrix}^T$.

33. Find the parametric equations of the line through P and Q.

34. Which of P, Q, and R are in the plane $x + 3y + 3z + 1 = 0$?

35. Find an equation of the plane through P, Q, and R.

36. Find an equation of the plane that contains the lines l_1 and l_4.

37. Find an equation of the plane through the point $X = (-4, 2, 7)$ with normal $\mathbf{n} = \begin{bmatrix} -3 & 2 & 1 \end{bmatrix}^T$.

38. Find an equation of the plane passing through $(2, 3, -1)$ and perpendicular to $\begin{bmatrix} -2 & 4 & 1 \end{bmatrix}^T$.

39. Find the equation of the plane passing through $(-1, -2, 5)$ and parallel to the plane $x - 6y + 2z - 3 = 0$.

40. Find the parametric equations for the line of intersection of the planes $x - y + z - 3 = 0$ and $-x + 5y + 3z + 4 = 0$.

41. Find an equation of the hyperplane in $\mathbf{R}^5$ passing through the point $(1, 2, 0, -1, 0)$ and normal $\begin{bmatrix} -1 & 3 & -2 & 8 & 4 \end{bmatrix}^T$.

42. Prove that the lines $m_1\colon \mathbf{x} = \mathbf{x}_1 + t\mathbf{n}_1$ and $m_2\colon \mathbf{x} = \mathbf{x}_2 + s\mathbf{n}_2$ intersect if and only if $\mathbf{x}_2 - \mathbf{x}_1$ is in $\operatorname{Span}\{\mathbf{n}_1, \mathbf{n}_2\}$.

43. Prove that a matrix transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ maps a straight line to a straight line or to a point.

44. Find an example of a matrix transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that maps all straight lines to straight lines.

45. Give an example of a matrix transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that maps some, but not all, straight lines to points.

46. Give an example of a matrix transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ that maps all straight lines to points.

47. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of nonzero vectors in $\mathbf{R}^n$, with $k \le n$, such that all possible pairs of these vectors are orthogonal. Prove that S is linearly independent.

48. (*Planes in* $\mathbf{R}^n$) A *plane* in $\mathbf{R}^n$ is the set of n-vectors of the form $\mathcal{P} = \mathbf{v}_0 + \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ are fixed vectors, and $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Consider the plane in $\mathbf{R}^4$ passing through the points $(1, 0, 0, 1)$, $(0, 1, 1, 0)$, and $(0, 0, 1, 1)$.

(a) Write this plane in the form $\mathcal{P} = \mathbf{v}_0 + \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
(b) Is $(1, -1, 1, 0)$ in this plane? Explain.

2.6 Application: Computer graphics

In this section, we study some special matrix and other transformations often used in computer graphics to transform images. These transformations are transformations of the plane $\mathbf{R}^2$ or the space $\mathbf{R}^3$.

2.6.1 Plane matrix transformations

We now study some basic matrix transformations of the plane $T : \mathbf{R}^2 \to \mathbf{R}^2$.

Reflections

Reflections can be defined about any line in the plane. We are interested in reflections about a line through the origin (Figure 2.28).

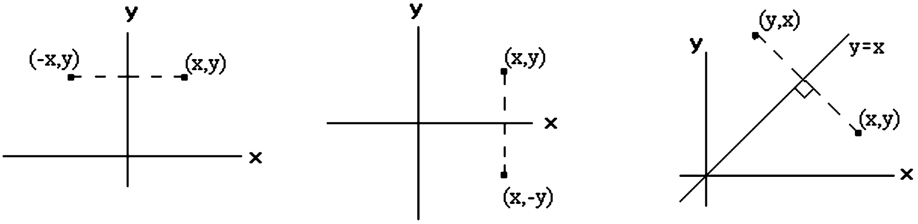

Figure 2.28: Reflections about lines throught the origin.

Example 2.6.1. Prove that left multiplication in $\mathbf{R}^2$ by $A = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ represents reflection about the y-axis.

Solution. For any $\mathbf{x} = [\, x \; y \,]^T$ in $\mathbf{R}^2$, we have

$$A\mathbf{x} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

Thus the point with coordinates (x, y) is mapped to the point with coordinates $(-x, y)$. This is the reflection of (x, y) about the y-axis.

Just as in Example 2.6.1, we can verify that reflections about the x-axis, the line $y = x$, and the origin are matrix transformations with respective standard matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Compressions and expansions

Compressions and expansions are scalings along the coordinate axes. If c and d are positive scalars, then (cx, y), (x, cy), and (cx, dy) represent scalings along the x-axis, along the y-axis, and along both axes. These scalings define matrix transformations with corresponding standard matrices

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}, \quad \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}.$$

If the scalars are less than 1, then we have *compressions*. If they are greater than 1, then we have *expansions*.

Example 2.6.2. Let $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right]$. Then $A\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2x \\ 3y \end{smallmatrix}\right]$ represents expansion by a factor of 2 along the x-direction and by a factor of 3 along the y-direction (Figure 2.29).

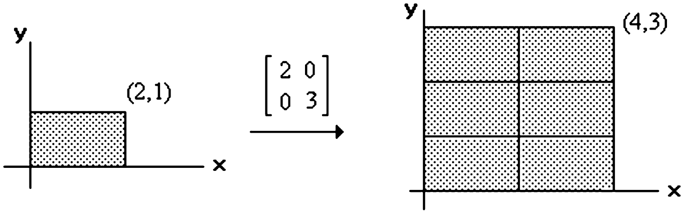

Figure 2.29: Scaling along the x-axis and y-axis.

Shears

A *shear along the x-axis* is a transformation of the form

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + cy \\ y \end{bmatrix}.$$

Each point is moved along the x-direction by an amount proportional to the distance from the x-axis. We also have shears along the y-axis:

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ cx + y \end{bmatrix}.$$

Example 2.6.3. Find the images of $(0, 0)$, $(2, 0)$, $(2, 1)$, and $(0, 1)$ under the shear with matrix $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$.

Solution. The image of any point (x, y) is computed by

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ y \end{bmatrix}.$$

Substitution of the given points yields $(0,0)$, $(2,0)$, $(4,1)$, and $(2,1)$. It appears that the rectangle with vertices the given points and its interior are mapped onto the parallelogram with vertices $(0,0)$, $(2,0)$, $(4,1)$, $(2,1)$ and its interior (Figure 2.30).

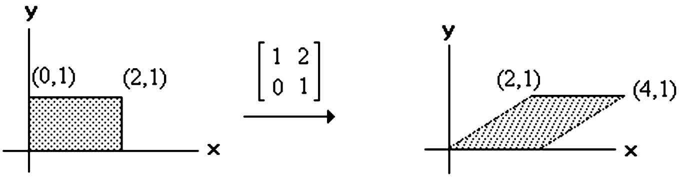

Figure 2.30: A shear along the x-axis.

Figure 2.31 shows the effect of the shear with matrix $\left[\begin{smallmatrix} 1 & 0 \\ -2 & 1 \end{smallmatrix}\right]$ on the unit square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$.

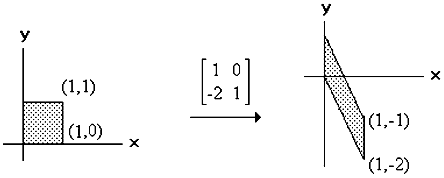

Figure 2.31: A shear along the y-axis.

Rotations

Another common plane transformation is rotation about any point in the plane. We are interested in rotations about the origin.

Example 2.6.4. Prove that the transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}$$

rotates each vector counterclockwise θ radians about the origin.

Solution. Let $\overrightarrow{OB}$ in Figure 2.32 be the rotation of $\overrightarrow{OA}$ by θ. Then $x = r\cos\phi$, $y = r\sin\phi$ and $x' = r\cos(\phi+\theta)$, $y' = r\sin(\phi+\theta)$. By the addition laws of sines and cosines we have

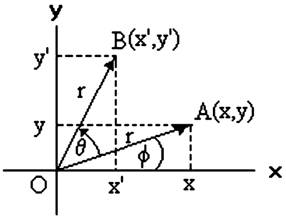

Figure 2.32: Rotation about the origin.

$$\begin{aligned} x' &= r\cos\phi\cos\theta - r\sin\phi\sin\theta = x\cos\theta - y\sin\theta, \\ y' &= r\sin\phi\cos\theta + r\cos\phi\sin\theta = y\cos\theta + x\sin\theta. \end{aligned}$$

Therefore

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix},$$

which shows that T is rotation by θ radians about the origin.

Projections

Projections of the plane onto a line are also transformations of the plane. We are interested in orthogonal projections onto lines through the origin, especially onto the axes (Figure 2.33).

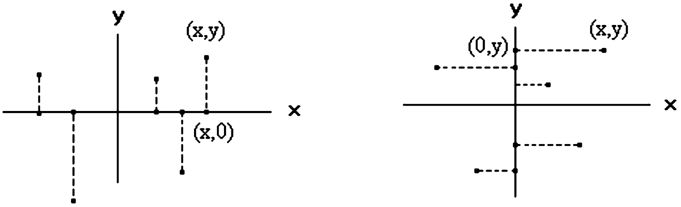

Figure 2.33: Orthogonal projections along the axes.

The *projections* onto the x-axis and the y-axis are matrix transformations given by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

2.6.2 Space matrix transformations

Special space matrix transformations $T : \mathbf{R}^3 \to \mathbf{R}^3$, such as reflections, rotations, and projections are also used in computer graphics. The extra coordinate allows for more variety of such transformations. For example, reflections are about the origin, the coordinate planes, the coordinate axes, the bisectors of the coordinate axes, and the bisector planes. These along with some projections are studied in the exercises. We give an example of one type of rotation. The verification of this is left as an exercise.

Example 2.6.5 (Rotation in 3D). The rotation transformation that rotates each 3-vector θ radians about the z-axis in the positive direction, determined by the *right-hand rule*, is given by the following standard matrix (Figure 2.34):

$$R_\theta^z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

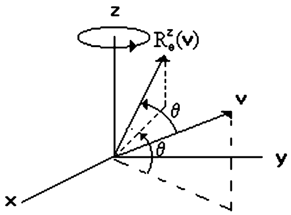

Figure 2.34: Rotation by θ radians about the z-axis.

Example 2.6.6 (Shear in 3D). An elementary shearing transformation in three dimensions moves every point along an axis direction by an amount proportional to the distance from that axis. The standard matrix of such transformation is obtained from the identity matrix by replacing one of the zero entries with nonzero entry. For example,

$$S_x = \begin{bmatrix} 1 & \sqrt{3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the shear matrix that stretches any vector by a factor of $\sqrt{3}$ along the x-axis (Figure 2.35).

2.6.3 Affine transformations

Not all interesting transformations that are used in computer graphics and other areas are matrix transformations. Perhaps the simplest nonmatrix transformation is translation by a fixed vector. A more general class of useful nonmatrix transformations consists of the composition of a matrix transformation followed by a translation. Such transformations are called affine transformations.

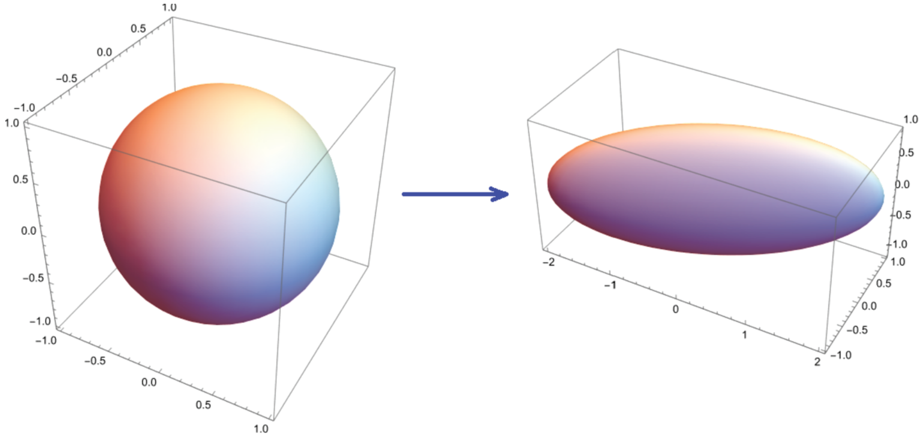

Figure 2.35: A shearing of a sphere along the *x*-axis.

Definition 2.6.7. Let A be an $m \times n$ matrix. An *affine matrix transformation* $T : \mathbf{R}^n \to \mathbf{R}^m$ is a transformation of the form

$$T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

for some fixed m-vector $\mathbf{b}$. This is a matrix transformation only if $\mathbf{b} = \mathbf{0}$. In the particular case where $m = n$ and $A = I$ we have

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{b}.$$

Such T is called the *translation* by the vector $\mathbf{b}$.

A translation by a vector $\mathbf{b} \neq \mathbf{0}$ translates a figure by adding $\mathbf{b}$ to all its points. An affine matrix transformation is a matrix transformation followed by a translation (Figure 2.36).

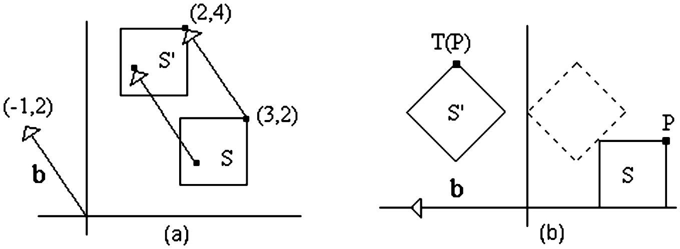

Figure 2.36: (a) Translation. (b) Affine transformation: First rotation, then translation.

Referring to Figure 2.37, the affine space matrix transformation T given by the rotation of $\pi/4$ about the z-axis in the positive direction followed by a translation by $[\,1\;1\;1\,]^T$ has been applied to the tetrahedron on the left and produced the tetrahedron on the right:

$$T(\mathbf{x}) = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

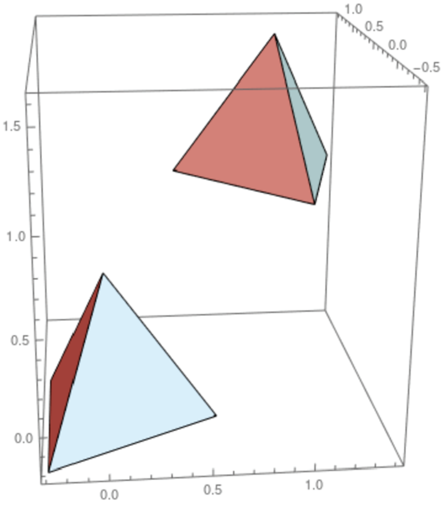

Figure 2.37: Rotation and translation of a tetrahedron.

Affine transformations are used in computer graphics and in fractal geometry.

Exercises 2.6

Plane matrix transformations

For Exercises 1–8, consider the matrix plane transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$, $T(x) = Ax$, with the given matrix A.

(i) Compute and draw the images of

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

(ii) Identify T as one of
  (a) reflection,
  (b) compression–expansion,

(c) shear,
(d) rotation,
(e) projection,
(f) none of the above.

1. (a) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; (b) $\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

3. (a) $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$.

4. (a) $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$; (b) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$.

5. (a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; (b) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$.

6. (a) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$; (b) $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$.

7. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$.

8. (a) $\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$; (b) $\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$.

9. Find a formula and the matrix for the reflection about the line $y = -x$.

10. Find a formula and the matrix for the reflection about the line $y = 2x$.

11. Find a formula and the matrix for the clockwise rotation θ radians about the origin.

12. Sketch the image of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ under the shear

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

13. Find a transformation that maps the unit square to the parallelogram shown in Figure 2.38.

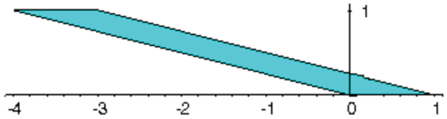

Figure 2.38: Unit square to parallelogram.

14. Find a transformation that maps the unit square to the rectangle shown in Figure 2.39.

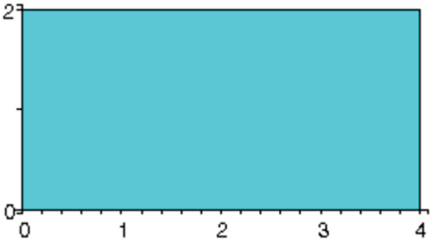

Figure 2.39: Unit square to rectangle.

Space matrix transformations

Reflections: In space the main reflections are about the *origin, coordinate planes, coordinate axes, bisectors of the coordinate axes*, and *bisector planes* (Figures 2.40 and 2.41).

15. In the following table, fill in the question marks.

Reflections about:	$T(\mathbf{x})$	Matrix A
?	?	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
?-plane	?	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
?-axis	?	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
first quadrant line bisecting the ?–plane	?	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
plane y =?	?	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

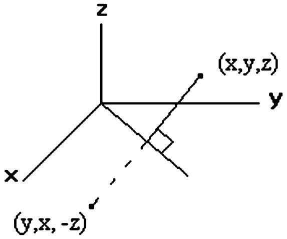

Figure 2.40: Reflection about the *xy*-bisector.

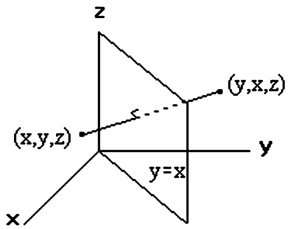

Figure 2.41: Reflection about the plane $y = x$.

16. Find formulas for the corresponding counterclockwise rotations in $\mathbf{R}^3$ by θ radians about (a) the x-axis and (b) the y-axis.

Projections: There are several orthogonal projections, mainly, onto the coordinate planes and the coordinate axes (Figures 2.42 and 2.43).

17. In the following table, fill in the question marks.

Projection onto:	$T(\mathbf{x})$	Matrix A
?-plane	?	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
?-axis	?	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

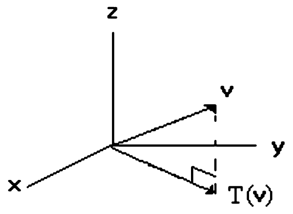

Figure 2.42: Projection onto the xy-plane.

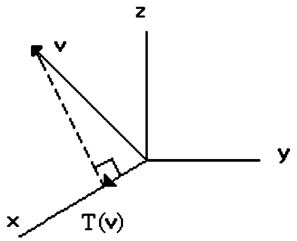

Figure 2.43: Projection onto the x-axis.

Affine transformations

In Exercises 18–21, find the images of the zero vector and the images of the standard basis vectors for the given affine transformations.

18. $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

19. $T(\mathbf{x}) = \begin{bmatrix} 4 & -3 \\ 2 & 5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

20. $T(\mathbf{x}) = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 2 & -1 \\ 0 & -4 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$

21. $T(\mathbf{x}) = \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & -4 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$

In Exercises 22–24, write the given affine transformations in the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$.

22. $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ -x + y - 1 \end{bmatrix}.$

23. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + 3y + 1 \\ x - z \\ x - 5y + z - 1 \end{bmatrix}.$

24. $T\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x - z - 9w + 1 \\ -3z - 6w \end{bmatrix}.$

In Exercises 25–28, find A and $\mathbf{b}$ for the given affine transformation $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$.

25. $T : \mathbf{R}^2 \to \mathbf{R}^2,\ T(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 3 \end{bmatrix},\ T(\mathbf{e}_2) = \begin{bmatrix} 4 \\ -7 \end{bmatrix},\ T(\mathbf{0}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

26. $T : \mathbf{R}^2 \to \mathbf{R}^2,\ T(\mathbf{e}_1) = \begin{bmatrix} 4 \\ -2 \end{bmatrix},\ T(\mathbf{e}_2) = \begin{bmatrix} -5 \\ 9 \end{bmatrix},\ T(\mathbf{0}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$

27. $T : \mathbf{R}^3 \to \mathbf{R}^3$ and

$$T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, \quad T(\mathbf{0}) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

28. $T : \mathbf{R}^3 \to \mathbf{R}^2$ and

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -7 \end{bmatrix}, \quad T(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \begin{bmatrix} 3 \\ -6 \end{bmatrix}.$$

29. Prove that if T is an affine transformation with $T(\mathbf{0}) = \mathbf{b}$, then $L(\mathbf{x}) = T(\mathbf{x}) - \mathbf{b}$ is a linear transformation.

30. Let $T : \mathbf{R}^n \to \mathbf{R}^m$, $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, be any affine transformation. Prove that T is uniquely determined by the values

$$T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n), \quad \text{and} \quad T(\mathbf{0}).$$

31. Prove that an affine transformation $T : \mathbf{R}^n \to \mathbf{R}^m$, $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, maps straight lines to straight lines or to points.

32. Let $T : \mathbf{R}^n \to \mathbf{R}^m$, $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, be an affine transformation. What is the relation between the set $\{\mathbf{x} \in \mathbf{R}^n, \ T(\mathbf{x}) = \mathbf{0}\}$ and the set of solutions of the system $A\mathbf{x} = -\mathbf{b}$?

33. Draw the image of the straight line $x - y = -1$ under $T(\mathbf{x}) = \begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

2.7 Applications: Averaging, dynamical systems

There are numerous applications of vectors and matrices to most areas of science and engineering. We discuss a few interesting, yet elementary, applications: smoothing of data, dynamical systems, and fractals.

2.7.1 Data Smoothing by Averaging

In measuring various quantities that depend on time, we often collect data that include sudden disturbances. For example, suppose we measure wind velocities and record some high velocities of sudden gusts that only last short time. We want to minimize the impact of these brief gusts that may affect our interpretation of the data. One way of doing this is by smoothing the data. Such a smoothing scheme is averaging. If we have a sequence of numbers

$$a, b, c, d, e, \dots,$$

then we transform it into the sequence of the successive *averages*

$$\frac{a}{2}, \frac{a+b}{2}, \frac{b+c}{2}, \frac{c+d}{2}, \frac{d+e}{2}, \dots,$$

starting with the average between a and 0 as the first new number. Averaging is in fact multiplication by a finite size square matrix of the form

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Suppose that we record the following wind velocities in tens of miles per hour with measurements one hour apart:[2]

$$2,\ 1,\ 3,\ 3,\ 4,\ 5,\ 3,\ 4,\ 3,\ 2,\ 1,\ 2.$$

Plotting the data as a function of time evaluated at times $t = 1, \dots, 12$ yields the graph in Figure 2.44. If $\mathbf{v}$ is the vector formed by the above data, then we can smooth the data vector $\mathbf{v}$ by computing $A\mathbf{v}$ (averaging). We find the sequence

$$1\,,\frac{3}{2},\ 2,\ 3,\ \frac{7}{2},\ \frac{9}{2},\ 4,\ \frac{7}{2},\ \frac{7}{2},\ \frac{5}{2},\ \frac{3}{2},\ \frac{3}{2}$$

with a smoother graph (Figure 2.45a).

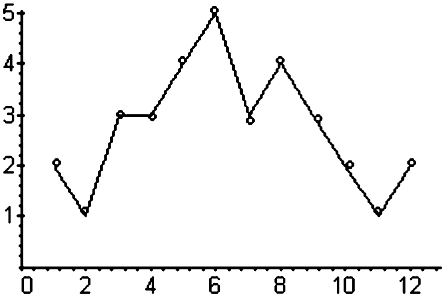

Figure 2.44: Initial data.

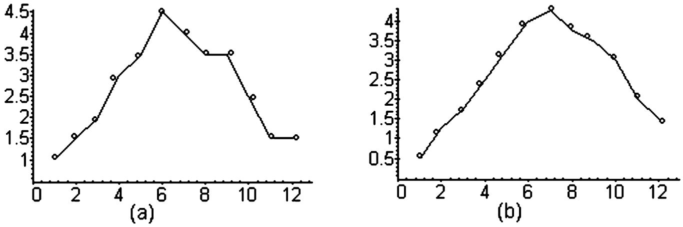

Figure 2.45: (a) Averaging once. (b) Averaging twice.

For further smoothing, we average again to get the new sequence (Figure 2.45b)

$$\frac{1}{2},\ \frac{5}{4},\ \frac{7}{4},\ \frac{5}{2},\ \frac{13}{4},\ 4,\ \frac{17}{4},\ \frac{15}{4},\ \frac{7}{2},\ 3,\ 2,\ \frac{3}{2}.$$

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[2] For this and other interesting examples of matrix transformations, see pp. 253–264 of Philip J. Davis' *The Mathematics of Matrices*, Blaisdell Publishing Co., 1965. See [24].

2.7.2 Discrete dynamical systems

We introduce discrete dynamical systems as applications of the product $A\mathbf{x}$. Because of its importance, we revisit this topic several times as our tools grow.

A dynamical system consists of one or more equations used to study time-dependent quantities. An example is the equation involving the balance P_t of an interest earning account at time t. In a *discrete* dynamical system the time variable takes on *integer* values.[3]

Suppose an account earns 8 % interest compounded annually. Let P_0 be the initial deposit, and let P_k be the balance at the end of the kth year. If $P_0 = \$1{,}000$, then $P_1 = 0.08 \cdot P_0 + P_0 = 1080$, $P_2 = 0.08P_1 + P_1 = 1166.4$, etc. At the end of the $(k+1)$th year the balance is

$$P_{k+1} = 0.08P_k + P_k = 1.08P_k. \tag{2.35}$$

Equation (2.35) is an example of a *discrete dynamical system*, or a *difference equation*. It gives the next value of P in terms of the current one. We can compute P_k by repeated applications of (2.35)

$$P_k = 1.08P_{k-1} = (1.08)^2 P_{k-2} = \cdots$$

to get

$$P_k = (1.08)^k P_0. \tag{2.36}$$

Equation (2.36) is the *solution* of this dynamical system.

2.7.3 A population growth model

Sometimes we have dynamical systems with several interrelated time-dependent quantities. These quantities are represented by a vector $\mathbf{v}_k$. It is often the case that the difference equation is of the form $\mathbf{v}_{k+1} = A\mathbf{v}_k$ for some matrix A. This is a *linear homogeneous discrete dynamical system*, or a *linear difference equation*. Let us study a specific example.

Suppose we have a population of insects divided into three age groups, A, B, C. Group A consists of insects 0–1 weeks old, group B consists of insects 1–2 weeks old, and group C consists of insects 2–3 weeks old. Suppose the groups have A_k, B_k, and C_k insects at the end of the kth week. We want to study how A, B, C change over time, given the following two conditions:

---

[3] For an excellent introduction, see James T. Sandefur's *Discrete Dynamical Systems, Theory and Applications*, Clarendon Press, Oxford, 1990. See [25].

(Survival Rate) Only 10 % of age group A survive a week. Hence

$$B_{k+1} = (1/10)A_k. \tag{2.37}$$

Also, only 40 % of age group B survive a week. So

$$C_{k+1} = (2/5)B_k. \tag{2.38}$$

(Birth Rate) Each insect from group A has 2/5 offspring, each insect from group B has 4 offsprings, and each insect from group C has 5 offsprings. In year $k+1$ the insects of group A are offsprings of insects in year k. Hence

$$A_{k+1} = (2/5)A_k + 4B_k + 5C_k. \tag{2.39}$$

Example 2.7.1. If the insect population starts out with 1,000 from each age group, then how many insects are in each group at the end of the third week?

Solution. Equations (2.37), (2.38), and (2.39) can be expressed in terms of vectors and matrices as follows:

$$\begin{bmatrix} A_{k+1} \\ B_{k+1} \\ C_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & 4 & 5 \\ \frac{1}{10} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \end{bmatrix} \begin{bmatrix} A_k \\ B_k \\ C_k \end{bmatrix}.$$

This matrix equation is the *dynamical system* of the problem. The condition on the initial population, called the *initial condition*, is

$$\begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}.$$

At the end of the first week, we have

$$\begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & 4 & 5 \\ \frac{1}{10} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \end{bmatrix} \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 9400 \\ 100 \\ 400 \end{bmatrix}.$$

At the end of the second week, we have

$$\begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & 4 & 5 \\ \frac{1}{10} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \end{bmatrix} \begin{bmatrix} 9400 \\ 100 \\ 400 \end{bmatrix} = \begin{bmatrix} 6160 \\ 940 \\ 40 \end{bmatrix},$$

and the end of the third week, we have

$$\begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & 4 & 5 \\ \frac{1}{10} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \end{bmatrix} \begin{bmatrix} 6160 \\ 940 \\ 40 \end{bmatrix} = \begin{bmatrix} 6424 \\ 616 \\ 376 \end{bmatrix}.$$

Therefore, after 3 weeks age, group A has 6424 insects, age group B has 616 insects, and age group C has 376 insects.

We are often interested in the long-term behavior of the dynamical system. This question is studied in Chapter 7. Here we can experiment using technology and compute more iterations in Example 2.7.1 to see that (A_k, B_k, C_k) seems to approach the vector $(6666.\overline{6}, 666.\overline{6}, 266.\overline{6})$. Thus the number of insects in age group C approaches $266.\overline{6}$. Observe the spiral trajectory in Figure 2.46.

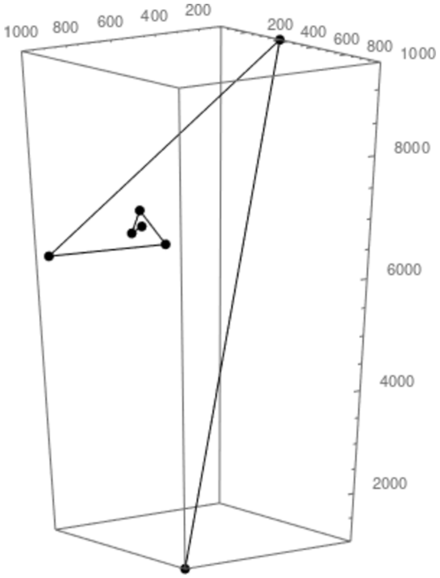

Figure 2.46: Population dynamics according to age category.

Exercises 2.7

Averaging

1. Plot the sequence and use matrices to average it twice. Plot each averaging.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 7 & 2 & 3 & 9 & 1 & 10 \end{array}$$

2. Plot the sequence and use matrices to average it twice. Plot each averaging.

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 25 & 15 & 45 & 15 & 20 & 30 & 20 & 50 \end{array}$$

Discrete dynamical systems

A species consists of two age groups, the young and the adult. Let Y_k and A_k be the numbers of individuals after k time units. The young have survival rate s. The birth rate of the young is y (i. e., one young individual has y offsprings), and the birth rate of the adult is a.

In Exercises 3–8, write in matrix form the dynamical system that models this population. Find the number of individuals in each group after 3 time units for the given values of s, y, a, Y_0, and $A_0 = 100$.

3. $s = 4/5, y = 2, a = 10, Y_0 = 100$.

4. $s = 1/2, y = 2, a = 6, Y_0 = 100$.

5. $s = 1/2, y = 4, a = 10, Y_0 = 100$.

6. $s = 1/4, y = 2, a = 12, Y_0 = 100$.

7. $s = 1/3, y = 3, a = 12, Y_0 = 300$.

8. $s = 1/5, y = 5, a = 30, Y_0 = 100$.

9. A population of flies is divided into 3 age groups A, B, C. Group A consists of flies 0–2 weeks old, group B consists of flies 2–4 weeks old, and group C consists of flies 4–6 weeks old. Suppose the groups have A_k, B_k, and C_k flies at the end of the $2k$th week. The survival rate of group A is 25 %, whereas the survival rate of group B is $33.\overline{33}\%$. Each fly from group A has .25 offspring, each fly from group B has 2.5 offspring, and each fly from group C has 1.5 offspring. If the original population consists of 4,800 flies in each age group, then write in matrix form the dynamical system that models this population. Find the number of individuals in each group after 6 weeks.

10. *(Long-term behavior of a dynamical system)* Suppose a species consists of two age groups, the young and the adult. Let A_k and B_k be the numbers of individuals after k time units. The young have survival rate 6/7. The birth rate of the young is 3 (i. e., one young individual has 3 offsprings), and the birth rate of the adult is 21.

(a) Write in matrix form the dynamical system that models this population.

(b) Find a formula in terms of A_0 and B_0 that computes the number of individuals in each group after 3 time units.

(c) Evaluate your formula for $A_0 = 700$ and $B_0 = 700$.

(d) Let $A_0 = 7$ and $B_0 = 1$, and let p_k be the ratio $A_k : B_k$. Find the long term value p of p_k as k grows large (i. e., find $p = \lim_{k\to\infty} p_k$). Justify your answer. (Knowledge of limits is not necessary.)

(e) Now let $A_0 = 8$ and $B_0 = 2$, let q_k be the ratio $A_k : B_k$, and let q be the long-term value q_k. Is it easy to predict q this time? Why or why not?

(f) It is a fact that $p = q$. Find the first value of k in q_k such that q_k is within 0.5 of p.

2.8 Special topic: Tessellations in weather models

Vectors and matrices are used in the mathematics of weather forecasting. The atmospheric air movement is studied by approximating solutions of complicated differential equations. The solution methods require that we subdivide the Earth into triangles and approximate the solution of the equations over each triangle. This is not an easy task, especially if we want to produce spherical triangles that are equal and are as closely as possible to being equilateral.

One approach that is currently used to triangularize a sphere is as follows: Starting with a polyhedron of triangular faces and vertices at distance from the center equal to the radius of the sphere, we subdivide each face into smaller triangles and project each new vertex onto the surface of the sphere. This way, we obtain a new polyhedron and use it to repeat the process. After several iterations, we get very fine triangularizations (Figure 2.47).

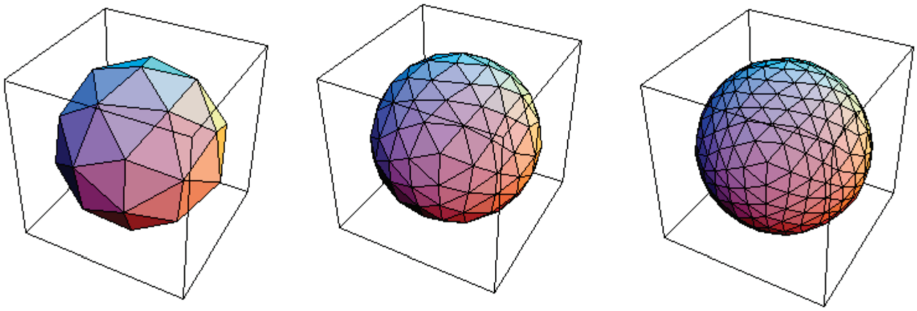

Figure 2.47: Triangularizations of a sphere.

The process just described can be easily implemented, provided that we can triangularize a single triangle.

Suppose we have a space triangle with vertices given vectors **a**, **b**, **c**. We may use a *regular tessellation* to triangularize it. This consists of partitioning each side of the triangle into n equal line segments and then joining with parallel lines the like numbered end points as shown in Figure 2.48.

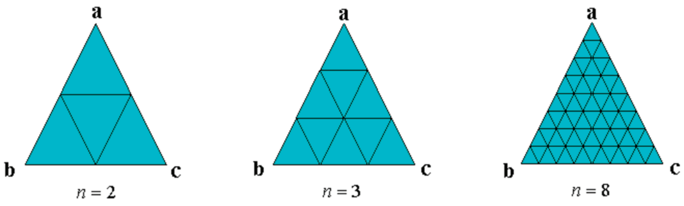

Figure 2.48: Regular triangular tessellations of a triangle.

We see now how we can use vector arithmetic to find the coordinates of the vertices of the smaller triangles in terms of the vertex vectors **a**, **b**, **c**.

We subdivide the side **ab** into n equal consecutive subintervals, and we label the endpoints as the vectors $\mathbf{p}_i$ for $0 \le i \le n$, so that $\mathbf{p}_0 = \mathbf{a}$ and $\mathbf{p}_n = \mathbf{b}$. We do the same with side **ac** to get vectors $\mathbf{q}_i$ for $0 \le i \le n$, so that $\mathbf{q}_0 = \mathbf{a}$ and $\mathbf{q}_n = \mathbf{c}$. Then we subdivide each side $\mathbf{p}_i\mathbf{q}_i$ into i equal subintervals and label the endpoints as the vectors $\mathbf{r}_{ij}$ for $0 \le j \le i$, so that $\mathbf{r}_{i0} = \mathbf{p}_i$ and $\mathbf{r}_{ii} = \mathbf{q}_i$ (Figure 2.49). The vectors $\mathbf{r}_{ij}$ are the vertices of the small triangles in the tessellation of the triangle **abc**. Our goal is to compute each $\mathbf{r}_{ij}$ in terms of **a**, **b**, and **c**.

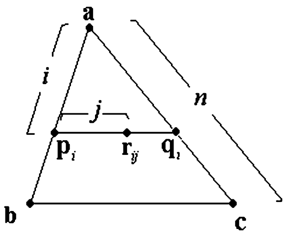

Figure 2.49: The (i,j) vertex of a regular triangular tessellation of a triangle.

The vectors $\mathbf{p}_i$ are computed from **a** and **b** by

$$\mathbf{p}_i = \frac{1}{n}((n-i)\mathbf{a} + i\mathbf{b}).$$

The verification of this formula is left as an exercise. Similarly, for the side **ac**, we get

$$\mathbf{q}_i = \frac{1}{n}((n-i)\mathbf{a} + i\mathbf{c}),$$

and for the side $\mathbf{p}_i\mathbf{q}_i$, we get

$$\mathbf{r}_{ij} = \frac{1}{i}((i-j)\mathbf{p}_i + j\mathbf{q}_i).$$

Therefore

$$\begin{aligned}
\mathbf{r}_{ij} &= \frac{1}{i}((i-j)\mathbf{p}_i + j\mathbf{q}_i) \\
&= \frac{1}{i}\left((i-j)\left[\frac{1}{n}((n-i)\,\mathbf{a} + i\mathbf{b})\right] + j\left[\frac{1}{n}((n-i)\,\mathbf{a} + i\mathbf{c})\right]\right) \\
&= \frac{1}{ni}((i-j)\,[((n-i)\,\mathbf{a} + i\mathbf{b})] + j[((n-i)\,\mathbf{a} + i\mathbf{c})]) \\
&= \frac{1}{ni}((n-i)\,i\mathbf{a} + i\,(i-j)\,\mathbf{b} + ij\mathbf{c}) \\
&= \frac{1}{n}((n-i)\,\mathbf{a} + (i-j)\,\mathbf{b} + j\mathbf{c}).
\end{aligned}$$

Thus we have proved the important formula that the (i,j) vertex of the tessellation is given by

$$\mathbf{r}_{ij} = \frac{1}{n}\left((n-i)\,\mathbf{a} + (i-j)\,\mathbf{b} + j\mathbf{c}\right), \quad 0 \le i \le n, \quad 0 \le j \le i. \tag{2.40}$$

Example 2.8.1. The triangle with vertices **i**, **j**, **k** is tessellated into 16 smaller triangles. Use formula (2.40) to compute the vertices of those triangles.

Solution. To have 16 triangles, n must be 4 (why?). We apply (2.40) with $\mathbf{a} = (1,0,0)$, $\mathbf{b} = (0,1,0)$, $\mathbf{c} = (0,0,1)$, and $n = 4$. For example, we get $\mathbf{r}_{32}$ by

$$\mathbf{r}_{32} = \frac{1}{4}\left((4-3)\,\mathbf{i} + (3-2)\,\mathbf{j} + 2\mathbf{k}\right) = (0.25, 0.25, 0.5)\,.$$

As $0 \le i \le 4$ and $0 \le j \le i$, we get all 15 vertices (Figure 2.50):

$$\begin{array}{lllll}
(1,0,0)\,, & (0.75,0,0.25), & (0.5,0,0.5), & (0.25,0,0.75), & (0,0,1)\,, \\
(0.75,0.25,0), & (0.5,0.25,0.25), & (0.25,0.25,0.5), & (0,0.25,0.75), & (0.5,0.5,0), \\
(0.25,0.5,0.25), & (0,0.5,0.5), & (0.25,0.75,0), & (0,0.75,0.25), & (0,1,0).
\end{array}$$

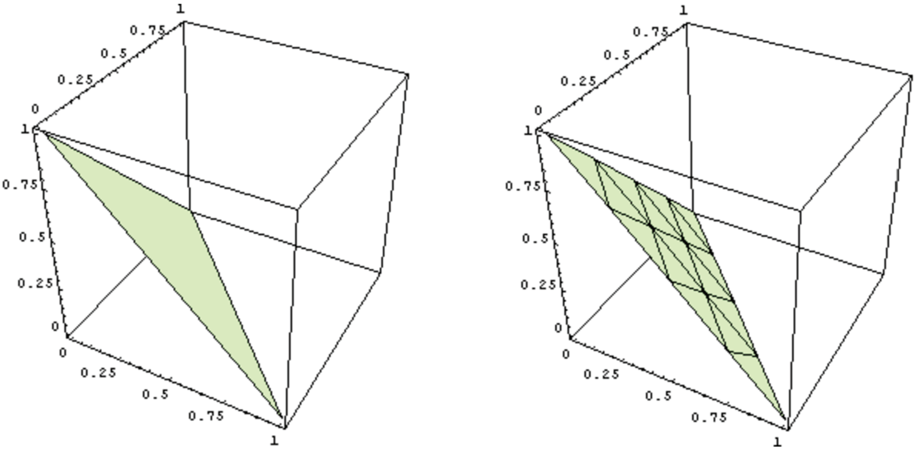

Figure 2.50: The regular tessellation with $n = 4$ of the triangle with vertices **i**, **j**, **k**.

Formula (2.40) is easily implemented in a computer language. We can compute the vertices of hundreds of thousands of triangles on a personal computer in just a few seconds.

2.9 Miniproject: Special affine transformations

In this project, we explore the basics of some special affine transformations, the similitudes. Similitudes are extensively used in computers graphics, dynamical systems, and fractals.

A *similarity transformation*, or *similitude* $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a special affine transformation of one of the forms

$$T(\mathbf{x}) = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix}\mathbf{x} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad T(\mathbf{x}) = \begin{bmatrix} r\cos\theta & r\sin\theta \\ r\sin\theta & -r\cos\theta \end{bmatrix}\mathbf{x} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

for some scalar $r \neq 0$, some angle θ, $0 \leq \theta < 2\pi$, and some scalars b_1 and b_2. Similitudes are scaled rotations followed by translations or scaled reflected rotations followed by translations. As such, they preserve angles, as we see later on.

Problem A. Prove that the following transformations are similitudes.

1. Any rotation about the origin.
2. Reflections about the axes, the diagonal, or the origin.
3. Any homothety of $\mathbf{R}^2$. A *homothety* is a transformation of the form $T(\mathbf{x}) = r\mathbf{x}$ for some fixed scalar r.

Problem B.

1. Are shears in general similitudes? Check the shears with standard matrices

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0.5 \\ -0.5 & 0 \end{bmatrix}.$$

2. Are projections onto the axes similitudes?
3. Are translations similitudes?

Problem C.

1. Find a formula for the similitude T that maps the triangle $(0,0), (1,0), (0,1)$ to the triangle $(1,1), (-1,1), (1,-1)$.
2. Let S_1 be the image under T of the rectangle S with vertices $(0,0), (2,0), (2,1), (0,1)$, and let S_2 be the image of S_1. Compute that areas (S), (S_1), and (S_2) and compare the ratios of the areas $(S_2) : (S_1)$ and $(S_1) : (S)$.
3. Find the formula of the similitude R that rotates any point 45° about the origin, then scales it by factor of 2, and, finally, translates it by $(1,1)$.
4. Find the image L_1 under R of the triangle L with vertices $(0,0), (1,0), (0,1)$ and find the image L_2 of L_1. Compute the area ratios $(L_2) : (L_1)$ and $(L_1) : (L)$. What did you observe?

2.10 Technology-aided problems and answers

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -4 \end{bmatrix},$$

$$M = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 6 & 8 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}.$$

1. Compute (a) $\mathbf{u} + \mathbf{v}$, (b) $\mathbf{u} - \mathbf{v}$, (c) $10\mathbf{u}$, (d) $\mathbf{u} - 2\mathbf{v} + 3\mathbf{w}$.

2. Check the identities (a) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ and (b) $10(\mathbf{u} + \mathbf{v}) = 10\mathbf{u} + 10\mathbf{v}$.

3. If possible, then write $\mathbf{v}$ as a linear combination of the columns of M.

4. If possible, then write $\mathbf{w}$ as a linear combination of the columns of N.

5. Plot $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in separate graphs and in the same graph.

6. Compute (a) $\|\mathbf{u}\| + \|\mathbf{v}\|$, (b) $\|\mathbf{u} + \mathbf{v}\|$, (c) $\mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$.

7. Compute the angle between $\mathbf{u}$ and $\mathbf{v}$.

8. Compute the orthogonal projection of $\mathbf{v}$ onto $\mathbf{w}$. Verify your answer.

9. Which of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are in the span of the columns of M?

10. Which of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are in the span of the columns of N?

11. Is the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ linearly independent? Does it span $\mathbf{R}^3$?

12. Do the columns of M span $\mathbf{R}^3$? Are they linearly independent?

13. Do the columns of N span $\mathbf{R}^3$? Are they linearly independent?

14. *True or False*?

$$\begin{aligned} &\text{(a)} \quad \{M\mathbf{x}, \quad \mathbf{x} \in \mathbf{R}^3\} = \mathbf{R}^3, \\ &\text{(b)} \quad \{N\mathbf{x}, \quad \mathbf{x} \in \mathbf{R}^4\} = \mathbf{R}^3. \end{aligned}$$

15. Write the fourth column of N as a linear combination of the first three.

16. Prove that the following system is consistent for all values of b_1, b_2, and b_3:

$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 \phantom{{}+x_5} &= b_1, \\ x_1 + 2x_2 + 2x_3 + x_4 \phantom{{}+x_5} &= b_2, \\ x_1 + 2x_2 + x_3 + 2x_4 + x_5 &= b_3. \end{aligned}$$

17. Find a solution for $b_1 = 1$, $b_2 = -1$, and $b_3 = 1$ and verify your answer by checking the corresponding vector equation.

18. Prove that the columns of N are linearly independent and span $\mathbf{R}^3$. How is this double property affected if you add a column of your choice? What happens if you delete a column of your choice?

19. If possible, compute and interpret the products $M\mathbf{u}$, $M\mathbf{r}$, $N\mathbf{u}$, and $N\mathbf{r}$.

20. Verify that $\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$ are orthogonal.

21. Verify *Jacobi's identity* $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{0}$.

22. Let l be the line with parametric equation

$$\mathbf{x} = (-1, 2, 1) + t(-4, 1, 5).$$

Which of $R(-9, 4, 11)$ and $S(7, 0, -10)$ is in l? Plot l from $t = -2$ to $t = 3$. Plot l from $P(-13, 5, 16)$ to $Q(-21, 7, 26)$.

23. Find a normal–point form equation for the plane through $P(1, 2, -3)$, $Q(-2, 4, 5)$, and $R(3, 3, 3)$. Plot this plane. Find two points one on and one off the plane.

24. Write and test the code for a function that computes the distance from a point to a plane.

25. Write and test the code for a function that computes the distance between two skew lines, given two points on each line.

2.10.1 Selected solutions with Mathematica

```
(* Problem A.*)                           (* Defining u,v,w as column vectors, or *)
(* DATA *)
u = {{1},{3},{2}}                  (* column matrices. u = {1,2,3}, etc. would *)
v = {{-1},{1},{2}}                           (* define them as "row vectors". *)
w = {{2},{1},{-4}}
M = {{1,3,5},{7,9,2},{4,6,8}}
n = {{1,2,3,4},{2,3,4,5},{3,4,5,6}} (* N. Symbol N is reserved. *)
(* To display a matrix m in row-column form use MatrixForm[m] . *)
(* Exercises 1,2. *)
u + v                           (* Sum. *)
u - v                           (* Difference. *)
10 u                            (* Scalar product. *)
u - 2 v + 3 w                   (* Linear combination. *)
(u+v)+w===u+(v+w)      (* Checking for equality. *)
10 (u+v) === 10 u + 10 v        (* Same. *)
(* Exercises 2,3. *)
am = Join[M, v, 2]                          (* The augmented matrix [M:v]. *)
rm = RowReduce[am] (* Reduction: The last column is non-pivot.*)
(* The system is consistent so yes v is a linear combination *)
(* in the columns of M. The coefficients are the entries of the last column  *)
rm[[1,3]] M[[All, 1]]+
rm[[2,3]] M[[All, 2]]+
rm[[3,3]] M[[All, 3]]       (* Indeed computing the lin. comb. yields v.*)
an = Join[n, w, 2]                   (* [N:w]. *)
RowReduce[an] (* The last column is pivot. No solutions.   *)
(* w is not a linear combination in the columns of N. *)
(* Exercises 5-8. *)
o={0,0,0}; u ={1,3,2}; v={-1,1,2}; w={2,1,-4};
p1=Line[{o,u}];           (* Define and name the line segments, but do not   *)
```

```
p2=Line[{o,v}];           (* display the graphs yet.                         *)
p3=Line[{o,w}];                          (* Next display p1 with labeled axes.*)
Show[Graphics3D[p1,Axes->True]]          (* Then repeat with p2 and p3...     *)
Show[Graphics3D[{p1,p2,p3},Axes->True]] (* Display all.                      *)
Norm[u] + Norm[v]
N[%]
Norm[u+v]
N[%]
u . v - u . w        (* The dot product of u and v is u.v or Dot[u,v].    *)
VectorAngle[u, v]    (* The angle between u and v *)
N[%]                 (* approximations in radians *)
pr=Projection[v,w]
vc = v - pr                     (* The vector component of v orthogonal to w. *)
pr . vc                         (* The dot product is zero and the            *)
pr + vc                         (* sum is v as expected.                      *)
(* Exercise 9 - Partial. *)
RowReduce[Join[M,{{1},{3},{2}},2]]   (* The last column is not pivot so u *)
                                       (* is in the span.*)
(* Exercise 11. *)
Join[{{1},{3},{2}},{{-1},{1},{2}},{{2},{1},{-4}},2]
RowReduce[%]           (* [u v w] has 3 pivots so the vecs. are independent. *)
(* Exercise 15. *)
RowReduce[n]                  (* From the last column: (-2)xcol1+3xcol2 = col4. *)
(* Exercise 16 - Hint:  Row reduce the coefficient matrix to see that its last  *)
(* column is pivot so the last column of the augmented matrix
is not a pivot column. *)
(* Exercise 19. *)
M . {{1},{3},{2}}                      (* Etc.. Mr and Nu are undefined.*)
(* Exercises 20,21. *)
u={1,2,3}; v={-1,-1,1}; w={2,1,-4};
uv = Cross[u,v]
u . uv
Cross[Cross[u,v],w]+Cross[Cross[v,w],u]+Cross[Cross[w,u],v]
(* Exercise 22. *)
Solve[{-1-4*t==-9,2+t==4,1+5*t==11},t]   (* t=2 so R is in l. *)
Solve[{-1-4*t==7,2+t==0,1+5*t==-11},t]  (* No solution. S is not in l.*)
ParametricPlot3D[{-1+3*t,-2+3*t,2-5*t},{t,-6,7}]
(* Exercise 25. *)
LineToLine[p_,q_,r_,s_] := Module[{u,v,w,cr},
u=q-p; v=s-r; w=r-p; cr = Cross[u,v];
Abs[w.cr] / Sqrt[cr[[1]]^2+cr[[2]]^2+cr[[3]]^2] ]
LineToLine[{1,-2,-1},{0,-2,1},{-1,2,0},{-1,0,-2}]          (* Testing *)
```

2.10.2 Selected solutions with MATLAB

```
% DATA
u = [1; 3; 2]                          % Defining u,v,w.
```

```
v = [-1; 1; 2]
w = [2; 1; -4]
M = [1 3 5; 7 9 2; 4 6 8]             % M.
N = [1 2 3 4; 2 3 4 5; 3 4 5 6]       % N.
% Exercises 1,2.
u + v                                 % Sum.
u - v                                 % Difference.
10 * u                                % Scalar product.
u-2*v+3*w                             % Linear combination.
(u+v)+w==u+(v+w)                      % Checking for equality. It returns
10*(u+v)==10*u+10*v                   % 1 (= TRUE) for each entry.
% Exercises 3,4.
am = [M v]                        % The augmented matrix [M:v].
rm = rref(am)                     % Reduction: The last column is nonpivot.
% The system is consistent so v is a lin. comb.
% in the columns of M. The coefficients of
rm(1,4)*M(:,1)+rm(2,4)*...        % the lin. com. are entries of the last column
M(:,2)+rm(3,4)*M(:,3)             % of rm. Indeed computing the lin. comb. yields v.
an = [N w]                        % [N:w].
rref(an)                          % The last column is pivot. No solutions.
% w is not a lin. com. in the columns of N.
% Exercises 5-8.
o=[0 0 0]; u=[1 3 2]; v=[-1 1 2]; w=[2 1 -4]
plot3([0 1], [0 3], [0 2])                    % Position vectors u,v,w.
plot3([0 -1], [0 1], [0 2])                   %
plot3([0 2], [0 1], [0 -4]),grid              % grid adds a grid to the graph.
plot3([0 1 0 -1 0 2], [0 3 0 1 0 1], [0 2 0 2 0 -4])   % u,v,w together.
norm(u) + norm(v)
norm(u+v)
dot(u, v) - dot(u, w)                % dot(a,b) is a.b .
acos(dot(u,v) / norm(u) / norm(v))   % Angle between u and v.
pr = (dot(v,w) / dot(w,w))*w    % The formula for the orthogonal projection.
vc = v - pr                     % The vector component of v orthogonal to w.
dot(pr, vc)                     % The dot product is (very close to) zero and
pr + vc                         %  the sum is v as expected.
% Exercise 9 - Partial.
rref([M [1;3;2]])               % The last column is not pivot so u is in the span.
% Exercise 11.
rref([u; v; w])                 % [u v w] has 3 pivots so the vecs. are independent.
% Exercise 15.
rref(N)                         % From the last column: (-2)xcol1+3xcol2 = col4.
% Exercise 16 - Hint:  Row reduce the coefficient matrix to see that its last
% column is pivot so the last column of the augmented matrix
% is not a pivot column.
% Exercise 19.
M*[1;3;2]                   % Also: M*[2;-3;1;-4], etc.. Mr and Nu are undefined.
% Exercises 20,21.
u=[1 2 3]; v=[-1 -1 1]; w=[2 1 -4];
uv = cross(u,v)
```

```
cross(cross(u,v),w)+cross(cross(v,w),u)+cross(cross(w,u),v)
% Exercise 22.
% R is in l since the system -1-4*t=-9,2+t=4,1+5*t=11 is consistent because
[roots([-4 -1+9]) roots([1 2-4]) roots([5 1-11])]    % returns [2 2 2].
[roots([-4 -1-7]) roots([1 2-0]) roots([5 1+10])]    % [.2,.2,-2.2] so the
t = -2:.25:3;                          % system has no solution and S is not in l.
plot3(-1-4*t,2+t,1+5*t)                % Plotting the line.
% Exercise 25.
function [A] = LnToLn(p,q,r,s)                          % In a file.
A = abs(dot(r-p,cross(q-p,s-r))) / norm(cross(q-p,s-r));
end
LnToLn([1 -2 -1],[0 -2 1],[-1 2 0],[-1 0 -2])  % In session.
```

2.10.3 Selected solutions with Maple

```
# DATA and Remarks.
with(LinearAlgebra);                  # Loading the linalg package.
u := Vector([1,3,2]);        # Defining u,v,w as vectors.
v := Vector([-1,1,2]);      # Or u := array([1,2,3]); , etc..
w := Vector([2,1,-4]);       # Vectors are not column matrices. A vector entry
# is determined by one number; a matrix entry by two.
M := Matrix(3,3,[1,3,5, 7,9,2, 4,6,8]);             # M.
N := Matrix(3,4,[1,2,3,4,2,3,4,5,3,4,5,6]);  # Yet another syntax for matrix.
# Exercises 1,2.
u+v;
u-v;
10*u;
u-2*v+3*w;             # Linear combination.
((u+v)+w)-(u+(v+w));      # Checking for equality.
(10*(u+v))-(10*u+10*v);   # Same.
# Exercises 2,3.
am:=<<M|v>>;              # The augmented matrix [M:v].
rm := ReducedRowEchelonForm(am);  # Reduction: The last column is nonpivot.
# The system is consistent so v is a lin. comb. in the columns of M.
#  The coefficients are the entries of the last column.
rm[1,4]*Column(M,1)+rm[2,4]*Column(M,2)+rm[3,4]*Column(M,3);
# Indeed computing the lin. comb. yields v.
an := <<N|w>>;                # [N:w].
ReducedRowEchelonForm(an);    # The last column is pivot. No solutions.
# w is not a lin. com. in the columns of N.
# Exercises 5-8.
with(plottools): with(plots):
p1:=line([0,0,0],[1,3,2]):    # Define and name the line segments, but do not
p2:=line([0,0,0],[-1,1,2]):    # display the graphs yet.
p3:=line([0,0,0],[2,1,-4]):
display(p1, axes=boxed);     # Display p1 in boxed axes. Repeat with p2 and p3.
display([p1,p2,p3],axes=boxed);       # Display all.
```

```
VectorNorm(u, 2)+VectorNorm(v, 2);
evalf(%);
VectorNorm(u+v, 2);
evalf(%);
u.v-u.w;
VectorAngle(u,v);
evalf(%);
pr:=((v.w)/(w.w))*w;                   # The orthogonal projection of v on w.
vc:=v-pr;                    # The vector component of v orthogonal to w.
pr.vc;                     # The dot product is zero and the
pr+vc;                       # sum is v as expected.
# Exercise 9 - Partial.
MM:=<<M|Vector([1,3,2])>>;   #Add vector u to M as a last column.
ReducedRowEchelonForm(MM); # The last column is not pivot so u is in the span.
# Exercise 11.
L := <<u|v|w>>;              # [u v w] has 3 pivots so the vecs. are independent.
ReducedRowEchelonForm(L);
# Exercise 15.
ReducedRowEchelonForm(N);      # From the last column: (-2)xcol1+3xcol2 = col4.
# Exercise 16 - Hint:  Row reduce the coefficient matrix to see that its last
# column is pivot so the last column of the augmented matrix
# is not a pivot column.
# Exercise 19.
M.u;                             # Etc.. Mr and Nu are undefined.
# Exercises 20, 21.
uv:=u &x v;                     # cross product uXv OR
uv:=CrossProduct(u,v);           # cross product uXv
u.(uv);
CrossProduct(CrossProduct(u,v),w)+CrossProduct(CrossProduct(v,w),u)
+CrossProduct(CrossProduct(w,u),v);
# Exercise 22.
solve({-1-4*t=-9,2+t=4,1+5*t=11},t);     # t=2 for R so Q is in l.
solve({-1-3*t=7,-2+3*t=0,2-5*t=-10},t);  # System has no solution so S is not in l.
with(plots):                             # Loading graphics package to use
spacecurve([-1-4*t,2+t,1+5*t],t=-2..3);  # the command spacecurve. Etc.
# Exercise 25.
LineToLine := proc(p,q,r,s) local u,v,w,cr;                     # Code
with(LinearAlgebra):
u:=Vector(q-p); v:=Vector(s-r); w:=Vector(r-p); cr := CrossProduct(u,v);
abs(w.cr) / VectorNorm(cr,2) end:
LineToLine([1,-2,-1],[0,-2,1],[-1,2,0],[-1,0,-2]);              # Testing.
```

3 Matrices

The seeds from Ramanujan's garden have been blowing on the wind and have been sprouting all over the landscape (Figure 3.1).

Freeman Dyson, British-American theoretical physicist and mathematician (1923–2020).

Figure 3.1: Srinivasa Ramanujan.
By Konrad Jacobs https://opc.mfo.de/detail?photoID=2328,
Public Domain,
https://commons.wikimedia.org/w/index.php?curid=111802441.
Srinivasa Ramanujan (1887–1920) was a brilliant Indian mathematician. With little formal training, he made extraordinary contributions to number theory, mathematical analysis, and modular forms. Ramanujan's work was characterized by its depth and originality. Ramanujan spent nearly five years in Cambridge collaborating with Hardy and Littlewood. He is widely regarded as one of the most significant mathematicians of the twentieth century.

Introduction

In this chapter, we introduce the most important matrix operation, *matrix multiplication*. Matrix multiplication and the inverse of a matrix were introduced by the English mathematician Sir Arthur Cayley in 1855 in his paper "Remarques sur la Notation des Fonctions Algébriques" for the *Journal für die reine und angewandte Mathematik*, also known as *Crelle's Journal*, now published by Walter De Gruyter [17] (Figure 3.2).

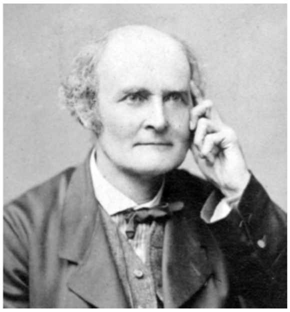

Figure 3.2: Sir Arthur Cayley.
Image by Herbert Berau, Public Domain,
https://commons.wikimedia.org/w/index.php?curid=904674.
Sir Arthur Cayley (1821–1895) was a renowned English mathematician. He made significant contributions to various fields of mathematics. He is one of the founders of modern algebra and is best known for his work in matrix theory and group theory. Cayley published over 900 papers and contributed to numerous mathematical areas, including geometry, algebra, and combinatorics. He made important discoveries in the theory of matrices.

Matrix multiplication is used in many areas of science, computer graphics, computer design, image recognition, economics, engineering, physics, electronic typography, printing, genetics, etc.

https://doi.org/10.1515/9783111331850-003

One useful consequence of Gauss elimination is that we can write a matrix as a product of simpler matrices, one in “lower” and one in “upper triangular form”.

In addition to the basic theory, we discuss a variety of applications including stochastic matrices, Markov chains, Leontief’s input–output economic models, and graph theory.

3.1 Matrix multiplication

In this section, we introduce matrix multiplication. This is the most important matrix operation. It is defined in terms of matrix–vector products.

Matrix multiplication is used in neural networks for weighting the input signals of a neuron. In the *feed forward pass*, the vector of the input signals is multiplied by the matrix of weights to produce an output signal. This output signal is then passed on to the next neuron in the network (Figure 3.3).

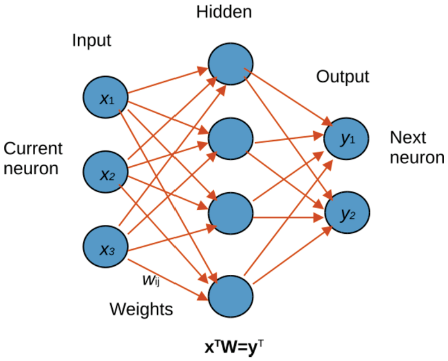

Figure 3.3: Neural networks use matrix multiplication.

Defining the product of two matrices as the matrix of the products of the corresponding entries is *not* very useful in applications. The following definition is far more useful.

Definition 3.1.1. Let A be an $m \times k$ matrix, and let B be a $k \times n$ matrix. Let $\mathbf{b}_i$ be the columns of B. The *product* AB is the $m \times n$ matrix with columns $A\mathbf{b}_i$ (Figure 3.4):

$$AB = [A\mathbf{b}_1\; A\mathbf{b}_1\; \cdots\; A\mathbf{b}_n].$$

Example 3.1.2. Compute AB, where

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 4 \\ -2 & 4 & 5 \\ 0 & 3 & -2 \end{bmatrix}.$$

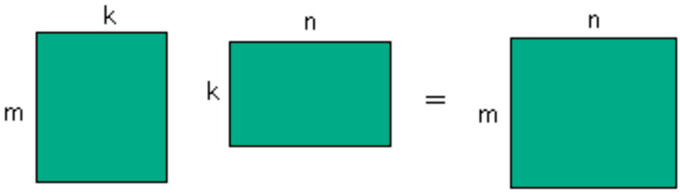

Figure 3.4: Matrix multiplication.

Solution. We have

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}.$$

Hence

$$AB = [A\mathbf{b}_1 \; A\mathbf{b}_2 \; A\mathbf{b}_3] = \begin{bmatrix} 6 & 7 & 6 \\ 4 & 14 & 9 \end{bmatrix}.$$

Example 3.1.3. Compute CD and DC, where

$$C = \begin{bmatrix} 5 & 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}.$$

Solution. We have $CD = \begin{bmatrix} 5 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} = [-2]$ and

$$DC = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}\begin{bmatrix} 5 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 1 & -1 \\ -15 & -3 & 3 \\ 20 & 4 & -4 \end{bmatrix}.$$

Note that in this case, $CD \neq DC$.

Matrix multiplication is only possible, if the number of columns of the first matrix equals the number of rows of the second matrix. Otherwise, the products $A\mathbf{b}_i$ cannot be defined.

The basic properties of matrix multiplication can be summarized in the following theorem.

Theorem 3.1.4 (Properties of matrix multiplication). *Let A be an $m \times n$ matrix. Let B and C be matrices of sizes such that the operations below can be performed. Let a be any scalar. Then*

1. $(AB)C = A(BC)$; *(Associativity law)*
2. $A(B + C) = AB + AC$; *(Left-distributivity law)*
3. $(B + C)A = BA + CA$ *(Right-distributivity law)*
4. $a(BC) = (aB)C = B(aC)$;
5. $I_m A = AI_n = A$; *(Multiplicative identity)*
6. $\mathbf{0}A = \mathbf{0}$ *and* $A\mathbf{0} = \mathbf{0}$;
7. $(AB)^T = B^T A^T$.

Proof. We prove Part 1, *associativity*, and leave the remaining proofs as exercises. First, we prove the particular case where C is a vector $\mathbf{v} = (v_1, \dots, v_k)$:

$$\begin{aligned}(AB)\mathbf{v} &= [A\mathbf{b}_1 \cdots A\mathbf{b}_k]\mathbf{v} \\ &= v_1(A\mathbf{b}_1) + \cdots + v_k(A\mathbf{b}_k) \\ &= A(v_1\mathbf{b}_1 + \cdots + v_k\mathbf{b}_k) \\ &= A(B\mathbf{v}).\end{aligned}$$

Now let C have l columns. Using the first case, we have

$$\begin{aligned}(AB)C &= [(AB)\mathbf{c}_1 \cdots (AB)\mathbf{c}_l] \\ &= [A(B\mathbf{c}_1) \cdots A(B\mathbf{c}_l)] \\ &= A[B\mathbf{c}_1 \cdots B\mathbf{c}_l] \\ &= A(BC).\end{aligned}$$ ☐

Just as with matrix addition, this associativity law allows us to drop parentheses from multiple products. For example, we have

$$(AB)(CD) = A\big((BC)D\big) = A\big(B(CD)\big) = ABCD.$$

Unlike addition, however, we are not allowed to change the order of the factors!

> As we saw in Example 3.1.3, *AB* may not equal *BA*. In fact, if *AB* is defined, then *BA* may not be defined. If *BA* is defined, then it may not have the same size as *AB*. If it does have the same size, then it may still not equal *AB*.

We say that matrix multiplication is *noncommutative*. If two matrices A and B satisfy $AB = BA$, then we say that they *commute*.

Example 3.1.5. $A = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 3 \end{smallmatrix}\right]$ commute. (Verify.)

Definition 3.1.6. The *commutator* $[A,B]$ of two $n \times n$ matrices A and B is the difference

$$[A,B] = AB - BA.$$

If two matrices commute, then their commutator is the zero matrix.

Example 3.1.7. The commutator of $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right]$ is

$$[A,B] = AB - BA = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

3.1.1 Another viewpoint of matrix multiplication

Just as in the case of the matrix–vector product, it is often useful to obtain AB one entry at a time. The (i,j) entry of AB can be computed as follows: we take the ith row of A and the jth column of B, multiply their corresponding entries together, and add up all the products.

Example 3.1.8. Compute the $(2,3)$ entry of AB.

Solution. We have

$$2 \cdot 4 + 1 \cdot 5 + 2 \cdot (-2) = 9,$$

$$\left[\begin{array}{ccc} 2 & 0 & 1 \\ \hline \multicolumn{1}{|c}{2} & 1 & \multicolumn{1}{c|}{2} \\ \hline \end{array}\right] \left[\begin{array}{cc|c|} \cline{3-3} 3 & 2 & 4 \\ -2 & 4 & 5 \\ 0 & 3 & -2 \\ \cline{3-3} \end{array}\right] = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 9 \end{bmatrix}.$$

In general, if $C = [c_{ij}] = AB$, then the entries c_{ij} are given by

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & \vdots & \vdots \\ \hline \multicolumn{1}{|c}{a_{i1}} & a_{i2} & \cdots & \multicolumn{1}{c|}{a_{ik}} \\ \hline \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{array}\right], \quad B = \left[\begin{array}{cc|c|cc} \cline{3-3} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kn} \\ \cline{3-3} \end{array}\right],$$

$$c_{ij} = a_{i1}\,b_{1j} \;+\; a_{i2}\,b_{2j} \;+ \cdots + a_{ik}\,b_{kj} = \sum_{r=1}^{k} a_{ir}b_{rj}.$$

In other words, the (i,j) entry of AB is the *dot product* of the ith row of A and the jth column of B, both considered as k-vectors.

3.1.2 Powers of a square matrix

Let A be a square matrix. The product AA is also denoted by A^2. Likewise, $AAA = A^3$ and $AA\cdots A = A^n$ for n factors of A. In addition, we write $A^1 = A$, and if A is nonzero, we write $A^0 = I$:

$$A^n = \underbrace{AA\cdots A}_{n\text{ factors}}, \quad A^1 = A, \quad A^0 = I.$$

Example 3.1.9. Let $A = \left[\begin{smallmatrix} 1 & -1 \\ -2 & 3 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix}\right]$, and $C = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Then

$$A^1 = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 3 & -4 \\ -8 & 11 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 11 & -15 \\ -30 & 41 \end{bmatrix}, \quad \dots,$$

$$B^1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad B^3 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad \dots,$$

$$C^1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \dots .$$

Theorem 3.1.10. *The following relations hold for any positive integers n and m:*

1. $A^nA^m = A^{n+m}$;
2. $(A^n)^m = A^{nm}$;
3. $(cA)^n = c^nA^n$.

Proof. Exercise. □

We have to be careful with matrix multiplication and not assume that it behaves much like ordinary multiplication!

1. $AB = \mathbf{0}$ does *not* necessarily imply that either $A = \mathbf{0}$ or $B = \mathbf{0}$.
2. $CA = CB$ does *not* necessarily imply that $A = B$.
3. $AC = BC$ does *not* necessarily imply that $A = B$.
4. $A^2 = I$ does *not* necessarily imply that $A = \pm I$.
5. In general, $(AB)^n \neq A^nB^n$.

3.1.3 Matrix multiplication with complex numbers

Matrix multiplication of matrices with complex numbers is defined exactly as in the case of real numbers. Background material may be reviewed in Appendix A.

For example, we have

$$\begin{bmatrix} 2i & -i \\ -1 & 0 \\ 1+i & 1 \end{bmatrix} \begin{bmatrix} i & 2-i & 0 \\ -2 & i & 1 \end{bmatrix} = \begin{bmatrix} -2+2i & 3+4i & -i \\ -i & -2+i & 0 \\ -3+i & 3+2i & 1 \end{bmatrix}.$$

3.1.4 Motivation for matrix multiplication

Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, and $\mathbf{z} = (z_1, z_2)$ be coordinate frames. Suppose we go from frame $\mathbf{y}$ to frame $\mathbf{z}$ by using the linear transformation

$$\begin{aligned} z_1 &= a_{11}y_1 + a_{12}y_2, \\ z_2 &= a_{21}y_1 + a_{22}y_2, \end{aligned}$$

and from frame $\mathbf{x}$ to frame $\mathbf{y}$ by the linear transformation

$$\begin{aligned} y_1 &= b_{11}x_1 + b_{12}x_2, \\ y_2 &= b_{21}x_1 + b_{22}x_2. \end{aligned}$$

If we want to go from frame $\mathbf{x}$ to frame $\mathbf{z}$, we substitute

$$\begin{aligned} z_1 &= a_{11}\left(b_{11}x_1 + b_{12}x_2\right) + a_{12}\left(b_{21}x_1 + b_{22}x_2\right), \\ z_2 &= a_{21}\left(b_{11}x_1 + b_{12}x_2\right) + a_{22}\left(b_{21}x_1 + b_{22}x_2\right) \end{aligned}$$

and rearrange to get

$$\begin{aligned} z_1 &= \left(a_{11}b_{11} + a_{12}b_{21}\right)x_1 + \left(a_{11}b_{12} + a_{12}b_{22}\right)x_2, \\ z_2 &= \left(a_{21}b_{11} + a_{22}b_{21}\right)x_1 + \left(a_{21}b_{12} + a_{22}b_{22}\right)x_2. \end{aligned}$$

Now if A and B are the coefficient matrices of the first two transformations and C is the coefficient matrix of the last one, then we see that

$$C = AB,$$

which leads us to the following conclusion.

Matrix multiplication represents the new linear transformation obtained from one linear transformation followed by another one (Figure 3.5).

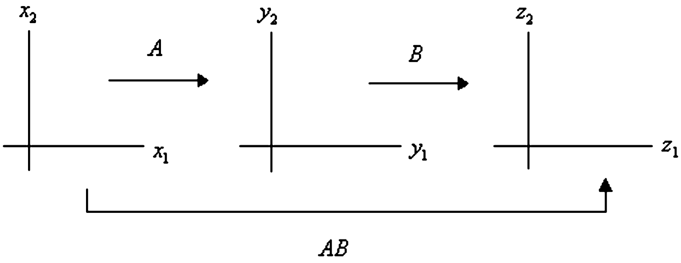

Figure 3.5: Matrix multiplication as one matrix transformation followed by another.

3.1.5 Computational considerations

Multiplication of large matrices is computationally expensive. There are particular cases where a clever choice reduces the amount of operations.

Calculation of A^k by squaring

The computation of matrix powers can be eased by squaring. For example, if we compute A^8 by using the definition

$$A^8 = ((((((AA)A)A)A)A)A)A$$

we use seven matrix multiplications. If, however, we first compute A^2, square it $A^2A^2 = A^4$, and, finally, square $A^4A^4 = A^8$, then we only need three matrix multiplications:

$$A^8 = ((A^2)^2)^2.$$

This method applies to any matrix power A^n. For example, we can compute A^{13} by computing A^2, squaring it to get A^4, and squaring it to get A^8. Now $A^{13} = A^8A^4A$. This takes only five matrix multiplications, instead of 12, if we use the definition.

Calculation of $A^k\mathbf{x}$ without matrix powers

If we want to compute the product $A^k\mathbf{x}$ for an $n \times n$ matrix A and an n-vector $\mathbf{x}$, then it is smart to avoid the calculation of A^k. We first compute $A\mathbf{x}$. Then compute the $A(A\mathbf{x})$ and continue this way. So

$$A(\dots(A(A\mathbf{x}))\dots) = A^k\mathbf{x}.$$

The advantage of this method is in that we always compute the product of a matrix and a vector and never the product of two matrices.

To illustrate, let $n = 3$ and $k = 2$. Computing A^2 first requires 27 multiplications. For $A^2\mathbf{x}$, we need other 9 multiplications. This is a total of 36 multiplications. If on the other hand, we first compute $A\mathbf{x}$, then we need 9 multiplications. Then for $A(A\mathbf{x})$, we need 9 more multiplications. This is a total of only 18 multiplications.

Reduction of computations in matrix products

Matrix multiplication is very important in applications. However, matrices in real-life problems have thousands or even millions of entries, so it is crucial that computations can be done efficiently.

In finding algorithms that improve efficiency in the number of numerical operations in calculations, we typically ignore the number of additions and only count the number of multiplications. This is because multiplications are more expensive in computer memory and speed than additions.

To compute the product AB for $n \times n$ matrices A and B, we need n^3 multiplications. This is because each dot product that computes one entry of AB requires n multiplications, and we have n^2 entries. This is provided that we use the definition of matrix product as stated in our text. The algorithm that follows this definition is known as the *Schoolbook algorithm*.

In 1969, Volker Strassen discovered a new algorithm that reduces the number of multiplications and increases the number of additions over the schoolbook algorithm. This is known as *Strassen's algorithm*. From this point on mathematicians and computer scientists have raced to find algorithms that reduce the number of multiplications even more.

The record as of 2022 is an algorithm announced by Duan, Wu, and Zhou, where the number of multiplications is approximately $Cn^{2.37188}$, where C is a constant. This slightly improves upon the algorithm published by Alman and Williams, where the above number was $Kn^{2.3728596}$, with K being constant.

To get an idea about the powers of n involved, for $n = 1000$, we have

$$1000^3 = 10^9 = 1{,}000{,}000{,}000 \quad \text{and} \quad 1000^{2.37188} \approx 1.30509 \times 10^7.$$

Although these algorithms are theoretically very interesting, in practice, they can be hard to use, because the constants C and K are large numbers. These algorithmic approaches may be useful in the cases of very large matrices.

3.1.6 Application to computer graphics

In Section 2.6, we used matrix–vector products to compute images of matrix transformations of the plane. Here we find a set of images by using matrix multiplication.

Example 3.1.11. Find the images of the vertices

$$(1.0, 0), (0.7, 0.7), (0, 1.0), (-0.7, 0.7), (-1.0, 0), (-0.7, -0.7), (0, -1.0), (0.7, -0.7)$$

of an octagon (Figure 3.6) under the shear transformation $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}.$$

Solution. We form a 2×8 matrix P with columns the vertices of the octagon and compute the product AP:

$$\begin{aligned} AP &= \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1.0 & 0.7 & 0 & -0.7 & -1.0 & -0.7 & 0 & 0.7 \\ 0 & 0.7 & 1.0 & 0.7 & 0 & -0.7 & -1.0 & -0.7 \end{bmatrix} \\ &= \begin{bmatrix} 1.0 & 1.05 & 0.5 & -0.35 & -1.0 & -1.05 & -0.5 & 0.35 \\ 0 & 0.7 & 1.0 & 0.7 & 0 & -0.7 & -1.0 & -0.7 \end{bmatrix}. \end{aligned}$$

The columns of AP are the transformed vertices of the octagon.

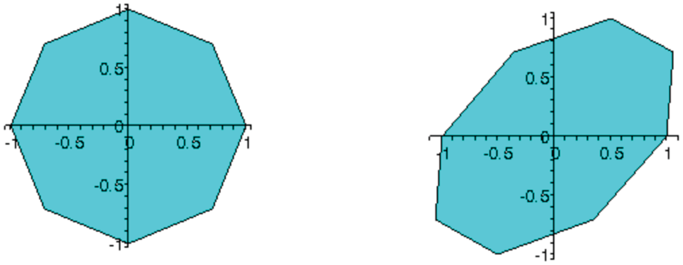

Figure 3.6: An octagon transformed by using matrix multiplication.

3.1.7 Application to manufacturing

Example 3.1.12. Each of the three appliance outlets receives and sells daily from three factories TVs and Game Consoles according to the following table:

	TVs	Game Consoles
Factory 1	40	50
Factory 2	70	80
Factory 3	60	65

Each outlet charges the following dollar amounts per appliance:

	Outlet 1	Outlet 2	Outlet 3
TVs	215	258	319
Game Consoles	305	282	264

If A and B are the matrices obtained from these tables, compute and interpret the product AB.

Solution. We have

$$AB = \begin{bmatrix} 40 & 50 \\ 70 & 80 \\ 60 & 65 \end{bmatrix} \begin{bmatrix} 215 & 258 & 319 \\ 305 & 282 & 264 \end{bmatrix} = \begin{bmatrix} 23\,850 & 24\,420 & 25\,960 \\ 39\,450 & 40\,620 & 43\,450 \\ 32\,725 & 33\,810 & 36\,300 \end{bmatrix}.$$

The $(1,1)$ entry $40 \cdot 215 + 50 \cdot 305 = 23{,}850$ is the first outlet's revenue from selling all appliances coming from the first factory. The remaining entries are interpreted similarly.

Exercises 3.1

1. Compute, if possible:

$$\text{(a)}\ \begin{bmatrix} 7 \\ 4 \end{bmatrix}\begin{bmatrix} 1 & 2 & -5 & 3 \end{bmatrix};\quad \text{(b)}\ \begin{bmatrix} 6 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix};$$

$$\text{(c)}\ \begin{bmatrix} 1 & -2 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & -4 & 3 & 0 \end{bmatrix};\quad \text{(d)}\ \begin{bmatrix} -1 & 0 \\ 2 & -5 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & 0 & b \\ 0 & c & d \end{bmatrix}.$$

2. Compute the third row of AB, where

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \\ 1 & 2 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 6 & 5 & 2 & 1 \end{bmatrix}.$$

3. Compute the $(2, 2)$ entry of

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}.$$

4. Compute the second column of AB, where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 6 & 5 & 2 \end{bmatrix},\quad B = \begin{bmatrix} 3 & 4 \\ 4 & 3 \\ 1 & 2 \end{bmatrix}.$$

5. Compute the $(3, 2)$ entry of AB, where

$$A = \begin{bmatrix} 4 & -8 & 0 & 5 & 7 \\ 9 & 5 & -6 & 3 & -4 \\ 9 & -5 & 8 & -2 & 5 \end{bmatrix},\quad B = \begin{bmatrix} -8 & -5 & -7 \\ 9 & 0 & 9 \\ 4 & 3 & 5 \\ 5 & -8 & -3 \\ 3 & -6 & 7 \end{bmatrix}.$$

6. Find A^2, where

$$A = \begin{bmatrix} 1 & 0 & 9 \\ 4 & 3 & -2 \\ 3 & -6 & 7 \end{bmatrix}.$$

7. Find $(2A)^3$, where

$$A^3 = \begin{bmatrix} 1 & 1 \\ -5 & -2 \end{bmatrix}.$$

8. Compute A^8, where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Guess a formula for A^n and use mathematical induction to prove it.

9. Compute

(a) $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}^3$;

(b) $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}^3$.

10. Let

$$H = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}.$$

(a) Explain the effect of H on A in the product HA.
(b) Explain the effect of H on A in the product AH.

11. Let

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}.$$

(a) Explain the effect of F on A in the product FA.
(b) Explain the effect of F on A in the product AF.

12. Let H and F be as in Exercises 10 and 11. Find the following:
(a) H^2, H^3, H^4;
(b) F^2, F^3, F^4;
(c) HF, FH.

13. Let H be as in Exercise 10. Verify that

$$a_0 I_4 + a_1 H + a_2 H^2 + a_3 H^3 = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \\ 0 & 0 & 0 & a_0 \end{bmatrix}.$$

14. Suppose that AB is a 9×5 matrix.
(a) How many columns does B have?
(b) How many rows does A have?
(c) What is the relation between the number of columns of A and the number of rows of B?

15. Prove that if both AB and BA are defined, then they are both square.

16. Explain why the product AB can be viewed as follows: the ith row of AB equals the ith row of A times the matrix B.

17. *True or False?* Explain.

(a) If AB is defined, then $(AB)^2$ is defined.

(b) If $(AB)^2$ is defined, then $(BA)^2$ is defined.

18. Assuming that AB is defined, mark your answer as true or false. Justify your choice.

(a) If B has a zero column, then AB has a zero column.

(b) If B has a zero row, then AB has a zero row.

(c) If A has a zero column, then AB has a zero column.

(d) If A has a zero row, then AB has a zero row.

19. Verify Theorem 3.1.4 for $a = -3$ and

$$A = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}.$$

20. Assuming that AB is defined, mark your answer as true or false. Justify your choice.

(a) If B has a repeated column, then AB has a repeated column.

(b) If A has a repeated row, then AB has a repeated row.

21. Prove Parts 2–6 of Theorem 3.1.4.

22. Prove Part 7 of Theorem 3.1.4.

23. Find a 2×2 matrix A such that $A^2 = \mathbf{0}$.

24. Find a 2×2 matrix $A \neq I$ such that $A^2 = A$.

25. Find two nonzero 2×2 matrices A and B such that $AB = \mathbf{0}$.

26. Find nonzero 2×2 matrices A, B, and C such that $AC = BC$ and $A \neq B$.

27. Find nonzero 2×2 matrices A, B, and C such that $CA = CB$ and $A \neq B$.

28. Recall that for real numbers, the equation $a^2 = 1$ has only two solutions $a = \pm 1$. An analogous statement is no longer true for matrices. Find four 2×2 matrices A such that $A^2 = I$.

29. Prove that $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ has no “square roots”. This means that there is no matrix B such that $B^2 = A$.

30. Find 2×2 matrices A and B such that

$$(AB)^2 \neq A^2B^2.$$

31. Find 2×2 nondiagonal and not equal matrices A and B that commute.

32. Prove that if A and B commute, then

$$(AB)^2 = A^2B^2$$

33. Let A and B be $n \times n$ matrices.

(a) Prove that if A and B commute, then

$$(A + B)^2 = A^2 + 2AB + B^2.$$

(b) Prove that if for matrices A and B,

$$(A+B)^2 = A^2 + 2AB + B^2,$$

then A and B commute.

(c) Find 2×2 matrices A and B such that

$$(A+B)^2 \neq A^2 + 2AB + B^2.$$

34. Find matrices A and B such that

$$(A+B)(A-B) \neq A^2 - B^2.$$

35. Find a relation between the matrices A and B such that

$$(A+B)(A-B) = A^2 - B^2$$

36. Find examples of 2×2 symmetric matrices whose product is not symmetric.

37. Prove that if A is symmetric, then A^n is symmetric, where n is a positive integer.

38. Let A and B be $n \times n$ symmetric matrices.
(a) Prove that if A and B commute, then the product AB is symmetric.
(b) Prove that if the product AB is symmetric, then A and B commute.
(c) Prove that the product ABA is symmetric.

39. *True or False?* If the matrix A is skew-symmetric, then A^2 is also skew-symmetric.

40. Prove that if A is skew-symmetric, then A^k is symmetric, if k is even and skew-symmetric, if k is odd.

41. Prove that if A and B are Hermitian, then ABA is also Hermitian.

42. Prove that the product of two Hermitian matrices A and B is Hermitian if and only if $AB = BA$.

43. *True or False?* If the matrix A is skew-Hermitian, then A^2 is also skew-Hermitian.

44. Prove that if A is skew-Hermitian, then A^k is Hermitian, if k is even and skew-Hermitian, if k is odd.

45. Verify Theorem 3.1.10 for $n = 2$, $m = 1$, $c = -3$, and

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}.$$

46. Prove Theorem 3.1.10.

47. Let A be a square matrix. Prove that if n is a positive integer, then

$$\left(A^T\right)^n = \left(A^n\right)^T.$$

48. Is the product of two $n \times n$ upper triangular matrices upper triangular? Explain.

49. Prove that if the product AB is defined and B has linearly dependent columns, then AB has also linearly dependent columns.

50. Is it true that if the product AB is defined and B has linearly independent columns, then AB has also linearly dependent columns? Explain.

51. Let $AB = I_n$. Prove that
(a) The system $B\mathbf{x} = \mathbf{0}$ has only the trivial solution;
(b) The columns of B are linearly independent.

52. Let A and B be $n \times n$ matrices. Prove that

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

53. Let $C = AB - BA$ for $n \times n$ matrices. Prove that $\operatorname{tr}(C) = 0$.

54. Prove that it is impossible to find $n \times n$ matrices A and B such that $AB - BA = I_n$.

55. Prove for 2×2 matrices that if $\operatorname{tr}(A^T A) = 0$, then A must be the zero matrix. Is this true for A of any size $m \times n$?

56. Prove the following properties of the commutator:
(a) $[A, A] = \mathbf{0}$;
(b) $[A, B] = -[B, A]$;
(c) $[A + B, C] = [A, C] + [B, C]$;
(d) The *Jacobi identity*

$$\big[A, [B, C]\big] + \big[B, [C, A]\big] + \big[C, [A, B]\big] = \mathbf{0}.$$

57. Consider the unit square in $\mathbf{R}^2$ with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$ (Figure 3.7). Compute and explain the geometric meaning of the product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

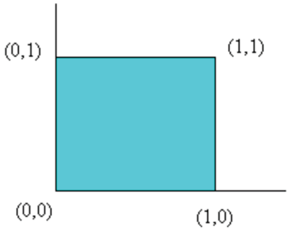

Figure 3.7: Transform the unit square.

58. Set up a matrix product that computes the images of the vertices of the unit cube in $\mathbf{R}^3$ under a 3×3 matrix transformation $T(\mathbf{x}) = A\mathbf{x}$.

59. For $A = \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}$, compute A^2, A^3, and A^4.

In Exercises 60–61, prove each identity and explain your answer geometrically by using rotations.

60. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}.$

61. For positive integer *n*,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos(n\theta) & -\sin(n\theta) \\ \sin(n\theta) & \cos(n\theta) \end{bmatrix}.$$

62. Each of the three department stores receives and sells weekly from three clothing factories slacks and jackets according to the table

	slacks	**jackets**
Factory 1	50	20
Factory 2	60	30
Factory 3	45	40

Each store charges in dollar amounts as follows:

	Store 1	**Store 2**	**Store 3**
slacks	100	85	75
jackets	350	400	250

If *A* and *B* are the matrices of these tables, compute and interpret the product *AB*.

3.2 Matrix inverse

Matrix inversion is the last basic matrix operation. It applies only to *square* matrices. The inverse of a matrix, if it exists, is the analogue of the reciprocal of a nonzero number.

Definition 3.2.1. An $n \times n$ matrix A is called *invertible* if there exists a matrix B such that

$$AB = I \quad \text{and} \quad BA = I.$$

In such case, B is called an *inverse* of A. If no such B exists for A, then we say that A is *noninvertible*. Another name for invertible is *nonsingular*, and another name for noninvertible is *singular*.

Note that the definition implies that *B* is square of size *n* (why?).

Example 3.2.2. Prove that B is an inverse of A, where

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Solution. We have

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Similarly, we verify that $BA = I$.

Theorem 3.2.3. *An invertible matrix has only one inverse.*

Proof. Suppose that the invertible matrix A has two inverses B and C. Then

$$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

Therefore $B = C$. ☐

The unique inverse of an invertible matrix A is denoted by A^{-1}. So

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I \tag{3.1}$$

3.2.1 Computation of the inverse

Next, we see how to compute the inverse of an invertible matrix A. The idea is simple: If A^{-1} has unknown columns $\mathbf{x}_i$, then $AA^{-1} = I$ takes the form

$$[A\mathbf{x}_1 \cdots A\mathbf{x}_n] = [\mathbf{e}_1 \cdots \mathbf{e}_n].$$

This matrix equation splits into n linear systems

$$A\mathbf{x}_1 = \mathbf{e}_1, \quad \dots, \quad A\mathbf{x}_n = \mathbf{e}_n,$$

which we solve to find each column $\mathbf{x}_i$ of A^{-1}. These systems have the same coefficient matrix A. Solving each system separately would amount into $n - 1$ unnecessary row reductions of A. It is smarter to solve the systems simultaneously by simply row reducing the matrix

$$[A : I].$$

If we get a matrix of the form $[I : B]$, then the ith column of B would be $\mathbf{x}_i$. Thus $B = A^{-1}$. So, to compute A^{-1}, we just row reduce $[A : I]$.

Example 3.2.4. Compute A^{-1} if $A = \left[\begin{smallmatrix} 1 & 0 & -1 \\ 3 & 4 & -2 \\ 3 & 5 & -2 \end{smallmatrix}\right]$.

Solution. We row reduce $[A : I]$:

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 3 & 4 & -2 & 0 & 1 & 0 \\ 3 & 5 & -2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & -3 & 1 & 0 \\ 0 & 5 & 1 & -3 & 0 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & -3 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{5}{4} & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & 5 & -4 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 5 & -4 \\ 0 & 4 & 0 & 0 & -4 & 4 \\ 0 & 0 & 1 & -3 & 5 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & 5 & -4 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -3 & 5 & -4 \end{bmatrix}.$$

Therefore

$$A^{-1} = \begin{bmatrix} -2 & 5 & -4 \\ 0 & -1 & 1 \\ -3 & 5 & -4 \end{bmatrix}.$$

What can go wrong when we row reduce $[A : I]$? If an echelon form of A has a row of zeros, then we can never reach a form $[I : B]$. In such case, A is noninvertible. We discuss the details of this in Section 3.3.

As an example, if $A = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 4 \end{smallmatrix}\right]$, then

$$[A : I_2] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}.$$

We cannot obtain I_2 on the left. The second row of the reduced form represents the equations $0x_1 + 0x_2 = 2$ and $0x_1 + 0x_2 = -1$. So A^{-1} does not exist.

Our discussion leads us to the following algorithm, which is analyzed in Section 3.3.

Algorithm 3.2.5 (Matrix inversion algorithm). To compute A^{-1}:
1. Find the reduced row echelon form of the matrix $[A : I]$, say $[B : C]$.
2. If B has a zero row, then stop. A is noninvertible. Otherwise, go to Step 3.
3. The reduced matrix is now in the form $[I : A^{-1}]$. Read off the inverse A^{-1}.

In the case of a 2×2 matrix A, there is an explicit formula for A^{-1}.

Theorem 3.2.6. *$A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is invertible if and only if $ad - bc \neq 0$, in which case*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{3.2}$$

Proof. Exercise. ☐

Example 3.2.7. Find A^{-1} if $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$.

Solution. By equation (3.2)

$$A^{-1} = \frac{1}{1 \cdot 4 - 2 \cdot 3} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

3.2.2 Relation of A^{-1} to square systems

If A^{-1} exists, then we may start with any square system of the form $A\mathbf{x} = \mathbf{b}$ and solve for $\mathbf{x}$ as follows:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ &\Rightarrow A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \\ &\Rightarrow I\mathbf{x} = A^{-1}\mathbf{b} \\ &\Rightarrow \mathbf{x} = A^{-1}\mathbf{b}. \end{aligned}$$

The solution is given by the formula $\mathbf{x} = A^{-1}\mathbf{b}$. Furthermore, the solution is unique: another solution would lead to the same formula!

We have proved Part 1 of the following theorem. The verification of Part 2 is left to the reader.

Theorem 3.2.8. *Let A be an invertible matrix. Then*

1. *For each n-vector* $\mathbf{b}$, *the square system* $A\mathbf{x} = \mathbf{b}$ *has a unique solution given by*

$$\mathbf{x} = A^{-1}\mathbf{b};$$

2. *The homogeneous system* $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

3.2.3 Properties of matrix inversion

Theorem 3.2.9. *Let A and B be invertible n × n matrices, and let c be a nonzero scalar. Then*

1. *AB is invertible, and*

$$(AB)^{-1} = B^{-1}A^{-1};$$

2. A^{-1} *is invertible, and*

$$(A^{-1})^{-1} = A;$$

3. *cA is invertible, and*

$$(cA)^{-1} = \frac{1}{c} A^{-1};$$

4. *A^T is invertible, and*

$$(A^T)^{-1} = (A^{-1})^T.$$

Proof of 1. Because A^{-1} and B^{-1} exist and $B^{-1}A^{-1}$ is a candidate for $(AB)^{-1}$, we only need to verify that $(AB)(B^{-1}A^{-1}) = I$ and $(B^{-1}A^{-1})(AB) = I$:

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1}\\ &= AIA^{-1}\\ &= AA^{-1}\\ &= I.\end{aligned}$$

The second equation is checked similarly. □

Note that Part 1 of Theorem 3.2.9 can be iterated. So if $A_1, \dots, A_n$ are invertible and of the same size, then so is the product $A_1 \dots A_n$. Its inverse is

$$(A_1A_2 \dots A_{n-1}A_n)^{-1} = A_n^{-1}A_{n-1}^{-1} \dots A_2^{-1}A_1^{-1}.$$

Recall from Section 3.1 that $CA = CB$ does not necessarily imply that $A = B$. This changes if C is invertible. We have the following theorem.

Theorem 3.2.10 (Cancellation laws). *If C is invertible, then*

1. *$CA = CB \Rightarrow A = B$;*
2. *$AC = BC \Rightarrow A = B$.*

Proof of 1. Since C^{-1} exists,

$$\begin{aligned}CA = CB &\Rightarrow C^{-1}(CA) = C^{-1}(CB)\\ &\Rightarrow (C^{-1}C)A = (C^{-1}C)B\\ &\Rightarrow IA = IB\\ &\Rightarrow A = B.\end{aligned}$$

□

3.2.4 Matrix powers with negative exponents

If A is invertible, then we define powers of A with negative exponents as follows. For a positive integer n, we have

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{n \text{ factors}}.$$

If A is invertible of size $k \times k$ and m and n are any integers, then

1. A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$;
2. $A^n A^m = A^{n+m}$;
3. $(A^n)^m = A^{nm}$;
4. $(cA)^n = c^n A^n$ for any nonzero scalar c.

3.2.5 Application to statics: Stiffness of elastic beam

We consider an elastic beam supported on the edges. We choose points $P_1, \ldots, P_n$ on which parallel forces $F_1, \ldots, F_n$ are applied causing displacements $y_1, \ldots, y_n$ (Figure 3.8). We assume the *principle of linear superposition*, or *Hooke's law*:

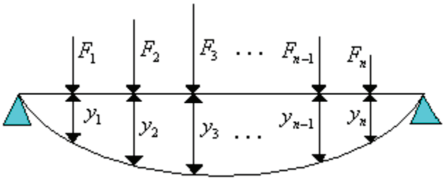

Figure 3.8: Elastic beam under parallel forces.

1. If two systems of forces are applied, then the corresponding displacements are added.
2. If the magnitudes of all forces are multiplied by a scalar c, then the displacements are multiplied by c.

Let a_{ik} be the displacement of P_i under the action of the unit force at P_k. Then under the action of all the forces, the displacements are given by the formulas

$$\sum_{k=1}^{n} a_{ik} F_k = y_i, \quad i = 1, \ldots, n. \tag{3.3}$$

If A is the matrix $A = [a_{ik}]$, $\mathbf{y}$ is the vector of displacements $\mathbf{y} = (y_1, \ldots, y_n)$, and $\mathbf{F}$ is the vector of forces, $\mathbf{F} = (F_1, \ldots, F_n)$, then equations (3.3) become

$$A\mathbf{F} = \mathbf{y}.$$

The matrix A is called the *flexibility matrix*. Given the flexibility matrix and the displacements, we can calculate the forces F_i by inverting A:

$$\mathbf{F} = A^{-1}\mathbf{y}.$$

The inverse A^{-1} is called the *stiffness matrix*.

Exercises 3.2

1. Use Theorem 3.2.6 to find the inverses of the following matrices:

$$\text{(a)} \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 7 & 5 \\ 4 & 4 \end{bmatrix}.$$

2. Use Theorem 3.2.6 to explain why the following matrices are noninvertible:

$$\text{(a)} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} -10 & 20 \\ 20 & -40 \end{bmatrix}.$$

3. Find a matrix B whose inverse is $A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}$.

4. Find the inverse of $10A$ if the inverse of A is $\begin{bmatrix} 4 & 4 \\ 8 & 6 \end{bmatrix}$.

5. For $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, assume that $a^2 + b^2 = 1$. Use Theorem 3.2.6 to prove that A is invertible and compute A^{-1}.

6. Compute $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^{-1}$ and verify your answer.

7. Compute $(2A)^{-3}$, given that $A^3 = \begin{bmatrix} 1 & 1 \\ -5 & -2 \end{bmatrix}$.

8. Compute the sum $A^3 + A^{-3}$, if $A = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$.

9. Find the inverses of the following matrices by inspection:

$$\text{(a)} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \text{(b)} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

10. Without actually computing, explain why

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is noninvertible.

In Exercises 11–15, use the matrix inversion algorithm to compute, if possible, the inverse of the given matrix.

11. (a) $\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & -4 \\ -2 & 8 \end{bmatrix}$.

12. (a) $\begin{bmatrix} -5 & 1 & 8 \\ 0 & -7 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & -2 & -1 \\ 1 & 6 & -1 \end{bmatrix}$.

13. (a) $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 3 & 3 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 2 & -1 & 0 \end{bmatrix}$.

14. $\begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

15. $\begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 2 \\ -1 & 1 & 1 & 1 \end{bmatrix}$.

In Exercises 16–17, solve the systems by computing the inverse of the coefficient matrix first.

16.
$$\begin{aligned} x + y - z &= 1, \\ x - z &= 2, \\ -x + y &= 3. \end{aligned}$$

17.
$$\begin{aligned} -x + y + z &= a, \\ -x + z &= a^2, \\ y - z &= a^3. \end{aligned}$$

18. Find A if

$$A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

19. Find all the values of a for which A^{-1} exists if

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & -2 & a \end{bmatrix}.$$

20. Let $c_1, \dots, c_4$ be nonzero scalars. Compute A^{-1}, where

$$A = \begin{bmatrix} 0 & 0 & 0 & c_4 \\ 0 & 0 & c_3 & 0 \\ 0 & c_2 & 0 & 0 \\ c_1 & 0 & 0 & 0 \end{bmatrix}.$$

21. Compute $(ABC)^{-1}$ if

$$A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}.$$

22. Compute $A^{-1}, A^{-2}, A^{-3}, A^{-24}$, and A^{-25}, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

23. Let A_n be the $n \times n$ matrix with zeros below the main diagonal and ones on and above the diagonal. For example,

$$A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find A_2^{-1}, A_3^{-1}, and A_4^{-1}. Guess a formula for A_n^{-1}.

24. If P is invertible and $P^{-1}AP = B$, then write A in terms of P, P^{-1}, and B.

25. Find matrices A and B such that

$$(A + B)^{-1} \neq A^{-1} + B^{-1}.$$

26. Find matrices A and B such that

$$(AB)^{-1} \neq A^{-1}B^{-1}.$$

27. If $AB = BA$ with A invertible, then prove that $A^{-1}B = BA^{-1}$.

28. Prove that a matrix with a row of zeros is noninvertible.

29. Prove that a matrix with a column of zeros is noninvertible.

30. Prove that a diagonal matrix A is invertible if and only if each element on the main diagonal is nonzero. What is A^{-1} in this case?

31. If A is invertible and $AB = \mathbf{0}$, then prove that $B = \mathbf{0}$.

32. Suppose $AB = \mathbf{0}$ and $B \neq \mathbf{0}$. Prove that A is noninvertible.

33. If A is a square matrix such that $A^2 = \mathbf{0}$, then prove that the inverse of $I - A$ exists and is equal to $A + I$.

34. If A is a square matrix such that $A^3 = \mathbf{0}$, then prove that the inverse of $I - A$ exists and is equal to $A^2 + A + I$.

35. Suppose $A^2 + 2A - I = \mathbf{0}$. Prove that $A^{-1} = A + 2I$.

36. Suppose A has size 3×2 and B has size 2×3. Prove that AB is *not* invertible.

37. Prove Theorem 3.2.6.

38. Prove Parts 2–4 of Theorem 3.2.9.

39. Prove Part 2 of Theorem 3.2.10.

40. Prove the claims of the note in Subsection 3.2.4.

41. Prove that if A is symmetric and invertible, then A^{-1} is symmetric.

42. Prove that if A is skew-symmetric and invertible, then A^{-1} is skew-symmetric.

43. Prove that if a matrix A is complex invertible, then $(A^H)^{-1} = (A^{-1})^H$.

Right and left inverses

Let A be an $m \times n$ matrix. A matrix B is called a *right inverse* of A if $AB = I$. Likewise, C is a *left inverse* of A if $CA = I$. For example, if $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, then P is a left inverse of Q, and Q is a right inverse of P.

44. Find two left inverses for the matrix Q above.

45. Find two right inverses for the matrix P above.

46. If an $m \times n$ matrix A has a right inverse B, then what is the size of B?

47. If an $m \times n$ matrix A has a left inverse C, then what is the size of C?

48. Prove that if A has a right inverse, then A^T has a left inverse.

49. Prove that if A has a left inverse, then A^T has a right inverse.

50. Let A be an $m \times n$ matrix. Prove that the following statements are equivalent:
(a) A has a right inverse;
(b) The system $A\mathbf{x} = \mathbf{b}$ is consistent for all m-vectors $\mathbf{b}$;
(c) Each row of A has a pivot;
(d) The columns of A span $\mathbf{R}^m$.

51. Let A be an $m \times n$ matrix. Prove that the following statement are equivalent:
(a) A has a left inverse;
(b) The system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution;
(c) Each column of A is a pivot column;
(d) The columns of A are linearly independent.

52. Prove that if an $m \times n$ matrix A has both a right inverse B and a left inverse C, then
(a) $m = n$;
(b) $B = C$;
(c) A is invertible.

3.3 Elementary matrices

This section is of theoretical nature. We study elementary matrices and then use them to characterize invertible matrices in several interesting ways. This characterization is outlined in Theorem 3.3.10. We also use elementary matrices to analyze the matrix inversion algorithm of Section 3.2.

Definition 3.3.1. An $n \times n$ matrix is called *elementary*, if it can be obtained from the identity matrix I_n by using only one elementary row operation: elimination, scaling, or interchange.

Definition 3.3.1 implies that an $n \times n$ elementary matrix is row equivalent to I_n.

Example 3.3.2. Prove that the following matrices are elementary:

$$E_1 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Solution. We have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boxed{R_1 - 3R_2 \to R_1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boxed{4R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boxed{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The key reason for studying elementary matrices is the observation that if we multiply a matrix A on the left by an elementary matrix E, then the product EA is the matrix obtained from A by using the same elementary row operation that produced E from I_n. To illustrate, we have

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} = \begin{bmatrix} a-3e & b-3f & c-3g & d-3h \\ e & f & g & h \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ 4e & 4f & 4g & 4h \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} = \begin{bmatrix} e & f & g & h \\ a & b & c & d \end{bmatrix}.$$

Elementary row operations are reversible or *invertible*. This means that for each such operation, there is another elementary row operation that reverses the effects of the first. For example, $(1/4)R_2 \to R_2$ cancels out the effect of $4R_2 \to R_2$. In general, we have the following correspondence:

Row Operation	**Inverse Row Operation**
$R_i \leftrightarrow R_j$	$R_i \leftrightarrow R_j$
$cR_i \to R_i$	$(1/c)R_i \to R_i$
$R_i + cR_j \to R_i$	$R_i - cR_j \to R_i$

Because elementary row operations are invertible, we can recover I_n from an elementary matrix by performing the inverse operation. For example, I_2 is obtained from E_1 by applying $R_1 + 3R_2 \to R_1$. This implies that elementary matrices are invertible: If

E is an elementary matrix obtained from I_n by applying an elementary operation and E' is the elementary matrix obtained from I_n by applying the inverse operation, then $EE' = I_n$. Similarly, $E'E = I_n$. Therefore E is invertible, and $E^{-1} = E'$. For example, E_1, E_2, and E_3 above are invertible, and

$$E_1^{-1} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We have proved the following theorem.

Theorem 3.3.3. *Every elementary matrix E has an inverse, which is also an elementary matrix. The inverse E^{-1} is obtained from I by performing the inverse of the elementary row operation that produced E from I.*

Example 3.3.4. Consider the following elementary matrices:

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The multiples KA, LA, and MA are the matrices obtained from A by using the operation

(a) $R_2 + cR_3 \to R_2$ for KA,
(b) $cR_2 \to R_2$ for LA, and
(c) $R_1 \leftrightarrow R_2$ for MA.

The inverses of these matrices are the elementary matrices

$$K^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}, \quad L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad c \neq 0, \quad M^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If matrices A and B are row equivalent, then B can be obtained from A by a finite sequence of elementary row operations, say, $O_1, \dots, O_k$. Let $E_1, \dots, E_k$ be the elementary matrices corresponding to these operations. The effect of operation O_1 on A is the same as the product E_1A. Likewise, the effect of O_2 is $E_2(E_1A) = E_2E_1A$. Continuing in the same manner, we get $B = E_k \dots E_2E_1A$. We have proved the following theorem.

Theorem 3.3.5. *Let A and B be two $m \times n$ matrices. The following statements are equivalent:*

1. *$A \sim B$, i. e., A and B are row equivalent.*
2. *There are elementary matrices $E_1, \dots, E_k$ such that*

$$B = E_k \dots E_1A.$$

In particular, Theorem 3.3.5 applies to a matrix A and an echelon form U of A. If U is obtained from A by using row operations with corresponding elementary matrices $E_1, \dots, E_k$, then

$$U = E_k \cdots E_1 A. \tag{3.4}$$

Solving for A yields

$$A = E_1^{-1} \cdots E_k^{-1} U. \tag{3.5}$$

Either of equations (3.4) or (3.5) *records the row reduction* of A.

Example 3.3.6. Let $A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 8 \\ 0 & 4 & 3 \end{bmatrix}$.

(a) Row reduce A to an echelon form U.
(b) Write U as a product of elementary matrices and A.
(c) Write A as a product of elementary matrices and U.

Solution.

(a) Let U be the echelon form matrix in the reduction

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 8 \\ 0 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & 0 & -6 \\ 0 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 3 \\ 0 & 0 & -6 \end{bmatrix} = U.$$

(b) The operations that produced U were $-2R_1 + R_2 \to R_2$ and $R_2 \leftrightarrow R_3$. So the corresponding elementary matrices are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence, according to our analysis, $U = E_2 E_1 A$,

$$\begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 3 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 8 \\ 0 & 4 & 3 \end{bmatrix}.$$

(c) We *factored* U as the product $E_2 E_1 A$. Because $A = E_1^{-1} E_2^{-1} U$ and

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

we have

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 6 & 8 \\ 0 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 3 \\ 0 & 0 & -6 \end{bmatrix}.$$

This is a factorization of A in terms of one of its echelon forms and the elementary matrices of the inverse operations that produced it.

3.3.1 Elementary matrices and invertible matrices

We now discuss relations between invertible and elementary matrices.

Let A be a square matrix, and let R be its reduced row echelon form. If each row of A has a pivot, then R is the identity matrix I. If A has at least a row that has no pivot, then R must have a row of zeros. So we have the following.

> The reduced row echelon form R of a square matrix A is either I, or it contains a row of zeros. **!**

Theorem 3.3.7. *Let A be an $n \times n$ matrix. The following statements are equivalent:*
1. *A is invertible.*
2. *$A \sim I_n$.*
3. *A is a product of elementary matrices.*

Proof.

$1 \Rightarrow 2$. Let A be invertible. Then $A\mathbf{x} = \mathbf{b}$ is consistent for all n-vectors $\mathbf{b}$ by Theorem 3.2.8. So each row of A has a pivot by Theorem 2.3.7. Hence the reduced row echelon form of A is I_n, because A is square. Therefore $A \sim I_n$.

$2 \Rightarrow 3$. Let $A \sim I_n$. Then there are elementary matrices $E_1, \dots, E_k$ such that $A = E_k \dots E_1 I_n = E_k \dots E_1$ by Theorem 3.3.5. So A is a product of elementary matrices.

$3 \Rightarrow 1$. Let A be a product of elementary matrices, say $A = E_k \dots E_1$. Then $E_1, \dots, E_k$ are invertible by Theorem 3.3.3. So A is invertible as a product of invertible matrices. □

An interesting implication of Theorem 3.3.7 is expressed in the next theorem, which says that for square matrices of the same size, *only one* of the two conditions $AB = I$ and $BA = I$ suffices to ensure that both A and B are invertible and that they are inverses of each other.

Theorem 3.3.8. *Let A and B be $n \times n$ matrices. If $AB = I$, then A and B are invertible, and $A^{-1} = B$, $B^{-1} = A$. In particular, $AB = I$ if and only if $BA = I$.*

Proof. Let R be the reduced row echelon form of A. If $R = I$, then we are done by Theorem 3.3.7. Otherwise, R has a row of zeros. Because $A \sim R$, by Theorem 3.3.5 there are elementary matrices $E_1, \dots, E_k$ such that $R = E_n \dots E_1 A$. Therefore $RB = E_n \dots E_1 AB = E_n \dots E_1$, since $AB = I$. Hence RB is invertible as a product of invertible matrices. But RB has a row of zeros, because R does. So RB does not reduce to I, and thus it cannot be invertible by Theorem 3.3.7. So R cannot have a row of zeros. Therefore A^{-1} exists, and $AB = I$ implies $A^{-1}AB = A^{-1}I$ or $B = A^{-1}$. Hence $B^{-1} = A$. □

3.3.2 The matrix inversion algorithm

Let us now use elementary matrices to explain why the matrix inversion algorithm (Section 3.2) works. Let A be an $n \times n$ matrix with reduced row echelon form R. There exist elementary matrices $E_1, \dots, E_k$ such that

$$A = E_k \dots E_1 R.$$

The row reduction of A using the operations that correspond to these elementary matrices can be described by

$$E_k^{-1} A = E_{k-1} \dots E_1 R, \quad E_{k-1}^{-1} E_k^{-1} A = E_{k-2} \dots E_1 R, \quad \dots, \quad E_1^{-1} \dots E_k^{-1} A = R.$$

So in the reduction of $[A : I]$ the matrix obtained from placing $I_n = I$ next to A yields

$$[E_1^{-1} \dots E_k^{-1} A : E_1^{-1} \dots E_k^{-1}] \quad \text{or} \quad [R : E_1^{-1} \dots E_k^{-1}].$$

If R has a row of zeros, then A is not invertible. Otherwise, R is I, and A is invertible. Hence $E_1^{-1} \dots E_k^{-1} A = R = I$ implies $A^{-1} = E_1^{-1} \dots E_k^{-1}$. So, in this case,

$$[R : E_1^{-1} \dots E_k^{-1}] = [I : A^{-1}].$$

We conclude that the reduction of $[A : I]$, which is the matrix inversion algorithm, either detects a noninvertible matrix, or it computes its inverse, as claimed in Section 3.2.

Example 3.3.9. Write $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right]$ and A^{-1} as products of elementary matrices.

Solution. We compute A^{-1}:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}.$$

The elementary matrices that correspond to the row operations $-R_1 + R_2 \to R_2$ and $R_1 - R_2 \to R_1$ are $E_1 = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]$ and $E_2 = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$. Hence

$$E_2 E_1 A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = I.$$

Therefore

$$A = (E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1} = E_2 E_1 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

are the required products.

3.3.3 Characterization of invertible matrices

We now offer several useful characterizations of an invertible matrix.

Theorem 3.3.10 (Characterization of invertible matrices). *Let A be an $n \times n$ matrix. The following statements are equivalent:*

1. *A is invertible.*
2. *A^T is invertible.*
3. *$A \sim I_n$.*
4. *A is a product of elementary matrices.*
5. *There is an $n \times n$ matrix B such that $AB = I_n$.*
6. *There is an $n \times n$ matrix C such that $CA = I_n$.*
7. *Each column of A is a pivot column.*
8. *Each row of A has a pivot.*
9. *The columns of A are linearly independent.*
10. *The rows of A are linearly independent.*
11. *The columns of A span $\mathbf{R}^n$.*
12. *The rows of A span $\mathbf{R}^n$.*
13. *The system $A\mathbf{x} = \mathbf{b}$ has at least one solution for each n-vector $\mathbf{b}$.*
14. *The system $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each n-vector $\mathbf{b}$.*
15. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

Proof.

(a) The equivalences $1 \Leftrightarrow 3 \Leftrightarrow 4$ are Theorem 3.3.7. The equivalence $1 \Leftrightarrow 2$ follows from Theorem 3.2.9 applied to A and A^T.

(b) The equivalences $1 \Leftrightarrow 5 \Leftrightarrow 6$ follow from Theorem 3.3.8. In addition, $3 \Leftrightarrow 7 \Leftrightarrow 8$, because A is square. Therefore 1–8 are all equivalent.

(c) Now $8 \Leftrightarrow 11 \Leftrightarrow 13$ by Theorem 2.3.7. Also, $11 \Leftrightarrow 12$ follows from the equivalences $1 \Leftrightarrow 2 \Leftrightarrow 11$. In addition, $1 \Leftrightarrow 9 \Leftrightarrow 15$, by Theorems 3.2.8 and 2.4.4. Also $9 \Leftrightarrow 10$ follows from the equivalences $1 \Leftrightarrow 2 \Leftrightarrow 9$. So we have proved the equivalences $1 \Leftrightarrow \cdots \Leftrightarrow 13 \Leftrightarrow 15$.

(d) As a final step, we prove that 13 ⇔ 14. Clearly, 14 ⇒ 13. We only need to prove that 13 ⇒ 14. Let us assume 13. Then for each n-vector $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has at least one solution, say $\mathbf{v}_1$. If $\mathbf{v}_2$ is another solution, then $A\mathbf{v}_1 = \mathbf{b} = A\mathbf{v}_2$. Hence $A(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$. Therefore $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$ or $\mathbf{v}_1 = \mathbf{v}_2$, because 13 ⇔ 15. So the solution of $A\mathbf{x} = \mathbf{b}$ is unique. This proves Statement 14. ☐

Theorem 3.3.10, has an interesting consequence.

Theorem 3.3.11. *Let A and B be $n \times n$ matrices. If either A or B is noninvertible, then AB is also noninvertible.*

Proof. Exercise. ☐

Exercises 3.3

In Exercises 1–4, indicate which of the matrices are elementary. For each elementary matrix, identify the elementary row operation that yielded the matrix from the identity matrix.

1. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$.

2. $C = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$.

3. $E = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $F = \begin{bmatrix} 1 & -9 \\ 0 & 1 \end{bmatrix}$.

4. $G = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$.

5. Is $-I_n$ $(n \geq 2)$ an elementary matrix? Explain.

In Exercises 6–7, determine the elementary row operation that yields the elementary matrices from the identity matrix of the same size.

6. $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

7. $L = \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$.

In Exercises 8–9, determine the row operation that yields an identity matrix from the given elementary matrix.

8. Matrices J, K of Exercise 6.

9. Matrices L, M of Exercise 7.

10. Write the inverses of the elementary matrix operations

$$\text{(a)}\quad R_1 \leftrightarrow R_3, \quad \text{(b)}\quad R_1 + 5R_3 \to R_1, \quad \text{(c)}\quad (1/2)R_4 \to R_4, \quad \text{(d)}\quad 10R_1 + R_3 \to R_3.$$

11. Multiply $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ on the left by a suitable elementary matrix to perform the following matrix operations:

$$\text{(a)}\quad R_1 \leftrightarrow R_3, \quad \text{(b)}\quad R_1 - 2R_3 \to R_1, \quad \text{(c)}\quad -2R_2 \to R_2, \quad \text{(d)}\quad 5R_1 + R_3 \to R_3.$$

12. Write A and A^{-1} as products of elementary matrices for

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

13. Show that the decomposition of an invertible matrix as a product of elementary matrices is *not unique* by finding a second factorization of the matrix A in Exercise 12.

14. Express each of the following matrices as products of elementary matrices:

$$A = \begin{bmatrix} 3 & -6 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

15. Matrices A and B are row equivalent. Find elementary matrices E_1 and E_2 such that $A = E_2E_1B$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -4 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 3 \\ -1 & -4 & -1 \\ 1 & 2 & 3 \end{bmatrix}.$$

16. Explain why the matrix

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

cannot be written as a product of elementary matrices.

17. Use Theorem 3.3.10 to prove that the following system has exactly one solution for any choices of b_1 and b_2:

$$\begin{aligned} x - y &= b_1, \\ x + 2y &= b_2. \end{aligned}$$

18. Use Theorem 3.3.10 to prove that the following system has infinitely many solutions:

$$\begin{aligned} x + y + z &= 0, \\ y \phantom{{}+z} &= 0, \\ x \phantom{{}+y} + z &= 0. \end{aligned}$$

19. Let c_1, c_2, c_3 be nonzero scalars, and let

$$A = \begin{bmatrix} 0 & 0 & c_1 \\ 0 & c_2 & 0 \\ c_3 & 0 & 0 \end{bmatrix}.$$

Write A and A^{-1} as products of elementary matrices

20. Let c_1, c_2 be nonzero scalars, and let

$$A = \begin{bmatrix} 0 & 0 & c_1 \\ 0 & c_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Write A as a product of elementary matrices and a noninvertible matrix in reduced row echelon form.

21. *True or False?* Explain.

(a) If $A \sim B$, then there is an invertible matrix P such that $PA = B$.

(b) If $A \sim B$, then there is an invertible matrix Q such that $QB = A$.

22. Use elementary matrices to prove the following statements about row equivalence of matrices:

(a) $A \sim A$.

(b) If $A \sim B$, then $B \sim A$.

(c) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Permutation matrices

The elementary matrix obtained from the identity matrix by interchanging two rows is called an *elementary permutation matrix*. For example,

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

are elementary permutation matrices, because P_1 is obtained from I_4 by switching rows 2 and 4 and P_2 by switching rows 1 and 2. Note that switching rows i and j in I_n is the same as *switching columns i* and *j*.

23. Let P be an elementary permutation matrix. Prove that $P^2 = I$. Deduce that $P^{-1} = P$.

24. Find all 3 × 3 elementary permutation matrices.

25. Let P_{ij} be the elementary permutation matrix obtained from I_n by interchanging rows i and j. Let $\mathbf{x}$ be an n-vector. Describe the product $P_{ij}\mathbf{x}$.

A *permutation matrix* is a product of elementary permutation matrices (one, two, or more). For example,

$$P_1P_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is a permutation matrix. Notice that a permutation matrix is obtained from I by permuting any number of rows (or columns).

26. Find two permutation matrices each of which is not an elementary permutation matrix.

27. Prove that the product of two permutation matrices is also a permutation matrix.

28. Use Theorem 3.3.10 to prove that a permutation matrix is invertible.

3.4 LU factorization

In Section 3.3, we factored a matrix as a product of elementary matrices and one of its echelon forms. In general, a factorization of a matrix can be very useful in understanding properties of the matrix. It can also be computationally efficient. For example, suppose that we know how to factor an $m \times n$ matrix A as

$$A = LU, \tag{3.6}$$

where L is $m \times m$ lower triangular, and U has size $m \times n$ and is in row echelon form. Then the system

$$A\mathbf{x} = \mathbf{b} \tag{3.7}$$

can be solved in two easy steps. First, we solve

$$L\mathbf{y} = \mathbf{b} \tag{3.8}$$

for $\mathbf{y}$, and then we solve

$$U\mathbf{x} = \mathbf{y} \tag{3.9}$$

for $\mathbf{x}$. Solving these two systems is in fact equivalent to solving the original system, because

$$LU\mathbf{x} = L(U\mathbf{x}) = L\mathbf{y} = \mathbf{b}.$$

The advantage of not solving (3.7) directly is that (3.8) is a lower triangular system and can be easily solved by a forward substitution and (3.9) is upper triangular and can be easily solved by a back-substitution.

Definition 3.4.1. A factorization $A = LU$ of an $m \times n$ matrix A is an *LU factorization* or *LU decomposition* if L is $m \times m$ lower triangular and U is $m \times n$ upper triangular.

Example 3.4.2. Solve the system $A\mathbf{x} = \mathbf{b}$,

$$\begin{bmatrix} 4 & -2 & 1 \\ 20 & -7 & 12 \\ -8 & 13 & 17 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 11 \\ 70 \\ 17 \end{bmatrix},$$

by using the following LU factorization of A:

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 20 & -7 & 12 \\ -8 & 13 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 7 \\ 0 & 0 & -2 \end{bmatrix} = LU \tag{3.10}$$

Solution. Let $\mathbf{y} = (y_1, y_2, y_3)$ be a new vector of unknowns. We first solve the lower triangular system $L\mathbf{y} = \mathbf{b}$,

$$\begin{aligned} y_1 &= 11, \\ 5y_1 + y_2 &= 70, \\ -2y_1 + 3y_2 + y_3 &= 17, \end{aligned}$$

by forward elimination to get $y_1 = 11$, $y_2 = 15$, and $y_3 = -6$. Then we solve the upper triangular system $U\mathbf{x} = \mathbf{y}$,

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 11, \\ 3x_2 + 7x_3 &= 15, \\ -2x_3 &= -6, \end{aligned}$$

by back-substitution to get $x_3 = 3$, $x_2 = -2$, and $x_1 = 1$. So the solution of the original system is $(1, -2, 3)$.

It is clear from Example 3.4.2 that once we have an LU factorization of A, then it is easy to solve the system $A\mathbf{x} = \mathbf{b}$. This is particularly useful when we have to solve *several* systems with the same coefficient matrix A.

Finding an LU factorization is essentially equivalent to Gauss elimination. Recall from Section 3.3 that any matrix A can be factored as

$$A = E_1^{-1} \cdots E_k^{-1} U,$$

where U is an echelon form of A, and $E_1, \dots, E_k$ are the elementary matrices corresponding to the elementary row operations used to reduce A to U.

Any matrix A can be reduced to *some* echelon form *without any scaling operations* by using only interchanges and eliminations. If A can be reduced by using only eliminations, which is not always possible, then the matrices $E_1, \dots, E_k$ are all *lower triangular* and have 1s on the main diagonal. Such square matrices are called *unit* lower triangular. It is easy to prove that $E_1^{-1} \cdots E_k^{-1}$ is also unit lower triangular. L now is given by

$$L = E_1^{-1} \cdots E_k^{-1}.$$

For example, the reduction

$$\begin{bmatrix} 4 & -2 & 1 \\ 20 & -7 & 12 \\ -8 & 13 & 17 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 7 \\ 0 & 9 & 19 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

yields the matrix U of (3.10), and the elementary row operations correspond to elementary matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}.$$

Hence

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}.$$

We compute L as a product

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}$$

to get the LU factorization

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 20 & -7 & 12 \\ -8 & 13 & 17 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & 7 \\ 0 & 0 & -2 \end{bmatrix} = LU.$$

A closer look at L shows that there is no need to compute inverses or products. In fact, L can be computed directly from the eliminations. It is unit square lower triangular. The entry $(1, 2)$ is 5, and it can be obtained from the operation $R_2 - 5R_1 \to R_2$ used to get a zero at $(1, 2)$. In L, we used 5 instead of -5, because we had to invert E_1. Likewise, -2 is obtained from the operation $R_3 + 2R_1 \to R_3$, and 3 from $R_3 - 3R_1 \to R_3$ in the second stage of the reduction.

Example 3.4.3. Find an LU factorization of

$$A = \begin{bmatrix} 2 & 3 & -1 & 4 & 1 \\ -6 & -6 & 5 & -11 & -4 \\ 4 & 18 & 6 & 14 & -1 \\ -2 & -9 & -3 & 4 & 9 \end{bmatrix}.$$

Solution. L is of size 4×4, and we have

$$A \sim \begin{bmatrix} 2 & 3 & -1 & 4 & 1 \\ 0 & 3 & 2 & 1 & -1 \\ 0 & 12 & 8 & 6 & -3 \\ 0 & -6 & -4 & 8 & 10 \end{bmatrix}, \quad \text{so } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & ? & 1 & 0 \\ -1 & ? & ? & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -1 & 4 & 1 \\ 0 & 3 & 2 & 1 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 10 & 8 \end{bmatrix}, \quad \text{so } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ -1 & -2 & ? & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -1 & 4 & 1 \\ 0 & 3 & 2 & 1 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} = U, \quad \text{so } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ -1 & -2 & 5 & 1 \end{bmatrix}.$$

Example 3.4.4. Find an LU factorization of

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -6 & -6 & 5 \\ 4 & 18 & 6 \\ -2 & -9 & -3 \end{bmatrix}.$$

Solution. L is of size 4×4, and

$$A \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 3 & 2 \\ 0 & 12 & 8 \\ 0 & -6 & -4 \end{bmatrix}, \quad \text{so } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & ? & 1 & 0 \\ -1 & ? & ? & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U\,, \quad \text{so } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ -1 & -2 & ? & 1 \end{bmatrix}.$$

In this case, there is no more elimination left, but because the $(3, 4)$ entry of L corresponds to the operation $R_4 - 0R_3 \to R_4$, we have

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}.$$

Our analysis so far yields the following theorem.

Theorem 3.4.5 (LU factorization). *Let A be an $m\times n$ matrix that can be reduced to the $m\times n$ echelon form U by using only eliminations. Then A has an LU factorization. In particular, A can be factored as*

$$A = LU,$$

where L is $m\times m$ lower triangular with only 1*s on the main diagonal (Figure* 3.9*). The (i,j) entry l_{ij} $(i > j)$ of L comes from the operation $R_i - l_{ij}R_j \to R_i$ used to get* 0 *at this position during the elimination process.*

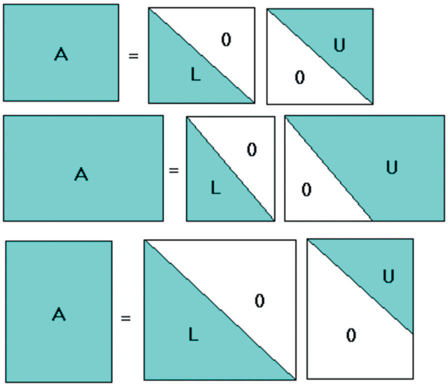

Figure 3.9: LU factorization.

1. The entries of *L* below the main diagonal are sometimes called *Gauss multipliers*.
2. If *A* is square, then the particular LU factorization we used is called *Doolittle*. There is another standard version, where the *upper* triangular matrix *U* has 1s on its main diagonal, called a *Crout factorization*.
3. Polish mathematician *Banachiewicz* in 1938 discovered the LU factorization of square matrices before *Crout* (1941) and also rediscovered factorization of the symmetric matrices after *Cholesky* (1924).[1]
4. Computer programs that find LU factorizations use *overwriting*. They compute *L* and *U* simultaneously and overwrite the original matrix so that the part of *A* below the diagonal becomes *L* and on and above the diagonal becomes *U*. Overwriting for large matrices is very important, because it saves memory storage. Additional saving is achieved by not explicitly storing the 1s on the main diagonal. Here is an example of LU reduction and gradual overwriting of the original matrix. The boxed numbers are the entries of *L* below the diagonal. The rest of the entries are those of *U* on and above the diagonal.

$$\begin{bmatrix}4&-2&1\\20&-7&12\\-8&13&17\end{bmatrix}\to\begin{bmatrix}4&-2&1\\\boxed{5}&3&7\\\boxed{-2}&9&19\end{bmatrix}\to\begin{bmatrix}4&-2&1\\\boxed{5}&3&7\\\boxed{-2}&\boxed{3}&-2\end{bmatrix}.$$

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[1] A. Schwarzenberg-Czerny, “On matrix factorization and efficient least squares solution”. Astronomy and Astrophysics Supplement Series, 1995, vol. 110, pp. 405–410. See [26].

3.4.1 Computational efficiency with LU

If A is a square matrix, say $n \times n$, then it can be shown that to solve the linear system $A\mathbf{x} = \mathbf{b}$ by using LU factorization, it takes approximately $2n^3/3$ operations for large n. This is exactly the number of operations in Gauss elimination. Now $2n^2$ of these operations are performed during the forward and backward elimination. To get an idea of how useful LU factorization can be, suppose we need to solve two systems with 500 equations and 500 unknowns and the same coefficient matrix A. If we use Gauss elimination, then it would take $2n^3/3 = 2 \cdot 500^3/3$ operations per system, a total of about 166 million operations. However, if we used an LU factorization of A to solve the first system ($2 \cdot 500^3/3$ operations), the second system would only require forward and backward elimination, an additional $2n^2 = 2 \cdot 500^2$ operations. This is a total of only about 83 million operations.

3.4.2 LU with interchanges

We have proved the existence of LU factorization in the case where the matrix A can be reduced to some echelon form by using only eliminations. Now we discuss the cases where interchanges are necessary. Recall that the interchange of two rows of a matrix A can be expressed as P_iA, where P_i is the elementary matrix that corresponds to the interchange. Such matrix P_i is called an *elementary permutation matrix*, and it was introduced in Exercises 3.3. Its effect is the permutation of two rows of I. If during a row reduction of A, we first perform all the interchanges $P_1, \dots, P_k$, then the matrix $P_k \cdots P_1A$ can be row reduced by using eliminations only. So it has an LU factorization. The matrix $P = P_k \cdots P_1$, which is a product of elementary permutation matrices, is called a *permutation matrix*. We compute P and then find an LU factorization for PA:

$$PA = LU.$$

To illustrate, let us consider the following Gauss elimination, which requires two interchanges:

$$A = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 1 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix}.$$

The elementary permutation matrices of the interchanges are

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

First, we compute the permutation matrix P:

$$P = P_2P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then we compute PA:

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Next, we find the LU decomposition of PA:

$$PA = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{bmatrix} = LU.$$

Even though A itself has no LU factorization, the LU factorization of PA is *just as useful.*

Example 3.4.6. Use $PA = LU$ above to solve the system $A\mathbf{x} = \mathbf{b}$,

$$\begin{bmatrix} 0 & 0 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 12 \\ 14 \\ 12 \end{bmatrix}.$$

Solution. First, we multiply $A\mathbf{x} = \mathbf{b}$ on the left by P to get the system $PA\mathbf{x} = P\mathbf{b}$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 14 \\ 12 \\ 12 \end{bmatrix}.$$

Now we use the LU factorization of PA to solve this system. The lower triangular system $L\mathbf{y} = P\mathbf{b}$ yields $y_1 = 14$, $y_2 = -2$, and $y_3 = 12$. The upper triangular system $U\mathbf{x} = \mathbf{y}$ gives us $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. This is the solution of the original system.

Exercises 3.4

In Exercises 1–3, find the solution of the system $A\mathbf{x} = \mathbf{b}$, where A is already factored as LU. There is no need to compute A explicitly.

1. $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 0 & -1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} -11 \\ 32 \end{bmatrix}.$

2. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}.$

3. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 \\ 0 & 5 & -1 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 6 \\ 22 \\ -13 \end{bmatrix}$.

In Exercises 4–8, find an LU factorization of the matrix.

4. $\begin{bmatrix} 4 & 1 \\ 12 & 2 \end{bmatrix}$.

5. $\begin{bmatrix} 2 & -2 & 1 \\ -8 & 11 & -5 \\ 4 & -13 & 3 \end{bmatrix}$.

6. $\begin{bmatrix} -1 & 2 & 1 \\ 4 & -5 & -5 \\ -7 & 5 & 5 \end{bmatrix}$.

7. $\begin{bmatrix} 4 & 1 & 1 & 2 & 1 \\ -12 & -1 & -4 & -4 & -1 \\ 4 & -3 & 3 & 0 & -4 \end{bmatrix}$.

8. $\begin{bmatrix} 4 & 1 & 1 \\ -12 & -1 & -4 \\ 0 & -4 & 5 \\ 20 & 3 & 6 \end{bmatrix}$.

In Exercises 9–11, find the solution of the system $A\mathbf{x} = \mathbf{b}$ by using an LU factorization of the coefficient matrix A.

9. $\begin{bmatrix} 5 & 1 \\ -10 & -3 \end{bmatrix}\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

10. $\begin{bmatrix} 2 & 1 \\ 14 & 2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$.

11. $\begin{bmatrix} 2 & 1 & 1 \\ 12 & 11 & 5 \\ -2 & 9 & 0 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 17 \\ 18 \end{bmatrix}$.

In Exercises 12–14, find a permutation matrix P and an LU factorization of PA.

12. $A = \begin{bmatrix} 0 & 3 \\ -5 & 4 \end{bmatrix}$.

13. $A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 2 & -4 \\ 2 & -5 & 1 \end{bmatrix}$.

14. $A = \begin{bmatrix} 0 & 0 & 2 \\ -1 & 5 & -2 \\ 3 & 6 & 7 \end{bmatrix}$.

In Exercises 15–17, solve the system $A\mathbf{x} = \mathbf{b}$ by using a $PA = LU$ factorization.

15. $\begin{bmatrix} 0 & 3 & -1 \\ 2 & 0 & 1 \\ 2 & -6 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -3 \\ -1 \\ -1 \end{bmatrix}.$

16. $\begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 1 \\ 2 & -6 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -10 \end{bmatrix}.$

17. $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & -4 \\ 2 & -5 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ -8 \end{bmatrix}.$

18. Prove that the product of two lower triangular matrices is lower triangular.

19. Prove that the product of two unit lower triangular matrices is unit lower triangular.

20. Prove that a lower triangular matrix is invertible if and only if all its diagonal entries are nonzero.

21. Prove that the inverse of an invertible lower triangular matrix is also lower triangular.

22. Prove that the inverse of a unit lower triangular matrix is also unit lower triangular.

23. (*Uniqueness of the LU factorization*) Suppose A is invertible with two LU factorizations LU and $L'U'$, where L and L' are *unit* lower triangular. Prove that $L = L'$ and $U = U'$.

Pascal matrices

Pascal's triangles, named after Blaise Pascal (Figure 3.10), are formed by two sides of 1s, and then each number is the sum of the numbers immediately above it.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

These triangles give rise to Pascal's $n \times n$ matrices P_n by truncation. For example,

$$P_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

Pascal's matrices have some very interesting properties.[2]

24. Find an LU factorization for P_3.

25. Prove the following LU factorization for P_4:

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[2] See *Pascal Matrices* by Alan Edelman and Gilbert Strang in the American Mathematical Monthly, Volume 111, Number 3, March 2004. See [27].

$$P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

26. Guess a LU factorization for P_n.

Figure 3.10: Blaise Pascal.
By Unknown artist, Public Domain,
https://commons.wikimedia.org/w/index.php?curid=194279.
Blaise Pascal (1623–1662) was a mathematician, physicist, and philosopher. He was a child prodigy. He initiated the theory of probability, and he formulated Pascal's law of pressure. At young age, he quit mathematics and devoted himself to religious philosophy, an area where he is widely known today.

3.5 Block and sparse matrices

In this section, we look at some special types of matrices that appear often in practice and whose properties may result in substantial reductions in computation and memory requirements. The types of matrices we discuss are block matrices and sparse matrices.

3.5.1 Block matrices

Often in applications, one encounters very large matrices that are hard to operate with. One way to get around this is to partition the matrices into submatrices and manipulate these. A *submatrix* is a matrix obtained by deleting rows and/or columns from the original matrix.

We partition a large matrix into blocks of submatrices. The resulted matrix is called a *block matrix*, or a *partitioned matrix*.

For example, let us partition the following 3×6 matrix A as follows:

$$A = \left[\begin{array}{ccccccccc} 1 & 2 & 0 & \vdots & 1 & -1 & \vdots & 1 \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots & \vdots & \cdots \\ 1 & 3 & 5 & \vdots & 0 & 1 & \vdots & 2 \\ 2 & 4 & 6 & \vdots & 1 & 0 & \vdots & 0 \end{array}\right].$$

We may view A as the 2×3 block matrix

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix},$$

where the blocks A_{ij} are

$$A_{11} = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \quad A_{12} = [1],$$
$$A_{21} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

We can perform matrix operations on block matrices by operating on the blocks provided that the sizes of the blocks and partitions are compatible.

3.5.2 Addition of block matrices

We may add two block matrices just as we would add regular matrices except that we operate on blocks instead of entries. For example, let

$$B = \begin{bmatrix} 1 & 2 & \vdots & 8 \\ 3 & 4 & \vdots & -1 \\ \cdots & \cdots & \vdots & \cdots \\ 1 & 1 & \vdots & 7 \\ 5 & 6 & \vdots & 9 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & \vdots & 3 \\ -3 & 4 & \vdots & -1 \\ \cdots & \cdots & \vdots & \cdots \\ 8 & 5 & \vdots & 2 \\ 1 & -2 & \vdots & 4 \end{bmatrix}.$$

If the blocks are denoted by

$$B = \begin{bmatrix} B_{11} & \vdots & B_{12} \\ \cdots & \vdots & \cdots \\ B_{21} & \vdots & B_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & \vdots & C_{12} \\ \cdots & \vdots & \cdots \\ C_{21} & \vdots & C_{22} \end{bmatrix},$$

then we may compute $B + C$ by adding the corresponding blocks:

$$\begin{bmatrix} B_{11} + C_{11} & \vdots & B_{12} + C_{12} \\ \cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ B_{21} + C_{21} & \vdots & B_{22} + C_{22} \end{bmatrix} = \begin{bmatrix} 1 & 4 & \vdots & 11 \\ 0 & 8 & \vdots & -2 \\ \cdots & \cdots & \vdots & \cdots \\ 9 & 6 & \vdots & 9 \\ 6 & 4 & \vdots & 13 \end{bmatrix} = B + C.$$

3.5.3 Multiplication of block matrices

We may also perform block matrix multiplication, provided that the sizes of the blocks are compatible. For example, if

$$C=\begin{bmatrix} 1 & 3 & -4 & \vdots & 0 & 0\\ 1 & -1 & 0 & \vdots & 0 & 0\\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots\\ 0 & 2 & 4 & \vdots & 1 & 0\\ 3 & 5 & 7 & \vdots & 0 & 1 \end{bmatrix}=\begin{bmatrix} C_{11} & \vdots & C_{12}\\ \cdots & \vdots & \cdots\\ C_{21} & \vdots & C_{22} \end{bmatrix}$$

and

$$D=\begin{bmatrix} -1 & 0\\ 3 & 1\\ 2 & 5\\ \cdots & \cdots\\ -4 & 0\\ 0 & 2 \end{bmatrix}=\begin{bmatrix} D_1\\ \cdots\\ D_2 \end{bmatrix},$$

then it is easy to check that

$$\begin{bmatrix} C_{11}D_1+C_{12}D_2\\ \cdots\cdots\cdots\cdots\cdots\\ C_{21}D_1+C_{22}D_2 \end{bmatrix}=\begin{bmatrix} 0 & -17\\ -4 & -1\\ \cdots & \cdots\\ 10 & 22\\ 26 & 42 \end{bmatrix}=CD.$$

For compatible block sizes of block matrices, we have analogous properties to matrix multiplication such as

$$\begin{bmatrix} A & \vdots & B \end{bmatrix}\begin{bmatrix} C\\ \cdots\\ D \end{bmatrix}=AC+BD,\qquad \begin{bmatrix} A\\ \cdots\\ B \end{bmatrix}C=\begin{bmatrix} AC\\ \cdots\\ BC \end{bmatrix},$$

$$\begin{bmatrix} A & \vdots & B\\ \cdots & \cdots & \cdots\\ C & \vdots & D \end{bmatrix}\begin{bmatrix} E & \vdots & F\\ \cdots & \cdots & \cdots\\ G & \vdots & H \end{bmatrix}=\begin{bmatrix} AE+BG & \vdots & AF+BH\\ \cdots & \cdots & \cdots\\ CE+DG & \vdots & CF+DH \end{bmatrix}.$$

3.5.4 Inversion of block matrices

Sometimes, block matrix multiplication is used to quickly invert special kinds of matrices. For example, suppose we need B^{-1}, where B is the block matrix

$$B = \begin{bmatrix} 3 & -4 & \vdots & 0 & 0 \\ -1 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \vdots & \cdots & \cdots \\ 2 & 4 & \vdots & 1 & 0 \\ 5 & -1 & \vdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} B_{11} & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ B_{21} & \vdots & I_2 \end{bmatrix}.$$

If B^{-1} has 2×2 blocks, $B^{-1} = \begin{bmatrix} C_{11} & \vdots & C_{12} \\ \cdots & \vdots & \cdots \\ C_{21} & \vdots & C_{22} \end{bmatrix}$, then

$$I_4 = BB^{-1} = \begin{bmatrix} B_{11}C_{11} & \vdots & B_{11}C_{12} \\ \cdots\cdots & \vdots & \cdots\cdots \\ B_{21}C_{11} + C_{21} & \vdots & B_{21}C_{12} + C_{22} \end{bmatrix} = \begin{bmatrix} I_2 & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & I_2 \end{bmatrix}.$$

This implies the following matrix equations:

$$B_{11}C_{11} = I_2, \quad B_{11}C_{12} = \mathbf{0}, \quad B_{21}C_{11} + C_{21} = \mathbf{0}, \quad B_{21}C_{12} + C_{22} = I_2.$$

Therefore

$$C_{11} = B_{11}^{-1} = \begin{bmatrix} 3 & -4 \\ -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1 \\ -\frac{1}{4} & -\frac{3}{4} \end{bmatrix}.$$

Hence

$$B_{11}C_{12} = \mathbf{0} \Rightarrow B_{11}^{-1}B_{11}C_{12} = \mathbf{0} \Rightarrow C_{12} = \mathbf{0}.$$

Thus

$$B_{21}C_{12} + C_{22} = I_2 \Rightarrow C_{22} = I_2$$

and

$$B_{21}C_{11} + C_{21} = \mathbf{0} \Rightarrow C_{21} = -B_{21}C_{11}$$
$$\Rightarrow C_{21} = -\begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -\frac{1}{4} & -\frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ -\frac{1}{4} & \frac{17}{4} \end{bmatrix}.$$

So we conclude that

$$B^{-1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -\frac{1}{4} & -\frac{3}{4} & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -\frac{1}{4} & \frac{17}{4} & 0 & 1 \end{bmatrix}.$$

3.5.5 Sparse matrices

A *sparse matrix* is a matrix in which a substantial proportion of its elements are zero. In contrast, a *dense matrix* has very few zero elements. These do not constitute precise mathematical definitions, but the concepts are useful nevertheless.

There are different types of sparsity patterns in sparse matrices, such as *rowwise sparsity* with many zero rows, *columnwise sparsity* with many zero columns, and, for *square* matrices, *band sparsity* with nonzero elements primarily along diagonals. Matrices with band sparsity are called *band matrices*.

We have the following examples, where each asterisk stands for any number:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \end{bmatrix}, \quad \begin{bmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & 0 & 0 \\ 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

Sparse matrices are space-efficient because they do not store zero values, saving memory and storage space. Also, operations involving sparse matrices can be faster because we only need to perform computations on the nonzero elements.

Sparse matrices have applications in various fields: finite element analysis, finite difference methods, and computational fluid dynamics often involve large sparse matrices. Also, in natural language processing, *term-document matrices* are often sparse.

One useful subcategory of band matrices is the *tridiagonal matrices*. These are matrices that have nonzero elements only on the main diagonal, the *subdiagonal*, i. e., the first diagonal below the main diagonal, and the *superdiagonal* i. e., the first diagonal above the main diagonal. For example,

$$M_1 = \begin{bmatrix} 9 & 1 & 0 & 0 & 0 \\ 1 & 8 & 2 & 0 & 0 \\ 0 & 2 & 7 & 3 & 0 \\ 0 & 0 & 3 & 6 & 4 \\ 0 & 0 & 0 & 4 & 5 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

Exercises 3.5

1. Find all 2 × 2 submatrices of

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

2. If

$$C = \left[\begin{array}{cccccc} 7 & 3 & -4 & \vdots & 0 & 0 \\ 1 & 1 & 2 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & \vdots & \cdots & \cdots \\ 0 & 2 & 3 & \vdots & 1 & 0 \\ 3 & 1 & -8 & \vdots & 0 & 1 \end{array}\right] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 5 & -5 \\ \cdots & \cdots \\ -3 & 0 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix},$$

then check that

$$CD = \begin{bmatrix} C_{11}D_1 + C_{12}D_2 \\ C_{21}D_1 + C_{22}D_2 \end{bmatrix}.$$

In Exercises 3–5, verify each identity by using

$$A = \begin{bmatrix} 0 & 5 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 \\ 1 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} -7 & 2 \end{bmatrix}.$$

3. $C\left[\begin{array}{ccc} A & \vdots & B \end{array}\right] = \left[\begin{array}{ccc} CA & \vdots & CB \end{array}\right].$

4. $\begin{bmatrix} A \\ \cdots \\ B \end{bmatrix} A = \begin{bmatrix} A^2 \\ \cdots \\ BA \end{bmatrix}$.

5. $\begin{bmatrix} A & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & B \end{bmatrix}^2 = \begin{bmatrix} A^2 & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & B^2 \end{bmatrix}$.

6. Use block matrices to find an inverse for

$$A = \begin{bmatrix} 0 & -4 & \vdots & 0 & 0 \\ -1 & -3 & \vdots & 0 & 0 \\ \cdots & \cdots & \vdots & \cdots & \cdots \\ 4 & 20 & \vdots & 4 & 0 \\ -1 & 17 & \vdots & 0 & 4 \end{bmatrix} = \begin{bmatrix} A_{11} & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ A_{21} & \vdots & 4I_2 \end{bmatrix}.$$

7. Let A and B be tridiagonal of the same size, and let c be any scalar.

(a) Prove that the sum $A + B$ is tridiagonal.

(b) Prove that the scalar product cA is tridiagonal.

(c) Is the product AB tridiagonal for all choices of A and B? Explain.

8. Let A and B have size $m \times n$, and let C and D have size $n \times k$.

(a) Prove that the product

$$\begin{bmatrix} A & \vdots & B \end{bmatrix} \begin{bmatrix} C \\ \cdots \\ D \end{bmatrix}$$

is defined. Find its size.

(b) Prove the following formula:

$$\begin{bmatrix} A & \vdots & B \end{bmatrix} \begin{bmatrix} C \\ \cdots \\ D \end{bmatrix} = [AC + BD] .$$

9. Let M_n be the $n \times n$ sparse matrix with 1 along each diagonal entry and a on each superdiagonal entry. Find M_4^{-1}.

$$M_4 = \begin{bmatrix} 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Guess a formula for M_n^{-1}.

3.6 Applications: Leontief models, Markov chains

In this section, we examine some new applications of matrices. More specifically, we study stochastic matrices, the Leontief input–output model in economics, transition probability matrices and Markov chains, and some applications to graph theory.

3.6.1 Stochastic matrices

Stochastic matrices are special square matrices that are useful in probability, statistics, economics, genetics, manufacturing, and several other areas.

Definition 3.6.1. A *stochastic matrix* is a square matrix with real nonnegative entries for which all entries of each column add up to 1. A stochastic matrix is *doubly stochastic* if, in addition, all entries of each row add up to 1.

Note that the entries of a stochastic matrix are numbers between 0 and 1.

Example 3.6.2. The following matrices are stochastic. Moreover, the matrices C and D are doubly stochastic.

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} \\ \frac{1}{4} & \frac{5}{6} \end{bmatrix}, \quad C = \begin{bmatrix} 0.25 & 0.75 \\ 0.75 & 0.25 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix}.$$

Theorem 3.6.3 (Properties of stochastic matrices). *Let A and B be two $n \times n$ stochastic matrices, and let k be a positive integer. Then*

1. *AB is stochastic;*
2. *A^k is stochastic.*

   If in addition, A and B are doubly stochastic, then
3. *AB is doubly stochastic;*
4. *A^k is doubly stochastic;*
5. *A^T is doubly stochastic.*

Proof. Exercise. □

3.6.2 Economics: Leontief input–output models

We now study two economic models introduced by the Harvard economist and Nobel laureate Wassily W. Leontief in the 1930s (Figure 3.11). These models are the Leontief closed model and the Leontief open model. They were introduced to study the U. S. economy. Now they are used to analyze the economy of any country, or even an entire geographic region.

Figure 3.11: Wassily Leontief, 1973 Nobel Prize in Economics. By Keystone [1], Public Domain, https://commons.wikimedia.org/w/index.php?curid=62223326. Wassily Leontief (1905–1999) was born in St. Petersburg, Russia. He studied in Leningrad and Berlin. In 1932, he joined the economics faculty at Harvard University. There he developed his theory of *input–output analysis* and applied it to study the productivity of the U. S. economy. Later, he became the director of the Institute for Economic Analysis of New York University.

The Leontief closed model

Let us consider an economy consisting of n industries each producing only one commodity needed by the others and possibly by itself. For example, suppose we have Coal, Steel and Auto which are interrelated in a way described by a 3×3 matrix as follows. Let c_{ij} be the dollar amount of the ith commodity needed to produce 1 dollar's worth of the jth commodity.

Suppose it takes 0.30 dollars of coal to produce 1 dollar's worth of steel. The value 0.30 is the $(1, 2)$ entry of the following matrix:

	Coal	**Steel**	**Auto**
Coal	0.10	0.30	0.25
Steel	0.25	0.20	0.45
Auto	0.05	0.15	0.10

According to this matrix, it also takes 0.45 dollars of steel to produce 1 dollar's worth of automobile. Note that Auto is the largest consumer of Steel and Steel is the largest consumer of Coal. Steel is most dependent on Auto to survive.

The above matrix is an example of an *input–output* matrix, or a *consumption* matrix. The (i, j) entry of a consumption matrix is the input of the ith industry needed by the jth industry to produce one unit of output. Consumption matrices describe the interdependency of the economic sectors.

The entries of a consumption matrix are numbers between 0 and 1. In addition, the sum of the entries of each column should be no more than 1 if each sector is to meet the demand of all sectors.

Suppose n producing economic sectors are interrelated in a way described by a consumption matrix $C = [c_{ij}]$. Let x_i be the total amount of output needed to be produced by the ith sector to satisfy the demands of all sectors. Then $c_{ij}x_j$ is the amount needed by commodity i to produce x_j units of commodity j. Because the total output of sector i equals the sum of the demands of all sectors, we have

$$\begin{aligned} x_1 &= c_{11}x_1 + \cdots + c_{1n}x_n, \\ &\vdots \qquad\qquad \vdots \\ x_n &= c_{n1}x_1 + \cdots + c_{nn}x_n. \end{aligned}$$

If $\mathbf{x} = (x_1, \ldots, x_n)$ is the *output vector*, then these relations can be expressed as the matrix equation

$$\mathbf{x} = C\mathbf{x}.$$

This is equivalent to the homogeneous system

$$(I - C)\,\mathbf{x} = \mathbf{0}. \tag{3.11}$$

If the coefficient matrix $I - C$ is invertible, then the system has only the trivial solution by Theorem 3.3.10. We are interested in nontrivial solutions, so $I - C$ has to be at least noninvertible. Actually, economists are interested in nontrivial solutions with all components being positive. Such a solution $\mathbf{x}$ is called a *positive solution*, and we write

$$\mathbf{x} \geq \mathbf{0}.$$

Let us now study the special case where the consumption matrix C is stochastic. So all entries are between 0 and 1, and all column sums are 1. In this case, we say that C is an *exchange matrix*. The case of C being an exchange matrix is economically significant. It indicates *economic equilibrium* among the producing sectors. These sectors produce exactly the necessary amounts to meet all demands from all producing sectors. The following theorem indicates why exchange matrices are important in economics.

Theorem 3.6.4. *Let C be an exchange matrix. Then*
1. *$I - C$ is noninvertible;*
2. *System* (3.11) *has a nontrivial solution;*
3. *System* (3.11) *has a positive solution.*

Proof of 1 and 2.
1. The column sums of $I - C$ are all 0, because the column sums of C are all 1. Therefore, if we add all the rows $\mathbf{r}_i$ of $I - C$, then we get $\mathbf{0}$:

   $$\mathbf{r}_1 + \cdots + \mathbf{r}_n = \mathbf{0}.$$

   This is a linear dependence relation, so the rows of $I - C$ are linearly dependent. Therefore $I - C$ cannot be invertible by Theorem 3.3.10.
2. This part follows from Part 1 by Theorem 3.3.10. ☐

Example 3.6.5. Find an equilibrium output $\mathbf{x}$ for Coal, Steel, and Auto if their exchange matrix C is given by the table

	Coal	**Steel**	**Auto**
Coal	0.2	0.4	0.5
Steel	0.4	0.4	0.1
Auto	0.4	0.2	0.4

Solution. We solve the system $(I - C)\mathbf{x} = \mathbf{0}$ by reducing $[I - C : 0]$:

$$\begin{bmatrix} 0.8 & -0.4 & -0.5 & 0 \\ -0.4 & 0.6 & -0.1 & 0 \\ -0.4 & -0.2 & 0.6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1.0625 & 0 \\ 0 & 1 & -0.875 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so we get the general solution $\mathbf{x} = [\,1.0625r \;\; 0.875r \;\; r\,]^T$ for $r \in \mathbf{R}$. For example, for $r = 10^4$ and output measured in tons, it takes 10625 tons of Coal, 8750 tons of Steel, and 10000 tons of Auto to meet exactly all demands.

Note that a positive solution vector $\mathbf{x}$ in System (3.11) for an exchange matrix C may be scaled to represent the *price* charged by each industry for its total output. In this case, $\mathbf{x}$ is called a *price vector*.

We have just studied the *Leontief closed model*. In the closed model, we consider demand for commodities from only producing economic sectors.

The Leontief open model

Often, there is also demand from nonproducing sectors such as consumers, government, etc. For instance, the government may demand coal, steel, and automobiles in our example. All nonproducing sectors form the *open sector*. Suppose that the demand of the open sector from the ith producing sector is d_i units. Then

$$x_i = c_{i1}x_1 + \cdots + c_{in}x_n + d_i.$$

If $\mathbf{d} = (d_1, \ldots, d_n)$, then

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}.$$

This matrix equation that takes into account the open sector is a *Leontief open model*. Again, $\mathbf{x}$ is called an *output vector*, and vector $\mathbf{d}$ is the *demand vector*. Economists are usually interested in computing the output vector $\mathbf{x}$ given the demand vector $\mathbf{d}$. This can be done by solving for $\mathbf{x}$:

$$\mathbf{x} = (I - C)^{-1}\mathbf{d},$$

provided that the matrix $I - C$ is invertible. If, in addition, $(I - C)^{-1}$ has nonnegative entries, then the entries of $\mathbf{x}$ are nonnegative, so they are acceptable as output values. In general, a matrix C is called *productive* if $(I - C)^{-1}$ exists and has nonnegative entries.

Example 3.6.6. Let C be the consumption matrix, and let $\mathbf{d}$ be the demand vector, in millions of dollars, for an open sector economy with three interdependent industries. Compute the output demanded by the industries and the open sector when

$$C = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}.$$

Solution. We have

$$I - C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{3}{4} & 0 \\ 0 & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}.$$

Therefore

$$\mathbf{x} = (I - C)^{-1}\,\mathbf{d} = \begin{bmatrix} \frac{9}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 55 \\ 45 \\ 70 \end{bmatrix}.$$

We conclude that to satisfy all demands, the output levels of the three industries should be 55, 45, and 70 million dollars.

Often analysts know the levels of production $\mathbf{x}$ and need the demand vector $\mathbf{d}$ placed on the producing sectors. Then

$$\mathbf{d} = \mathbf{x} - C\mathbf{x}.$$

The producing sectors are usually some key industries, such as agricultural goods, steel, chemicals, coal, livestock, etc. For the US national input–output matrix, the open sectors are the federal, state, and local governments.

3.6.3 Probability matrices and Markov processes

If an event is certain to occur, then we say that its probability to occur is 1. If it will not occur, then its probability to occur is 0. Other values of probabilities are numbers between 0 and 1. The larger the probability of occurrence of an event, the more likely the event will occur. If an event has n equally likely to occur outcomes, then the *probability* that one of m of these outcomes occurs is m/n. If we roll a die, then there 6 possible outcomes. The probability to get a 2 is 1 out of 6, or 1/6.

Because the elements of a stochastic matrix are numbers between 0 and 1, they can be viewed as probabilities of outcomes of events. In such case, we talk about a *transition matrix of probabilities*. Its entries are numbers p_{ij}, called *transition probabilities*. They express the probability that if a system is in state j currently, then it will be in state i at the next observation.

Let us look at the following study of the smoking habits of a group of people. Suppose that the probability of a smoker to continue smoking a year later is 65 %. So there is a 35 % probability of quitting. Also suppose that the probability of a nonsmoker to continue nonsmoking is 85 %. Thus there is a 15 % probability of switching to smoking. This information can be tabulated by the stochastic transition probabilities matrix defined by the table

		initial **smoker**	**state** **nonsmoker**
final	smoker	0.65	0.15
state	nonsmoker	0.35	0.85

Example 3.6.7. Suppose that when the study started, 70 % of the group members were smokers and 30 % nonsmokers. What are the percentages of smokers and nonsmokers after (a) one year? (b) four years?

Solution. (a) After one year, the percentage of smokers consists of those who were initially smokers $70\,\% \cdot 65\,\% = 0.455$ plus those who picked up smoking during the year $30\,\% \cdot 15\,\% = 0.045$; this is a total of $0.5 = 50\,\%$. Likewise, the percentage of nonsmokers is $0.7 \cdot 0.35 + 0.3 \cdot 0.85 = 0.5$ or 50 %. Both numbers can be computed by the matrix product

$$\begin{bmatrix} 0.65 & 0.15 \\ 0.35 & 0.85 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

(b) For four years, we multiply the initial vector $(0.7, 0.3)$ by the probability matrix four times to get

$$\begin{bmatrix} 0.65 & 0.15 \\ 0.35 & 0.85 \end{bmatrix}^4 \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.325 \\ 0.675 \end{bmatrix}.$$

So in four years, there are 32.5 % smokers and 67.5 % nonsmokers. In general, after k years, the percentages can be computed as the initial vector $[\,0.7\;0.3\,]^T$ times the kth power of the transition matrix of probabilities,

$$\begin{bmatrix} 0.65 & 0.15 \\ 0.35 & 0.85 \end{bmatrix}^k \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}.$$

This is an example of a *discrete linear dynamical system*. If we use large values for k, we see that the product approaches the vector $[\,0.3\ 0.7\,]^T$. It seems that in the long run the smokers will be about 30 % versus 70 % of nonsmokers.

The process described above is an example of a *Markov process*, or *Markov chain*, named after Andrei Markov (Figure 3.12). In a Markov process the next state of a system depends only on its current state. In our case the percentages of smokers and nonsmokers depend only on the percentages of the previous year.

Figure 3.12: Andrei Andreevitch Markov.
By Unknown author, Public Domain,
https://commons.wikimedia.org/w/index.php?curid=448609.
Andrei Andreevitch Markov (1856–1922) was born in Ryazan, Russia, and died in St. Petersburg, Russia. He was a student of Chebyshev. He did important research in mathematical analysis and became a professor of mathematics at St. Petersburg University. He is best known for his contributions to probability theory, particularly to the study of the processes, now known as Markov chains.

Exercises 3.6

Stochastic matrices

In Exercises 1–4, test the matrices for stochastic and doubly stochastic.

1. $A = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & 1 \end{bmatrix}$, $B = \begin{bmatrix} -0.1 & 1.1 \\ 1.1 & -0.1 \end{bmatrix}$.

2. $C = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & 1 \end{bmatrix}$, $D = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{3} & 1 \\ \frac{1}{6} & 0 \end{bmatrix}$.

3. $E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $F = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}$.

4. $G = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$, $H = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$.

5. Is $\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}^{185}$ stochastic?

6. Is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{277}$ doubly stochastic?

7. Find x, y, if possible, such that the matrices below are doubly stochastic.

$$A = \begin{bmatrix} x & 0.2 \\ 0.2 & y \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{4} & x \\ \frac{3}{4} & y \end{bmatrix}.$$

8. Find x, y, z, if possible, such that the matrix

$$A = \begin{bmatrix} x & 0 & 1 \\ \frac{1}{4} & y & 0 \\ \frac{1}{4} & \frac{1}{4} & z \end{bmatrix}$$

is stochastic.

9. *True or False?* Explain.

(a) The sum of two $n \times n$ stochastic matrices is stochastic.

(b) If A is stochastic, then $2A$ is also stochastic.

10. Let A be an $n \times n$ stochastic matrix, and let $\mathbf{x}$ be an n-vector. Prove that the sum of the components of $\mathbf{x}$ equals the sum of the components of $A\mathbf{x}$.

11. Prove Theorem 3.6.3.

Input–output models

12. Can an exchange matrix be productive? Explain.

13. Why is the matrix $C = \begin{bmatrix} 0.5 & 0.4 \\ 0.1 & 0.6 \end{bmatrix}$ productive? Let $\mathbf{d} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$ denote a demand vector. Find the output vector of production.

14. Which of the following consumption matrices are productive? Explain.

$$A = \begin{bmatrix} 0.5 & 0.4 \\ 0.3 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.9 & 0.8 \\ 0.1 & 0.9 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{bmatrix}.$$

15. Find an equilibrium output $\mathbf{x}$ for Electric, Gas, and Coal if their exchange matrix C is given by the table

	Electric	**Gas**	**Coal**
Electric	0.3	0.3	0.3
Gas	0.2	0.4	0.2
Coal	0.5	0.3	0.5

16. Find the output vector in millions of dollars demanded by the economic sectors Farming, Oil, and Steel and by the open sector. The table for the consumption matrix of the industries and the demand vector **d** of the open sector are

	Farming	Oil	Steel
Farming	0.3	0.4	0.3
Oil	0.3	0.3	0.2
Steel	0.3	0.1	0.3

and

$$\mathbf{d} = \begin{bmatrix} 40 & 30 & 10 \end{bmatrix}^T .$$

Markov processes

17. Statistical data for a city and its surrounding suburban area shows the following yearly residential moving trends. For example, the probability that a person living in the city will move to the suburban area in a year is 55 %.

	Initial City	State Suburb
City	45 %	35 %
Suburb	55 %	65 %

If 3 million people live in the city and 0.5 million in the suburbia, what is the distribution likely to be in (a) one year? (b) three years?

18. Voting trends of successive elections are given in the following matrix. For example, the probability that a Democrat will vote Republican next elections is 20 %.

	Initial Dem.	State Rep.	Ind.
Democrat	70 %	25 %	60 %
Republican	20 %	70 %	30 %
Independent	10 %	5 %	10 %

If there are 4 million Democrat voters, 4.5 million Republican voters, and 0.5 million independent voters, what is the distribution likely to be (a) in the next election? (b) in two elections?

3.7 Graph theory

Matrix algebra has important applications to graph theory. Graphs are main tools in operations research, electrical engineering, computer programming and networking, business administration, sociology, economics, marketing, and communications networks. The entire internet can be represented by a very complicated graph.[3]

3.7.1 Graphs

Definition 3.7.1. A *graph* consists of a set of points $P_1, \dots, P_n$, called *vertices*, together with a set of lines, called *edges*, connecting some vertices. The *adjacency matrix* $A(G)$ of a graph G is the matrix whose (i, j) entry is 1 if there is at least one edge connecting the ith to the jth vertex and zero otherwise.

Example 3.7.2. Write an adjacency matrix of the graph G_1 of Figure 3.13.

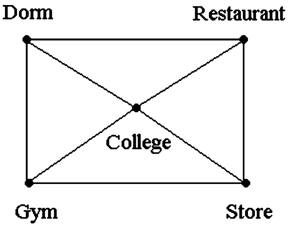

Figure 3.13: Graph G_1.

Solution. We label College, Gym, Store, Restaurant, and Dorm as vertices $1, 2, 3, 4, 5$, respectively. Then we have the following adjacency matrix:

$$A(G_1) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}. \tag{3.12}$$

Definition 3.7.3. Let G be a graph. A *walk of length m* from the ith to the jth vertex in G is a sequence of $m + 1$ vertices that starts at i, ends at j, and all consecutive vertices are connected by an edge.

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[3] For a graph of the internet on 1 April 2003, see the cover of the magazine *Notices of the American Mathematical Society 51*(4), April 2004.

In Figure 3.13 the sequence College, Gym, Store, Restaurant, College defines a walk of length 4 from College back to College.

The following useful theorem from graph theory is stated without proof.

Theorem 3.7.4. *The number of walks of length m from vertex i to vertex j in a graph G is equal to the* (i, j) *entry of* $A(G)^m$.

Example 3.7.5. Consider the graph G_1 of Figure 3.13.
(a) Find a matrix that displays all numbers of walks of length 3.
(b) Determine the number of walks of length 3 from Dorm to Restaurant.

Solution. (a) By Theorem 3.7.4 and equation (3.12) the number of walks of length 3 is given by

$$A\left(G_1\right)^3 = \begin{bmatrix} 8 & 8 & 8 & 8 & 8 \\ 8 & 4 & 8 & 4 & 8 \\ 8 & 8 & 4 & 8 & 4 \\ 8 & 4 & 8 & 4 & 8 \\ 8 & 8 & 4 & 8 & 4 \end{bmatrix}.$$

(b) The number of walks of length 3 from Dorm to Restaurant is the $(5, 4)$ entry of $A(G_1)^3$, which is 8. Can you find all 8 walks?

Directed graphs

Definition 3.7.6. A *directed graph* or *digraph* is a graph whose edges are *directed* line segments. The *adjacency matrix* $A(D)$ of a digraph D is the matrix whose (i, j) entry is 1 if there is at least one directed edge connecting the ith to the jth vertex and zero otherwise.

Example 3.7.7. Write the adjacency matrix of the digraph D_1 of Figure 3.14.

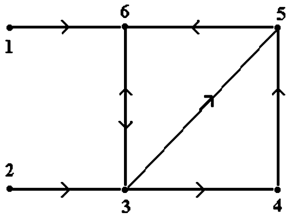

Figure 3.14: Digraph D_1.

Solution. We have

$$A(D_1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Example 3.7.8. Write the adjacency matrix of the digraph D_2 of Figure 3.15.

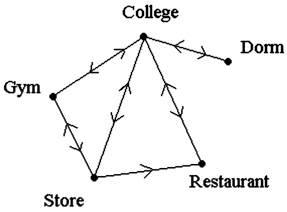

Figure 3.15: Digraph D_2.

Solution. We number College, Gym, Store, Restaurant, and Dorm as vertices $1, 2, 3, 4, 5$, respectively. Then we have the following adjacency matrix:

$$A(D_2) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.13}$$

Definition 3.7.9. For a digraph D, a *walk of length* m from the ith to the jth vertex is a sequence of $m + 1$ vertices that starts at i, ends at j, and all consecutive vertices are connected by an edge.

In Figure 3.14 the sequence $2, 3, 4, 5$ is a walk of length 3 from 2 to 5. In Figure 3.15 the sequence College, Gym, Store, Restaurant, College defines a walk of length 4 from College back to College.

The following useful theorem from graph theory is stated without proof.

Theorem 3.7.10. *The number of walks of length m from vertex i to vertex j in a digraph D is equal to the (i, j) entry of $A(D)^m$.*

Example 3.7.11. Consider the digraph D_2 of Figure 3.15 and its adjacency matrix $A(D_2)$ (3.13). Use Theorem 3.7.10 to answer the following questions:

(a) How many walks of length 2 from College back to College are there?

(b) How many walks of length 3 from College to Store are there?
(c) Are there are any walks of length 3 from Restaurant to Dorm?

Solution. We have

$$A(D_2)^2 = \begin{bmatrix} 4 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad A(D_2)^3 = \begin{bmatrix} 3 & 5 & 5 & 5 & 4 \\ 6 & 2 & 3 & 2 & 1 \\ 5 & 4 & 3 & 4 & 2 \\ 4 & 1 & 1 & 1 & 0 \\ 4 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

(a) By Theorem 3.7.10 entry $(1,1)$ of $A(D_2)^2$ represents the number of walks of length 2 from College back to College. So there are four such walks.
(b) By Theorem 3.7.10 entry $(1,3)$ of $A(D_2)^3$ represents the number of walks of length 3 from College to Store. So there are five such walks.
(c) There are no such walks, because entry $(4,5)$ of $A(D_2)^3$ is zero.

For graphs with many vertices and edges, it may be difficult to count the number of walks, so Theorem 3.7.10 can be quite useful.

3.7.2 Sociology and psychology: Dominance graphs

Sociologists and psychologists use graphs to examine various kinds of relationships such as influence, dominance, and communication in groups.

Suppose that in a group, for every pair of members V_i and V_j, either V_i influences (or dominates) V_j, or V_j influences V_i, or there is no direct influence between V_i and V_j. This situation can be described by a digraph D that has *at most one directed edge* connecting any two vertices. Such a digraph is called a *dominance* digraph.

Figure 3.16 displays the dominance relationships among seven individuals $V_1, \ldots, V_7$.

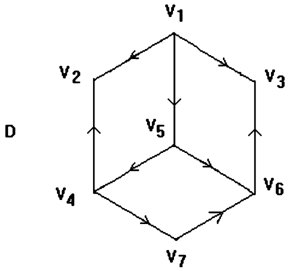

Figure 3.16: A dominance graph.

The adjacency matrix of a dominance digraph gives information about the influence relationships of a group. Rows with the most ones represent group members with greatest influence. Walks of length one represent direct influence, whereas walks of length greater than one represent indirect influence.

Example 3.7.12. In Figure 3.16, find the most influential member in
(a) a direct influence;
(b) a 2-stage influence.

Solution. We have

$$A(D) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad A(D)^2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) V_1 is the most influential member in direct influence, because the first row of $A(D)$ has the most nonzero entries.

(b) V_5 is the most influential member in 2-stage influence, because the first row of $A(D)^2$ has the most nonzero entries. It is clear from the graph, for instance, that V_5 is more influential than V_1. This is because V_5 influences V_2, V_3, and V_7 in two stages, whereas V_1 only influences V_4 and V_6.

Exercises 3.7

1. Find the adjacency matrix of the graph. Use Theorem 3.7.4 to determine the number of 3-walks from node 5 to node 1. Find these walks on the graph of Figure 3.17.

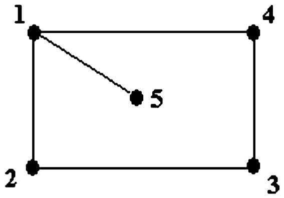

Figure 3.17: Find $A(G)^3$.

2. Find the adjacency matrix of the “wheel”graph. Number its vertices and use Theorem 3.7.4 to determine the number of 3-walks from the center to one of the pentagon vertices. Find these walks on the graph of Figure 3.18.[4]

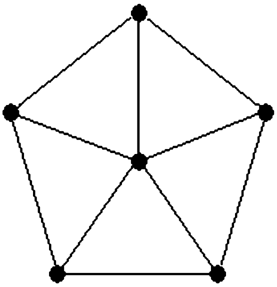

Figure 3.18: Adjacency matrix for number of walks.

Graph theory can be used to enumerate isomers of *saturated hydrocarbons* C_nH_{2n+2}, where n is the number of carbon atoms. C_nH_{2n+2} can be represented by a graph with nodes the atoms and edges the connections among the atoms. For example, for methane, we have the graph in Figure 3.19.

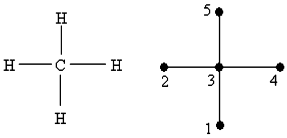

Figure 3.19: Methane.

3. Use Theorem 3.7.4 to find the number of walks of length 4 from the carbon atom C back to itself for the graph of methane.

4. Find a graph representing propane (Figure 3.20). Use Theorem 3.7.4 to find the number of walks of length 3 from the first carbon atom to the third one. Find these walks on the graph.

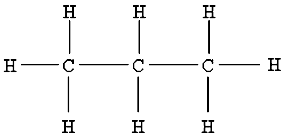

Figure 3.20: Propane.

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[4] Wheel graphs were studied by the distinguished graph theorist W. T. Tutte.

In 1847, G. R. Kirchhoff lay the foundations of the study of electrical circuits, known since as *Kirchhoff's laws*. Several of the formulas Kirchhoff found depend only the geometry of the circuit and not on the resistors, inductors, or voltage sources present. To study geometric properties, Kirchhoff replaced the electrical circuit with the underlying graph. For example, the electrical circuit of Figure 3.21.

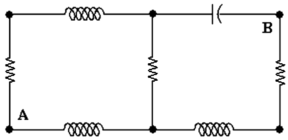

Figure 3.21: Electrical circuit.

is represented by the graph of Figure 3.22.

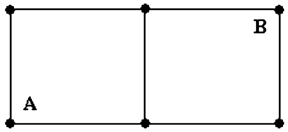

Figure 3.22: The underlying graph.

5. Use Theorem 3.7.4 to find the number of walks of length 3 from *A* to *B* in the above electrical circuit. Find these walks on the graph of Figure 3.22.

6. Find a graph representing the electrical circuit of Figure 3.23. Then use Theorem 3.7.4 to find the number of walks of length 3 from the center point back to itself.

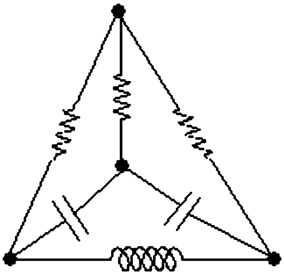

Figure 3.23: Use of the graph of circuit.

7. Find the adjacency matrix of the digraph of Figure 3.24. Use Theorem 3.7.10 to determine the number of 3-walks from node 4 back to itself. Find these walks on the digraph.

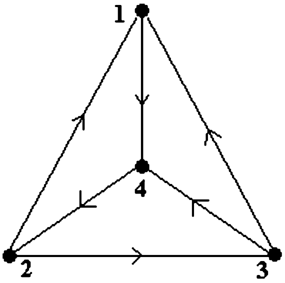

Figure 3.24: Number of walks on digraph.

8. Find the adjacency matrix of the digraph of Figure 3.25. Use Theorem 3.7.10 to determine the number of 3-walks from node 5 back to itself. Find these walks on the digraph.

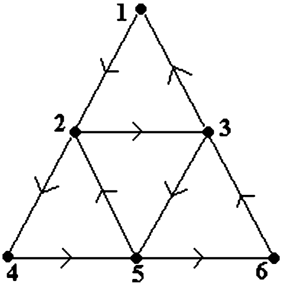

Figure 3.25: Number of walks on digraph.

9. In the dominance graph of Figure 3.26, use matrices to discuss if there is (a) one-stage and (b) two-stage dominance of one, or more nodes.

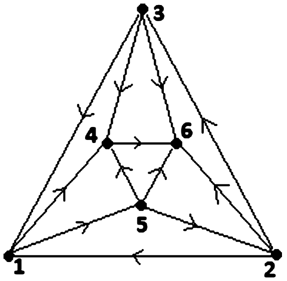

Figure 3.26: Dominance graph.

3.8 Miniprojects

3.8.1 Cryptology: The Hill cipher

Governments, security agencies, and companies are interested in the transmission of encrypted messages that are hard to decrypt, if intercepted, yet easily decrypted by the intended receiver who possesses a decryption key. There are many mathematical ways of encryption and decryption, which are studied in cryptology.

Here we discuss an encryption–decryption scheme that uses an invertible matrix. This is a *simplified version* of the *Hill cipher*.

Let us start out with an invertible matrix M that is known only to the transmitting and receiving ends, say

$$M = \begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix}.$$

Suppose we want to code the message

$$A\ T\ T\ A\ C\ K \quad N\ O\ W.$$

We replace each letter with the number that corresponds to the letter's position in the alphabet, and we use 0 for an empty space:

$$\begin{array}{cccccccccc} A & T & T & A & C & K & & N & O & W \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 20 & 20 & 1 & 3 & 11 & 0 & 14 & 15 & 23 \end{array}$$

The message has now been converted into a sequence of numbers, which we group as a sequence of column vectors

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix}, \begin{bmatrix} 20 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 14 \end{bmatrix}, \begin{bmatrix} 15 \\ 23 \end{bmatrix}$$

and multiply each on the left by M

$$\begin{bmatrix} 77 \\ 39 \end{bmatrix}, \begin{bmatrix} -56 \\ -18 \end{bmatrix}, \begin{bmatrix} 35 \\ 19 \end{bmatrix}, \begin{bmatrix} 56 \\ 28 \end{bmatrix}, \begin{bmatrix} 47 \\ 31 \end{bmatrix}$$

to get the sequence of numbers $77, \ldots, 31$. This is the coded message. To decode it, the receiving end needs to compute

$$M^{-1} = \begin{bmatrix} -1 & 2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

and multiply it by the coded vectors to get the original numbers back.

Problem A (Decoding message). Based on the above approach, decode the message given by the numbers

$$17, 15, 29, 15, 17, 29, 16, 31, 47, 6, 19, 20, 35, 24, 39, 14, 19, 19$$

if

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Problem B (Code breaking). Suppose that you intercepted the following coded stock market message: $1156, -203, 624, -84, -228, 95, 1100, -165, 60, 19$. Your sources inform you that the message was coded by using a 2×2 symmetric matrix. Your intuition tells you that the first word of the message is very likely to be either “sell” or “buy”. Can you break the code?

3.8.2 Transition of probabilities

Problem A. A group of people buys cars every four years from one of three automobile manufacturers, A, B, and C. The transition of probabilities of switching from one manufacturer to another is given by the following matrix:

$$\begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.1 \end{bmatrix}.$$

Suppose that at a given year manufacturer A sold 1000 cars, B sold 800 cars, and C sold 400 cars.

(a) How many cars are sold four years later?

(b) How many cars are sold seven years later?

(c) Will one manufacturer eventually dominate the market over the others?

Problem B. Consider the stochastic matrix of transition probabilities

$$T = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix},$$

expressing the flow of customers from and to markets A and B after one purchase. Suppose that the first time around $2/3$ of the customers buy from A and $1/3$ from B.

(a) What are the market shares after the first purchase?

(b) What are the market shares after the second purchase?

(c) Is there a *market equilibrium* (i. e., a vector of shares (a, b) that remains the same from one purchase to the next)? If yes, then compute it. (Keep in mind that the customers can only buy from A or from B. This means that $a + b = 1$.)

3.8.3 Digraph walks

Royal Ice Cream Company makes deliveries to four stores. The stores along with delivery routes (some one way) form a digraph with adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

1. Draw the digraph D.
2. Compute the matrices representing the number of the routes that can be traveled from one store to another so that a delivery truck would pass
   (a) by exactly one store,
   (b) by exactly two stores,
   (c) by at most one store,
   (d) by at most two stores.
   (e) In how many ways can one go from store 3 to store 4 passing by exactly one other store?
3. Can A be the adjacency matrix of a graph (as opposed to a digraph)?

3.8.4 A theoretical problem

Problem. Let A, B, C be $n \times n$ matrices, and let r be any nonzero scalar.[5]

If

$$A + B + rAB = 0,$$
$$B + C + rBC = 0,$$
$$C + A + rCA = 0,$$

then prove that $A = B = C$. Do not *assume* that the matrices are invertible!

---

[5] This problem with $r = 1996$ appeared in the 1996 National College Entrance Exams of Greece (the equivalent of SATs).

3.9 Technology-aided problems and answers

Let

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & -2 & -3 & -4 \\ 5 & 6 & 7 & 8 \\ -5 & -6 & -7 & -8 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -2 & 3 \\ 3 & -3 & 3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 2 & 3 & 6 \\ 3 & 3 & 6 \\ 6 & 6 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 4 \\ 0 & 4 \\ 1 & 4 \end{bmatrix}.$$

First, enter these matrices and name them as above. If a letter is already used as a command by your software, then change it. Use your program to solve the following exercises.

1. Display the fourth row, the third column, and the (2, 3)th entry of M.

2. Display the matrix obtained from M by using the first three columns.

3. Display the matrix obtained from M by using the last two rows.

4. Display the portion of M obtained by deleting the first row and the first two columns.

5. Display the matrix obtained from M by adding the numbers 4, 3, 2, 1 as a (a) first row, (b) last column.

6. Display the matrix obtained by writing A and B next to each other.

7. Display the matrix obtained by writing A above B.

8. Display a diagonal matrix with diagonal entries 1, 2, 3, 4.

For Exercises 9–11, let T be the matrix obtained by reversing the rows of M. Hence the last row becomes first, the fourth becomes second, and so on.

9. Compute $M - T$.

10. Compute $15M - 35T$.

11. Solve the matrix equation $17X - 51M = 62T$ for X.

12. Compute the products AB, BA, $(AB)C$, and $A(BC)$.

13. Compute the products $(AB)^2$, A^2B^2, $(A^3)^4$, and A^{12}.

14. Compute and compare the expressions (a) $(A + B)^2$, (b) $A^2 + 2AB + B^2$.

15. Compute A^{-1}.

16. Solve the matrix equations for X. (a) $AX = B$, (b) $XA = B$.

Consider the sequence of matrices of size ≥ 3 with 0s on the diagonal and 1s elsewhere:

$$A_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \dots .$$

17. Write an one argument function, called diagzero, that displays these matrices according to size. So diagzero(3) is A_3, and so on.

18. Use diagzero to display A_3, A_4, and A_5 and compute A_3^{-1}, A_4^{-1}, and A_5^{-1}.

19. Guess the formula for A_n^{-1}.

20. Write and test the code of three functions that produce elementary matrices of a given size that are obtained from each of elementary row operations.

21. Suppose a graph has the adjacency matrix A_4 above. Compute the matrix that yields (a) the number of walks of length 4; (b) the number of walks of lengths 1, or 2, or 3, or 4.

22. Write the code for a function, called sumpower, that takes two arguments, a square matrix A, and a positive integer n. The value of the function is the matrix

$$A + A^2 + \cdots + A^n.$$

Apply sumpower with $A = A(G)$ and $n = 4$ to verify your answer from the second part of the last exercise.

3.9.1 Selected solutions with Mathematica

```
M = {{1,2,3,4},{-1,-2,-3,-4},{5,6,7,8},{-5,-6,-7,-8}} (* Matrix       *)
A = {{1,-2,3},{2,-2,3},{3,-3,3}}                      (* definitions. *)
B = {{2,3,6},{3,3,6},{6,6,6}}
C1 = {{2,4},{0,4},{1,4}}       (* C is reserved for differential constants.*)
MatrixForm[A]                                    (* Displays A as a matrix. *)
(* Exercises 1-5.*)
M[[4]]                       (* The fourth row.                            *)
M[[All, 3]]                  (* The third column.                          *)
M[[2,3]]                     (* The (2,3) entry.                           *)
M[[All,1;;3]]                (* The first three columns.                   *)
M[[3;;4,All]]                (* The last two rows.                         *)
M[[2;;4,3;;4]]               (* Submatrix.                                 *)
v={4,3,2,1}
Join[{v}, M]                 (* Adds (4,3,2,1) to M as a first ROW.        *)
Prepend[M, v]                (* Another way.                               *)
(* Append[M,v] would add v to M as a last row.                             *)
Join[M,{{4},{3},{2},{1}},2]        (* Add v to M as the last column.       *)
Transpose[Append[Transpose[M],v]]                (* Or in an indirect way. *)
(* Exercises 6,7.*)
Join[A,B,2]                               (* A and B next to each other.  *)
Join[A,B]                                 (* A above B.                   *)
(*  Exercise 8. *)
DiagonalMatrix[{1,2,3,4}]
(* Exercises 9-11. *)
T=Reverse[M]    (* Reverses the order of the elements (rows) of the list. *)
M-T
15M-35T     (* Can use 15M or 15*M since the first factor is a number.    *)
1/17(51M+62T)
```

```
(* Exercises 12-14. *)
A.B                     (* Matrix multiplication is denoted by a dot . *)
B.A
(A.B).C1
A.(B.C1)
MatrixPower[A.B, 2]              (* A^n is denoted by MatrixPower[A,n] *)
MatrixPower[A,2].MatrixPower[B,2]
(A.A).(B.B)                                         (* Same as above. *)
(* Warning typing A^2 will yield the squares of the list elements
of A and not the matrix A^2. *)
MatrixPower[MatrixPower[A,3],4]
MatrixPower[A,12]
(A+B).(A+B)
A.A+2A.B+B.B       (* Not equal. *)
(* Exercise 15. *)
Inverse[A]                           (* Computes A^(-1).                *)
(* Exercise 16. *)
Inverse[A].B                           (* (a) X=A^(-1)B                *)
B.Inverse[A]                           (* (b) X=BA^(-1)                *)
(* Exercises 17-19. *)
diagzero[n_] := Table[If[i==j,0,1], {i,1,n},{j,1,n}]
(* The 2 dimensional table = matrix with entries 0 on the                *)
(* diagonal and 1 elsewhere.                                             *)
(* Also we can use the nxn matrix of 1's minus I_n.                      *)
diagzero[n_] := Table[1, {n},{n}] - IdentityMatrix[n]
diagzero[3]
diagzero[4]
diagzero[5]
Inverse[diagzero[3]]
Inverse[diagzero[4]]
Inverse[diagzero[5]]
(*A_n^(-1) has -(n-2)/(n-1) on the main diagonal and 1/(n-1), elsewhere.*)
(* Exercise 21. *)
AA = diagzero[4]        (* Use diagzero or enter the matrix directly. *)
MatrixPower[AA,4]       (* Matrix yielding the number of walks of length 4.*)
AA+MatrixPower[AA,2]+MatrixPower[AA,3]+MatrixPower[AA,4]
(* Exercise 22. *)
sumpower[A_, n_] := Sum[MatrixPower[A,i], {i,1,n}]
sumpower[AA,4]
```

3.9.2 Selected solutions with MATLAB

```
M = [1 2 3 4; -1 -2 -3 -4; 5 6 7 8; -5 -6 -7 -8]     % Matrix
A = [1 -2 3; 2 -2 3; 3 -3 3]                         % definitions.
B = [2 3 6; 3 3 6; 6 6 6]
C = [2 4; 0 4; 1 4]
% Exercises 1-5.
```

```
M(4,:)               %  Fourth row.
M(:,3)               %  Third column.
M(2,3)               %  (2,3)th entry.
M(:,1:3)             %  First three columns.
M(3:4,:)             %  Last two rows.
M(2:4,3:4)           %  Deleting first row, first two columns.
v = [4 3 2 1]        %  Define the vector (4,3,2,1).
[v ; M]              %  Adds v to M as a first row.
[M v.']              %  Adds v to M as a last column. v.' is v in column form.
[M [4;3;2;1]]        %  Same but entering v directly as a column vector.
% Exercises 6,7.
[A,B]                    % A and B next to each other.
[A;B]                    % A above B.
% Exercise 8.
diag([1,2,3,4])
% Exercises 9-11.
T=flipud(M)          %  flips M upside down. fliplr flips left to right.
M-T
15*M-35*T
1/17*(51*M+62*T)
% Exercises 12-14.
A*B                         % Matrix multiplication is denoted by * .
B*A
(A*B)*C
A*(B*C)
(A*B)^2                     % Matrix powers are denoted by ^ .
(A*B)*(A*B)                 % Same as above.
A^2*B^2
(A^3)^4
A^12
(A+B)^2
A^2+2*A*B+B^2
% Exercise 15.
inv(A)                                      % Computes A^(-1).
A^(-1)                                      % Another way.
% Exercise 16.
A^(-1)*B                                 % (a) X=A^(-1)B
B*A^(-1)                                 % (b) X=BA^(-1)
% Exercises 17-19.
function [A] = diagzero(n)            % Write the code on the left in an  m-file.
A = ones(n)-eye(n);                   % Matrix with 1's minus I_n.
end
% It is also useful to know how to use fuller code such as:
function [A] = diagzero(n)
for i=1:n, for j=1:n,
if i==j A(i,j)=0; else A(i,j)=1; end; end; end;
A;
end
% Then in MATLAB session in the same directory call these functions:
```

```
diagzero(3)
diagzero(4)
diagzero(5)
inv(diagzero(3))
inv(diagzero(4))
inv(diagzero(5))
% A_n^(-1) has -(n-2)/(n-1) on the main diagonal and 1/(n-1), elsewhere.
% Exercise 21.
AA = diagzero(4)       % Use diagzero or enter the matrix directly.
AA^4                   % Matrix yielding the number of walks of length 4.
AA+AA^2+AA^3+AA^4
% Exercise 22.
function [B] = sumpower(A,n)                        % Code in an file.
B = A;
for i=1:(n-1), B = B*A + A; end       % A->A^2+A->A^3+A^2+A->...
end
sumpower(AA,4)                                       % Type in session
```

3.9.3 Selected solutions with Maple

```
with(LinearAlgebra);                     #  Loading the package LinearAlgebra.
M := Matrix(4,4,[1,2,3,4,-1,-2,-3,-4,5,6,7,8,-5,-6,-7,-8]); #Matrix definitions.
A := Matrix(3,3,[1,-2,3,2,-2,3,3,-3,3]);
# or A := Matrix(3,3,[[1,-2,3],[2,-2,3],[3,-3,3]]);
# or A := Matrix([[1,-2,3],[2,-2,3],[3,-3,3]]);
B := Matrix(3,3,[2,3,6,3,3,6,6,6,6]);
C := Matrix(3,2,[2,4,0,4,1,4]);
# Exercises 1-5.
Row(M,4);                #  Fourth row.
Column(M,3);                #  Third column as a vector.
M[2,3];                  #  (2,3)th entry.
SubMatrix(M,[1..4],[1..3]);  #  First three columns as a matrix.
Column(M,[1..3]);        #  First three columns as a sequence of vectors.
SubMatrix(M,[3..4],[1..4]);  #  Last two rows as a matrix.
Row(M,[3..4]);             #  Last two rows as a sequence of vectors.
SubMatrix(M,[2..4],[3..4]);  #  Deletion of first row and first two columns.
v := Matrix(1,4,[4,3,2,1]);    # Define (4,3,2,1) as 1-column matrix.
<v,M>;                       #  Add v to M as the first row.
<M|Transpose(v)>;            # Add v to M as the last column.
# Exercises 6,7.
<A|B>;             # A and B next to each other.
<A,B>;              # A above B.
# Exercise 8.
DiagonalMatrix([1,2,3,4]);
# Exercises 9-11.
T:=Matrix(4,4,(i,j)->M[5-i,j]);    # The row entries are flipped while
# the column entries remain in tact.
```

```
M-T;
15*M-35*T;
1/17*(51*M+62*T);
# Exercises 12-14.
A.B;                    # Matrix multiplication is denoted by &* .
B.A;
(A.B).C;
A.(B.C);
(A.B)^2;                   # Matrix powers are denoted by ^ .
(A^2).(B^2);
(A^3)^4;
A^12;
(A+B)^2;
A^2+2*A.B+B^2;
# Exercise 15.
A^(-1);                                 # Computes A^(-1).
MatrixInverse(A);                           # Another way.
# Exercise 16.
A^(-1) . B;              # X = A^(-1)*B
B . A^(-1);              # X = B*A^(-1)
# Exercise 17-19.
diagzero := proc(n) local i;     # Matrix of 1's minus I_n.
Matrix(n, n, [seq(1, i = 1 .. n^2)]) - IdentityMatrix(n);
end proc:
# It is also useful to know how to use fuller code such as:
diagzero := proc(n) local i,j,a;
a := array(1..n,1..n):
for i from 1 to n do
for j from 1 to n do
if i=j then a[i,j]:=0 else a[i,j]:=1 fi od: od:
eval(Matrix(a));
end:
diagzero(3);
diagzero(4);
diagzero(5);
diagzero(3)^(-1);
diagzero(4)^(-1);
diagzero(5)^(-1);
# A_n^(-1) has -(n-2)/(n-1) on the main diagonal and 1/(n-1), elsewhere.
# Exercise 21.
AA := diagzero(4);       # A_4 using diagzero, or enter it directly.
AA^4;             # Matrix yielding the number of walks of length 4.
AA+AA^2+AA^3+AA^4;
# Exercise 22.
sumpower := proc(A,n) sum('A^i', 'i'=1..n) end:
sumpower(AA,4);
```

4 Vector spaces

> You have to spend some energy and effort to see the beauty of math.
>
> Maryam Mirzakhani, Iranian mathematician, Figure 4.1.

Figure 4.1: Maryam Mirzakhani, Fields Medalist.
(Image by Maryeraud9 – Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=117626026.)
Maryam Mirzakhani (1977–2017), Iranian mathematician, professor at Stanford. She won gold medals in the International Mathematical Olympiads. In 2014, she was awarded the Fields Medal for "her outstanding contributions to the dynamics and geometry of Riemann surfaces" (quoted from the International Mathematical Union). She was the first woman to win this prestigious prize.

Introduction

In this chapter, we generalize the basic concepts of Chapter 2: vectors, span, and linear independence. The common features of matrix and vector arithmetic become defining properties for a set of *abstract* or *generalized vectors*, called a *vector space*. Vector spaces were introduced by *H. Grassmann* in 1844 (Figure 4.2), and their defining axioms by *G. Peano*.

The major advantage of such generalizations is labor savings, because the properties of abstract vectors automatically apply to all particular examples. Therefore, we do not need to reprove essentially the same properties for each particular example. This abstraction also brings clarification and highlights the essential requirements in a proof of a property.

Figure 4.2: Hermann Günter Grassmann.
(Image by Unknown author, Public Domain, https://commons.wikimedia.org/w/index.php?curid=20262197.)
Grassmann (1809–1877) was born and died in Stettin, Prussia (now Szczecin, Poland). He taught at the Gymnasium (high-school) in Stettin. His most important mathematical work is *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*, where he developed the idea of an algebra representing geometric objects and defined the concepts of vector space and linear independence.

We discuss vector spaces and subspaces, spanning, linear independence, basis, dimension, the null space, the column space, and the row space. We conclude with an interesting application to *coding theory*.

https://doi.org/10.1515/9783111331850-004

4.1 Vector space

In this section, we generalize $\mathbf{R}^n$ and its operations. We consider sets where abstract operations of “addition” and “scalar multiplication” can be defined not through any specific direct rules, but by the requirement that they satisfy the basic properties of addition and scalar multiplication. These properties, expressed in Theorem 2.1.5, are valid for $m \times n$ matrices and hence for n-vectors.

4.1.1 Definition and properties

Definition 4.1.1. Let V be a set equipped with two operations named *addition* and *scalar multiplication*. Addition is a map that associates any two elements $\mathbf{u}$ and $\mathbf{v}$ of V with a third one, called the *sum* of $\mathbf{u}$ and $\mathbf{v}$ and denoted by $\mathbf{u} + \mathbf{v}$:

$$V \times V \to V, \quad (\mathbf{u}, \mathbf{v}) \to \mathbf{u} + \mathbf{v}.$$

Scalar multiplication is a map that associates any real scalar c and any element $\mathbf{u}$ of V with another element of V, called the *scalar multiple* of $\mathbf{u}$ by c and denoted by $c\mathbf{u}$:

$$\mathbf{R} \times V \to V, \quad (c, \mathbf{u}) \to c\mathbf{u}.$$

Such a set V is called a (real) *vector space* if the two operations satisfy the following properties, known as *axioms* for a vector space.

Addition

(A1) $\mathbf{u} + \mathbf{v}$ belongs to V for all $\mathbf{u}, \mathbf{v} \in V$.

(A2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$. (*Commutativity law*)

(A3) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. (*Associativity law*)

(A4) There exists a unique element $\mathbf{0} \in V$, called the *zero* of V, such that for all $\mathbf{u}$ in V,

$$\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}.$$

(A5) For each $\mathbf{u} \in V$, there exists a unique element $-\mathbf{u} \in V$, called the *negative* or *opposite* of $\mathbf{u}$, such that

$$\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}.$$

Scalar multiplication

(M1) $c\,\mathbf{u}$ belongs to V for all $\mathbf{u} \in V$ and all $c \in \mathbf{R}$.

(M2) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ for all $\mathbf{u}, \mathbf{v} \in V$ and all $c \in \mathbf{R}$. (*Distributivity law*)

(M3) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ for all $\mathbf{u} \in V$ and all $c, d \in \mathbf{R}$. (*Distributivity law*)

(M4) $c(d\mathbf{u}) = (cd)\mathbf{u}$ for all $\mathbf{u} \in V$ and all $c, d \in \mathbf{R}$.

(M5) $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.

The elements of a vector space are called *vectors*. Axioms (A1) and (M1) are also expressed by saying that V *is closed under addition* and *closed under scalar multiplication*. Note that *a vector space is a nonempty set*, because it contains the zero vector by (A4).

Note that the operations in the definition of a vector space are not specified. Acceptable operations are *any* operations that satisfy the axioms.

The *difference* $\mathbf{u} - \mathbf{v}$ between $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

Axioms (A1), (A2), (A3), and (M1) allow us to add any *finite* set of scalar multiples of vectors without worrying about the order or grouping of terms. If $\mathbf{v}_1, \dots, \mathbf{v}_n$ are vectors and $c_1, \dots, c_n$ are scalars, then the expression

$$c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n$$

is well defined and is called a *linear combination* of $\mathbf{v}_1, \dots, \mathbf{v}_n$. If not all c_i are zero, then we have a *nontrivial linear combination*. If all c_i are zero, then we have the *trivial linear combination*. The trivial linear combination represents the zero vector.

Theorem 4.1.2. *Let V be a vector space. Let $\mathbf{u} \in V$ and $c \in \mathbf{R}$. Then*
1. $0\,\mathbf{u} = \mathbf{0}$.
2. $c\,\mathbf{0} = \mathbf{0}$
3. *If $c\mathbf{u} = \mathbf{0}$, then either $c = 0$, or $\mathbf{u} = \mathbf{0}$.*
4. $(-c)\,\mathbf{u} = -(c\,\mathbf{u})$.

Proof of 1 and 4.
1. We have

$$\begin{array}{rlll}
 & 0\,\mathbf{u} + 0\,\mathbf{u} = (0+0)\,\mathbf{u} = 0\,\mathbf{u} & \text{by} & \text{(M3)} \\
\Rightarrow & (0\,\mathbf{u} + 0\,\mathbf{u}) + (-0\,\mathbf{u}) = 0\,\mathbf{u} + (-0\,\mathbf{u}) & \text{by} & \text{adding } -0\,\mathbf{u} \\
\Rightarrow & 0\,\mathbf{u} + \big(0\,\mathbf{u} + (-0\,\mathbf{u})\big) = \mathbf{0} & \text{by} & \text{(A3) and (A5)} \\
\Rightarrow & 0\,\mathbf{u} + \mathbf{0} = \mathbf{0} & \text{by} & \text{(A5)} \\
\Rightarrow & 0\,\mathbf{u} = \mathbf{0} & \text{by} & \text{(A4).}
\end{array}$$

4. By (M3), Part 1, and (A2) we have

$$\begin{aligned}
c\,\mathbf{u} + (-c)\,\mathbf{u} = \big(c + (-c)\big)\,\mathbf{u} = 0\,\mathbf{u} = \mathbf{0} \\
\Rightarrow (-c)\,\mathbf{u} + c\,\mathbf{u} = \mathbf{0}.
\end{aligned}$$

Therefore $(-c)\,\mathbf{u} = -\,c\;\mathbf{u}$ by (A5). □

The axioms allow us to do n-vector–like arithmetic in a vector space. For example, to show that $\mathbf{u}+\mathbf{u}=2\,\mathbf{u}$, we have

$$\mathbf{u}+\mathbf{u}=1\,\mathbf{u}+1\,\mathbf{u}=(1+1)\,\mathbf{u}=2\,\mathbf{u}.$$

4.1.2 Examples of vector spaces

To verify that a given set is a vector space, we need to either be given or to define or specify explicitly the *two operations* and the *zero element* and then *verify the axioms*.

Note that once a scalar multiplication is defined, then the opposite of $\mathbf{v}$ can be defined as $(-1)\,\mathbf{v}$, so there is no need to define opposites separately.

Example 4.1.3. The set $\mathbf{R}^n$ of all n-vectors with real components.

1. *Operations:* The usual vector addition and scalar multiplication.
2. *Zero:* The zero n-vector $\mathbf{0}$.
3. *Axioms:* See Theorem 2.1.5.

Example 4.1.4. The set M_{mn} of all $m \times n$ matrices with real entries.

1. *Operations:* The usual matrix addition and scalar multiplication.
2. *Zero:* The $m \times n$ zero matrix $\mathbf{0}$.
3. *Axioms:* See Theorem 2.1.5.

Example 4.1.5. The set P of all polynomials with real coefficients.

1. *Operations:* Let x be the indeterminant of the polynomials.
   (a) *Addition:* The sum of two polynomials is formed by adding the coefficients of the same powers of x of the polynomials. Explicitly, if

   $$p_1=a_0+a_1x+\cdots+a_nx^n, \quad p_2=b_0+b_1x+\cdots+b_mx^m, \quad n \geq m,$$

   then we write $p_2=b_0+b_1x+\cdots+b_nx^n$ by adding zeros, if necessary, and form the sum

   $$p_1+p_2=(a_0+b_0)+(a_1+b_1)\,x+\cdots+(a_n+b_n)\,x^n.$$

   (b) *Scalar multiplication:* This is multiplication of a polynomial through by a constant:

   $$cp_1=(ca_0)+(ca_1)\,x+\cdots+(ca_n)\,x^n.$$

2. *Zero:* The zero polynomial $\mathbf{0}$ is the polynomial with zeros as coefficients.
3. *Axioms:* The verification of the axioms is left as an exercise.

Example 4.1.6. The set $F(\mathbf{R})$ of all real-valued functions defined on $\mathbf{R}$.

1. *Operations:* Let f and g be two real-valued functions with domain $\mathbf{R}$, and let c be any scalar.
   (a) *Addition:* We define the sum $f+g$ of f and g as the function whose values are given by

$$(f+g)(x) = f(x) + g(x) \quad \text{for all } x \in \mathbf{R}$$

   (Figure 4.3(a)).
   (b) *Scalar multiplication:* The scalar product cf is defined by

$$(cf)(x) = cf(x) \quad \text{for all } x \in \mathbf{R}$$

   (Figure 4.3(b)).
2. *Zero:* The zero function $\mathbf{0}$ is the function whose values are all zero:

$$\mathbf{0}(x) = 0 \quad \text{for all } x \in \mathbf{R}.$$

3. *Axioms:* The verification of the axioms is left as an exercise.

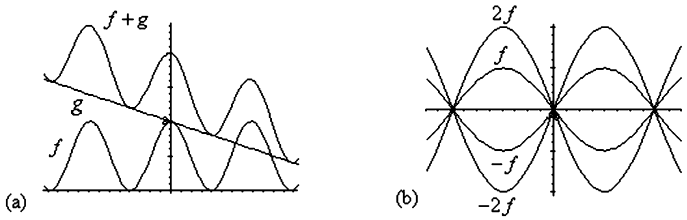

Figure 4.3: (a) The graphs of f, g, and $f+g$. (b) Scalar multiples of f.

The set $F(X)$ of all real-valued functions defined on *any* set X is also a vector space. The operations, zero, and negatives are defined as in Example 4.1.6. **!**

Example 4.1.7. Is $\mathbf{R}^2$ with the usual addition and the following scalar multiplication, denoted by $\odot$, a vector space?

$$c \odot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ a_2 \end{bmatrix}.$$

Solution. It is *not* a vector space, because

$$(c+d) \odot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} (c+d)\,a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 + da_1 \\ a_2 \end{bmatrix}$$

and

$$c \odot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + d \odot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} da_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} ca_1 + da_1 \\ 2a_2 \end{bmatrix}.$$

So, $(c + d) \odot \mathbf{v} \neq c \odot \mathbf{v} + d \odot \mathbf{v}$, and Axiom (M3) fails.

4.1.3 Subspaces

Definition 4.1.8. A subset W of a vector space V is called a *subspace* of V if W itself is a vector space under the same addition and scalar multiplication as in V. A subspace is *nonempty*, because it contains the zero element of V.

In verifying that W is a subspace of V, it is sufficient to only check the closure of operations, because the remaining properties are inherited from V.

Theorem 4.1.9 (Criterion for subspace). *Let W be a nonempty subset W of a vector space V. Then W is a subspace of V if and only if it is closed under addition (Axiom (A1)) and scalar multiplication (Axiom (M1)), that is, if and only if*

1. *If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W;*
2. *If c is any scalar and $\mathbf{u}$ is in W, then $c\,\mathbf{u}$ is in W.*

Proof.

⇒ If W is a subspace of V, then all axioms hold for W. In particular, Axioms (A1) and (M1) hold.

⇐ Conversely, let W be a subset that satisfies Conditions 1 and 2. Therefore (A1) and (M1) hold. Axioms (A2–3), (M2–5) hold, because they are valid in V. It remains to verify (A4) and (A5). Condition 2 implies that $0\,\mathbf{u} = \mathbf{0}$ is in W for $\mathbf{u}$ in W by letting $c = 0$. Likewise, $(-1)\mathbf{u} = -\mathbf{u}$ is in W for any $\mathbf{u}$ in W by choosing $c = -1$. So Axioms (A4) and (A5) hold as well. ☐

Example 4.1.10. The set $W = \{c\mathbf{v},\ c \in \mathbf{R}\}$ of all scalar multiples of the fixed vector $\mathbf{v}$ of a vector space V is a subspace of V.

Solution. W is nonempty, because it contains $\mathbf{v}$ (why?). Also,

$$c_1\mathbf{v} + c_2\mathbf{v} = (c_1 + c_2)\mathbf{v} \quad \text{and} \quad r(c\mathbf{v}) = (rc)\mathbf{v}.$$

So W is closed under addition and scalar multiplication. Therefore W is a subspace of V.

Example 4.1.11. $\{\mathbf{0}\}$ and V are subspaces of V. (Verify.)

Definition 4.1.12. The subspaces $\{\mathbf{0}\}$ and V are called the *trivial subspaces* of V. $\{\mathbf{0}\}$ is called the *zero subspace* of V.

Example 4.1.13. Prove that the set P_n that consists of all polynomials of degree $\leq n$ and the zero polynomial is a subspace of P.

Solution. P_n is nonempty. Recall that the *degree* of a nonzero polynomial is the highest power of x with nonzero coefficient. A polynomial in P_n can be written in the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n.$$

The sum of two such polynomials is again a polynomial of degree $\leq n$ or zero. Also, a constant multiple of such a polynomial is a polynomial of degree $\leq n$ or zero. So Conditions 1 and 2 of Theorem 4.1.9 are satisfied. Therefore P_n is a subspace of P.

Example 4.1.14. Let $\mathbf{a}$ be a fixed vector in $\mathbf{R}^3$, and let W be the set of all vectors orthogonal to $\mathbf{a}$. Then W is a subspace of $\mathbf{R}^3$.

Solution. W is nonempty, because it contains $\mathbf{0}$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in W. Then $\mathbf{a}\cdot\mathbf{u} = 0$ and $\mathbf{a}\cdot\mathbf{v} = 0$. Hence

$$\mathbf{a}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{a}\cdot\mathbf{u} + \mathbf{a}\cdot\mathbf{v} = 0 + 0 = 0.$$

Therefore $\mathbf{u}+\mathbf{v}$ is orthogonal to $\mathbf{a}$, so $\mathbf{u}+\mathbf{v}$ is an element of W. If c is any scalar, then

$$\mathbf{a}\cdot(c\,\mathbf{u}) = c(\mathbf{a}\cdot\,\mathbf{u}){=}c\,0 = 0$$

implies that $c\,\mathbf{u}$ is in W. So W is closed under addition and scalar multiplication. So it is a subspace of $\mathbf{R}^3$.

Theorem 4.1.9 can be also used for the remaining examples of subspaces. The details of verification are left as an exercise.

Example 4.1.15 (Requires calculus). The set $C(\mathbf{R})$ of all continuous real-valued functions defined on $\mathbf{R}$ is a subspace of $F(\mathbf{R})$.

Example 4.1.16. The set D_n of size n diagonal matrices is a subspace of M_{nn}.

Example 4.1.17. The set of all $n \times n$ symmetric matrices is a subspace of M_{nn}.

4.1.4 Complex vector spaces

A *complex vector space* or *a vector space over* $\mathbf{C}$ is a vector space as defined above with the difference that the scalars are complex numbers. Similarly, we have the notion of (*complex*) *subspace* of a complex vector space.

Example 4.1.18. The following are examples of complex vector spaces.

(a) $\mathbf{C}^n$, the set of all n-vectors with complex components.

(b) $M_{mn}(\mathbf{C})$, the set of all $m \times n$ matrices with complex entries.

(c) $P(\mathbf{C})$, the set of all polynomials with complex coefficients.

Exercises 4.1

Unless stated otherwise, all subsets of $\mathbf{R}^n$, P, M_{mn}, or $F(\mathbf{R})$ considered here are equipped with the ordinary addition or scalar multiplication.

1. Prove that M_{23} is a vector space.

2. Prove that P is a vector space.

3. Prove that the set of matrices of the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ is a vector space.

4. Prove that the set of matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a + b = c + d$ is a vector space.

5. Prove that the set of matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ab = cd$ is not a vector space.

6. Is the set of matrices of the form $\begin{bmatrix} a & 1 & 0 \\ b & 0 & 0 \end{bmatrix}$ a vector space? Explain.

7. Is the set of matrices of the form $\begin{bmatrix} a & a \\ a & a \\ a & a \end{bmatrix}$ a vector space? Explain.

8. Is $\mathbf{R}^2$ with the usual addition and the following scalar multiplication, denoted by $\odot$, a vector space?

$$c \odot (x, y) = (0, 0).$$

9. Is $\mathbf{R}^3$ with the usual addition and the following scalar multiplication, denoted by $\odot$, a vector space?

$$c \odot (a_1, a_2, a_3) = (a_1, a_2, ca_3)\,.$$

10. What is wrong with the following claim? "The sum of two polynomials of degree 2 is a polynomial of degree 2." Use your explanation to prove that the set of all polynomials of degree n is *not* a vector space.

In Exercises 11–16, determine whether the set S is a subspace of P_2. Use the notation $p = a_0 + a_1 x$.

11. $S = \{p \in P_2,\ a_1 = 0\}$.

12. $S = \{p \in P_2,\ a_0 < 0\}$.

13. $S = \{p \in P_2,\ a_0 + a_1 = 1\}$.

14. $S = \{p \in P_2,\ a_0 = 2a_1\}$.

15. $S = \{p \in P_2,\ a_0 a_1 = 0\}$.

16. $S = \{p \in P_2,\ a_0^2 = a_1^2\}$.

17. *True or False?* Explain. The set P_E that consists of all even degree polynomials plus the zero polynomial is a subspace of P.

In Exercises 18–20, determine whether the given subset of M_{mn} is a subspace of M_{mn}.

18. All matrices $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$ such that $a + b = 0$, $c + d = 0$, and $e + f = 0$.

19. All matrices $\begin{bmatrix} a & 0 \\ 0 & a^2 \end{bmatrix}$.

20. All matrices $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with $a > 0$.

In Exercises 21–29, determine whether the given subset of M_{nn} is a subspace of M_{nn}.

21. The upper triangular matrices.

22. The lower triangular matrices.

23. The diagonal matrices (Example 4.1.16).

24. The scalar matrices.

25. The invertible matrices.

26. The noninvertible matrices.

27. The matrices of trace zero.

28. The set of symmetric matrices (Example 4.1.17).

29. The set of skew-symmetric matrices.

30. Prove that for a fixed $m \times n$ matrix A, all $n \times k$ matrices B such that $AB = \mathbf{0}$ is a subspace of M_{nk}.

31. Let A be a fixed $n \times n$ matrix. Prove that all matrices B such that $[A, B] = \mathbf{0}$ is a subspace of M_{nn}.

32. A function f in $F(\mathbf{R})$ is called *even* if for all real x,

$$f(x) = f(-x).$$

Is the set of even functions a subspace $F(\mathbf{R})$? Explain.

33. A function f in $F(\mathbf{R})$ is called *odd* if for all real x,

$$f(x) = -f(-x)$$

Is the set of odd functions a subspace $F(\mathbf{R})$? Explain.

34. Prove that the set

$$S = \{c_1 \cos t + c_2 \sin t,\ c_1, c_2 \in \mathbf{R}\}$$

is a subspace of $F(\mathbf{R})$. S actually represents all possible displacements $x = x(t)$ at time t of a mass attached to a spring of spring constant $k = 1$ (Figure 4.4).

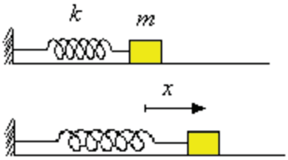

Figure 4.4: Displacements as a subspace.

35. Is S a subspace of $F(\mathbf{R})$? Explain.
(a) $S = \{f \in F(\mathbf{R}), f(0) = f(1)\}$.
(b) $S = \{f \in F(\mathbf{R}), f(3) = 0\}$.
(c) $S = \{f \in F(\mathbf{R}), f(0) = 1\}$.

36. Let V be a vector space, and let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $r \in \mathbf{R}$. Use the vector space axioms to prove the following:
(a) If $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$.
(b) If $\mathbf{u} + \mathbf{v} = \mathbf{v}$, then $\mathbf{u} = \mathbf{0}$.
(c) If $r\mathbf{u} = r\mathbf{v}$, then either $r = 0$ or $\mathbf{u} = \mathbf{v}$.
(d) $(-1)\mathbf{u} = -\mathbf{u}$.

37. Prove Parts 2 and 3 of Theorem 4.1.2.

38. Let U and W be subspaces of a vector space V. Prove that the intersection $U \cap W$ is a subspace of V.

39. Find an example of two subspaces V_1 and V_2 of some vector space V such that the union $V_1 \cup V_2$ is not a subspace of V.

40. *True or False?* The described set is a vector space.
(a) The solution set of the system $A\mathbf{x} = \mathbf{0}$.
(b) The solution set of the system $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$.
(c) The line through $(1,1,1)$ and $(2,2,2)$.
(d) The line through $(1,1,1)$ and $(3,-1,2)$.

41. (*Sum of subspaces*) Let V be a vector space. The *sum* of two subspaces V_1 and V_2 of V is the set $V_1 + V_2$ that consists of all vectors $\mathbf{v}_1 + \mathbf{v}_2$ with $\mathbf{v}_1$ in V_1 and $\mathbf{v}_2$ in V_2:

$$V_1 + V_2 = \{\mathbf{v}_1 + \mathbf{v}_2 : \ \mathbf{v}_1 \in V_1, \ \mathbf{v}_2 \in V_2\}$$

Prove that the sum $V_1 + V_2$ is a subspace of V.

42. (*Direct sum of subspaces*) Let V be a vector space. The sum of two subspaces V_1 and V_2 of V as defined in Exercise 41 is called a *direct sum* if the intersection of V_1 and V_2 is the zero subspace, $V_1 \cap V_2 = \{\mathbf{0}\}$. In this case, we write $V_1 \oplus V_2$ instead of $V_1 + V_2$.
(a) Prove that in $V_1 \oplus V_2$, we have

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0} \Rightarrow \mathbf{v}_1 = \mathbf{0} \quad \text{and} \quad \mathbf{v}_2 = \mathbf{0}.$$

(b) Prove that in $V_1 \oplus V_2$, if

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_2,$$

then

$$\mathbf{v}_1 = \mathbf{w}_1 \quad \text{and} \quad \mathbf{v}_2 = \mathbf{w}_2.$$

43. Let $\mathcal{P}$ be the xy-plane, and l be the z-axis of $\mathbf{R}^3$. Referring to Exercise 42, find the direct sum $\mathcal{P} \oplus l$.

44. Prove that the Cartesian product $V_1 \times V_2$, of two vector spaces V_1 and V_2 defined by

$$\left\{(\mathbf{v}_1, \mathbf{v}_2), \text{ where, } \mathbf{v}_1 \in V_1, \ \mathbf{v}_2 \in V_2\right\}$$

with the addition and scalar multiplication

$$(\mathbf{u}_1, \mathbf{u}_2) + (\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{v}_1, \mathbf{u}_2 + \mathbf{v}_2), \quad c(\mathbf{v}_1, \mathbf{v}_2) = (c\mathbf{v}_1, c\mathbf{v}_2)$$

is a vector space.

45. (*Requires calculus*) Work out the details of Example 4.1.15.

46. (*Requires calculus*) Prove that the set of twice differentiable functions $y(x)$ that satisfy $y'' + y = 0$ is a vector space.

47. (*Requires calculus*) Explain why the set of differentiable functions $y(x)$ that satisfy $y' - y = -x$ is *not* a vector space. (*Hint:* Look at the functions $y_1(x) = 1 + x + e^x$ and $y_2(x) = 1 + x$.)

4.2 Span, linear independence

In this section, we generalize to abstract vectors the important notions of a spanning set and of linear independence for n-vectors.

4.2.1 Span

Definition 4.2.1. Let S be a nonempty subset of a vector space V. The set of *all finite linear combinations* of vectors in S is called the *span* of S and is denoted by $\text{Span}(S)$. If $V = \text{Span}(S)$, then we say that S *spans* V and that S is a *spanning set* of V.

If S is a finite set, say $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, then we write $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for $\text{Span}(S)$. We also define the span of the empty set $\emptyset$ to be the zero subspace:

$$\text{Span}(\emptyset) = \{\mathbf{0}\}.$$

Example 4.2.2. Let V be a vector space, and let $\mathbf{v}_1$, $\mathbf{v}_2$ be in V. The following vectors are in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$:

$$\mathbf{0}, \quad \mathbf{v}_1, \quad \mathbf{v}_2, \quad \mathbf{v}_1 + \mathbf{v}_2, \quad -2\mathbf{v}_1, \quad 3\mathbf{v}_1 - 2\mathbf{v}_2.$$

Example 4.2.3. Let V be a vector space, and let $\mathbf{v}$ be in V. Span$\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$:

$$\mathrm{Span}\{\mathbf{v}\} = \{c\mathbf{v}\,,\ c \in \mathbf{R}\}.$$

Example 4.2.4. Let $p = -1 + x - 2x^2$ in P_3. Prove that $p \in \mathrm{Span}\{p_1, p_2, p_3\}$, where

$$p_1 = x - x^2 + x^3, \quad p_2 = 1 + x + 2x^3, \quad p_3 = 1 + x.$$

Solution. Let c_1, c_2, c_3 be scalars such that

$$-1 + x - 2x^2 = c_1(x - x^2 + x^3) + c_2(1 + x + 2x^3) + c_3(1 + x).$$

Then

$$-1 + x - 2x^2 = (c_2 + c_3) + (c_1 + c_2 + c_3)\,x - c_1x^2 + (c_1 + 2c_2)\,x^3.$$

Equating the coefficients of the same powers of x yields the linear system

$$c_2 + c_3 = -1, \quad c_1 + c_2 + c_3 = 1, \quad -c_1 = -2, \quad c_1 + 2c_2 = 0$$

with solution $c_1 = 2$, $c_2 = -1$, $c_3 = 0$. Therefore p is in the span of p_1, p_2, p_3.

Example 4.2.5. Prove that $P = \mathrm{Span}\{1, x, x^2, \dots, x^n, \dots\}$.

Solution. This is true, because every polynomial $p = a_0 + a_1x + \cdots + a_kx^k$ in P is a linear combination of elements of $\{1, x, x^2, \dots, x^n \dots\}$ with coefficients $a_0, a_1, \dots, a_k$.

Let E_{ij} be the matrix in M_{mn} whose (i,j) entry is 1 and the rest of the entries are zero. For example, in M_{23}, we have

$$E_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example 4.2.6. Prove that $\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}\}$ spans M_{32}.

Solution. This is true, because

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = a\,E_{11} + bE_{12} + cE_{13} + dE_{21} + eE_{22} + fE_{23}.$$

Example 4.2.7. Describe the span of $\{A, B\}$ in M_{22}, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Solution. Any linear combination of A and B is a diagonal matrix

$$aA + bB = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & -b \end{bmatrix}.$$

Conversely, any diagonal matrix can be written as a linear combination of A and B, because

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = (a+b)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-b)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore $\mathrm{Span}(\{A, B\}) = D_2$, the set of all 2×2 diagonal matrices.

Example 4.2.8. Is it true that $\mathrm{Span}\{1 + x^2, 1 - x^2, 5\} = P_2$?

Solution. Let $p = a_0 + a_1x + a_2x^2$ be any polynomial of P_2. For all choices of a_i, we seek scalars c_i such that

$$\begin{aligned} a_0 + a_1x + a_2x^2 &= c_1(1 + x^2) + c_2(1 - x^2) + c_3 5 \qquad \Rightarrow \\ a_0 + a_1x + a_2x^2 &= (c_1 + c_2 + 5c_3) + 0x + (c_1 - c_2)x^2. \end{aligned}$$

Therefore $a_1 = 0$. So a polynomial p with $a_1 \neq 0$ cannot be in the span. Hence the span is not all of P_2.

> The following statements can be easily verified and are left as exercises.
> 1. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ spans $\mathbf{R}^n$.
> 2. $\{1, x, x^2, \ldots, x^n\}$ spans P_n.
> 3. $\{E_{11}, E_{12}, \ldots, E_{ij}, \ldots, E_{mn}\}$ spans M_{mn}.
> 4. $\{E_{11}, E_{22}, \ldots, E_{ii}, \ldots, E_{nn}\}$ spans D_n.

Theorem 4.2.9. *Let S be a subset of a vector space V. Then*
1. $\mathrm{Span}(S)$ *is a subspace of V;*
2. $\mathrm{Span}(S)$ *is the smallest subspace of V that contains S.*

Proof. The Span of any subset S of V is nonempty. (Why?)
1. Consider two linear combinations of vectors $\mathbf{u}_i$ $(1 \leq i \leq n)$ and $\mathbf{v}_j$ $(1 \leq j \leq m)$ in S,

$$\mathbf{u} = c_1\,\mathbf{u}_1 + \cdots + c_n\,\mathbf{u}_n \quad \text{and} \quad \mathbf{v} = d_1\,\mathbf{v}_1 + \cdots + d_m\,\mathbf{v}_m.$$

The sum

$$\mathbf{u} + \mathbf{v} = c_1\,\mathbf{u}_1 + \cdots + c_n\,\mathbf{u}_n + d_1\,\mathbf{v}_1 + \cdots + d_m\,\mathbf{v}_m$$

is again a linear combination of vectors in S. If c is any scalar, then

$$c\,(c_1\,\mathbf{u}_1 + \cdots + c_n\,\mathbf{u}_n) = (cc_1)\,\mathbf{u}_1 + \cdots + (cc_n)\,\mathbf{u}_n$$

is also a linear combination of vectors in S. Therefore Span(S) is closed under the addition and scalar multiplication of V. Hence it is a subspace of V by Theorem 4.1.9.

2. Let W be a subspace that contains S. As a subspace, W contains all linear combinations of its elements. In particular, it contains all linear combinations of elements of S. But these are the elements of Span(S). Therefore Span(S) $\subseteq W$. In addition, Span(S) is a subspace by Part 1. Hence Span(S) is the subspace contained in any subspace W that contains S. So it is the smallest subspace that contains S. ☐

Theorem 4.2.10 (Reduction of spanning set). *If one of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ of the vector space V is a linear combination of the remaining vectors, then the span does not change if we remove this vector.*

Proof. See the proof of Theorem 2.3.10. ☐

4.2.2 Linear dependence

The notion of linear dependence and independence of m-vectors extends to vectors of a general vector space.

Definition 4.2.11. A subset S of a vector space V is called *linearly dependent* if there are vectors in S, say, $\mathbf{v}_1, \dots, \mathbf{v}_k$, and scalars $c_1, \dots, c_k$ *not all zero* such that

$$c_1\,\mathbf{v}_1 + \cdots + c_k\,\mathbf{v}_k = \mathbf{0}. \tag{4.1}$$

So *there are nontrivial linear combinations that represent the zero vector*. Equation (4.1) with not all c_i zero is called a *linear dependence relation* of the $\mathbf{v}_i$s.

Example 4.2.12. Prove that the set $\{2 - x + x^2, 2x + x^2, 4 - 4x + x^2\}$ is linearly dependent in P_3.

Solution. This is true, because

$$2(2 - x + x^2) + (-1)(2x + x^2) + (-1)(4 - 4x + x^2) = \mathbf{0}.$$

Example 4.2.13. Is the set $\{A, B, C\}$ linearly dependent in M_{22}?

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}.$$

Solution. Yes, because $(-1)A + B + C = \mathbf{0}$.

Example 4.2.14. Is the set $\{1, \cos(2x), \cos^2(x)\}$ linearly dependent in $F(\mathbf{R})$?

Solution. The half-angle formula $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ implies that

$$1 \cdot 1 + 1 \cdot \cos(2x) + (-2) \cdot \cos^2(x) = 0 \quad \text{for all } x \in \mathbf{R}.$$

Hence the set is linearly dependent in $F(\mathbf{R})$.

Note:

1. *If the set of* vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *contains the zero vector, then it is linearly dependent*. This is because if one vector, say $\mathbf{v}_1 = \mathbf{0}$, then
$$1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \cdots + 0\mathbf{v}_k = \mathbf{0}$$
is a linear dependence relation.
2. *Two vectors* $\mathbf{v}_1$, $\mathbf{v}_2$ *are linearly dependent if and only if one is a scalar multiple of the other*. Indeed, if $\mathbf{v}_1 = k\mathbf{v}_2$, then
$$1\mathbf{v}_1 + (-k)\,\mathbf{v}_2 = \mathbf{0}$$
is a linear dependence relation. Conversely, if the vectors are linearly dependent, then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ for some c_1, c_2 not both zero. If $c_1 \neq 0$, then $\mathbf{v}_1 = (-c_2/c_1)\mathbf{v}_2$. So $\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_2$.

Theorem 4.2.15. *Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a subset of a vector space V. We have*

1. *If $k = 1$, then S is linearly dependent if and only if $\mathbf{v}_1 = \mathbf{0}$.*
2. *If $k \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$, then S is linearly dependent if and only if at least one vector, say, $\mathbf{v}_i$ $(i \geq 2)$, is a linear combination of the vectors that precede it, i. e., $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}$.*

Proof. See the proof of Theorem 2.4.6. ☐

Theorem 4.2.15 implies that $\mathbf{v}_1, \dots, \mathbf{v}_k$ is linearly dependent in V if and only if at least one of the vectors is in the span of the remaining vectors.

4.2.3 Linear independence

Definition 4.2.16. A subset S of a vector space V is called *linearly independent* if it is not linearly dependent. This means that for every finite subset of vectors in S, say, $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$, if

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

then

$$c_1 = 0, \quad \dots, \quad c_k = 0.$$

This is equivalent to saying that *there are no linear dependence relations* among the elements of S.

Example 4.2.17. Prove that set $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ is linearly independent in M_{22}.

Solution. We have

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $c_1 = c_2 = c_3 = c_4 = 0$. So the set is linearly independent.

Example 4.2.18. Is the set $\{1, x, \dots, x^n\}$ linearly independent in P_n?

Answer. Yes: If a linear combination $p = a_0 + a_1 x + \cdots + a_n x^n$ in $\{1, x, \dots, x^n\}$ is the zero polynomial, then

$$a_0 + a_1 r + \cdots + a_n r^n = 0 \quad \text{for all } r \in \mathbf{R}.$$

Recall now the basic fact from algebra that a *nonzero polynomial of degree n has at most n roots*. Since p has more than n roots, it must be the zero polynomial. Therefore $a_0 = \cdots = a_n = 0$. So the set is linearly independent.

Example 4.2.19. Are $1 + x$, $-1 + x$, $4 - x^2$, $2 + x^3$ linearly independent in P_3?

Answer. If a linear combination in these polynomials is the zero polynomial, then

$$c_1(1 + x) + c_2(-1 + x) + c_3(4 - x^2) + c_4(2 + x^3) = 0 \Rightarrow$$
$$(c_1 - c_2 + 4c_3 + 2c_4) + (c_1 + c_2)x + (-c_3)x^2 + c_4 x^3 = 0.$$

Equating coefficients yields

$$c_1 - c_2 + 4c_3 + 2c_4 = 0, \quad c_1 + c_2 = 0, \quad -c_3 = 0, \quad c_4 = 0.$$

Solving this linear system, we get $c_1 = c_2 = c_3 = c_4 = 0$. So the vectors are linearly independent in P_3.

Example 4.2.20. Prove that $\{\cos x, \sin x\}$ is linearly independent in $F(\mathbf{R})$ (Figure 4.5).

Solution. If a linear combination $c_1 \cos x + c_2 \sin x$ is the zero function, then

$$c_1 \cos x + c_2 \sin x = 0 \quad \text{for all } x \in \mathbf{R}.$$

If $x = 0$, then

$$c_1 \cos 0 + c_2 \sin 0 = 0 \Rightarrow c_1 = 0.$$

If $x = \frac{\pi}{2}$, then

$$c_1 \cos\frac{\pi}{2} + c_2 \sin\frac{\pi}{2} = 0 \Rightarrow c_2 = 0.$$

Thus $\{\cos x, \sin x\}$ is linearly independent in $F(\mathbf{R})$.

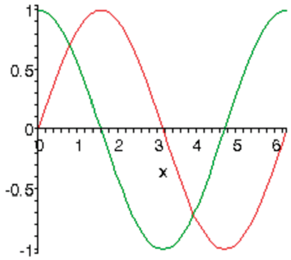

Figure 4.5: $\{\cos x, \sin x\}$ is linearly independent in $F(\mathbf{R})$.

Example 4.2.21. The following sets are linearly independent. The verification of linear independence is left as an exercise.

(a) $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \subseteq \mathbf{R}^n$.

(b) $\{E_{11}, E_{12}, \ldots, E_{ij}, \ldots, E_{mn}\} \subseteq M_{mn}$.

Theorem 4.2.22. *Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a linearly independent subset of a vector space V, and let $\mathbf{v}$ be a vector of V.*

1. *If $\mathbf{v} \in \operatorname{Span}(S)$, then $\mathbf{v}$ is a linear combination of vectors in S with unique coefficients. This means that the relations*

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v} = d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n$$

   imply

$$c_1 = d_1, \quad c_2 = d_2, \quad \ldots, \quad c_n = d_n.$$

2. *If $\mathbf{v} \notin \operatorname{Span}(S)$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}\}$ is linearly independent.*

Proof. See the proof of Theorem 2.4.8. ☐

4.2.4 Linear dependence for sequences

Sometimes, we need to work with *sequences* or *lists* of vectors. These are not sets. A sequence is *ordered* and can contain *duplicates*. In a set, on the other hand, all elements are distinct. Also, the order of the elements in a set is irrelevant.

For example, $1, 2, 3$ and $3, 2, 1$ are different sequences, but $\{1, 2, 3\}$ and $\{3, 2, 1\}$ represent the same set. Likewise, $1, 2, 1$ and $1, 2$ are different sequences, whereas the only set that contains these elements is $\{1, 2\}$.

If we work with sequences and need the concepts of linear dependence or independence, then one issue that may come up is when the sequence has duplicates.

A *sequence* of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is called *linearly dependent* if there are scalars $c_1, \dots, c_k$ not all zero such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

If this sequence is not linearly dependent, it is called *linearly independent*.

According to these definitions, *a sequence of vectors that contains duplicates is always linearly dependent*. For example, the sequence $\mathbf{u}, \mathbf{v}, \mathbf{u}$ is linearly dependent because $1\mathbf{u} + 0\mathbf{v} + (-1)\mathbf{u} = \mathbf{0}$. If, however, we were examining the set $\{\mathbf{u}, \mathbf{v}, \mathbf{u}\}$, which is the same as the set $\{\mathbf{u}, \mathbf{v}\}$, we would not have the same linear dependence relation. For instance, if the two vectors are not multiples of each other, then the *set* is linearly independent.

This problem may appear when, for example, the columns of a matrix are checked for linear dependence. If there are repeated columns, then the columns are a linearly dependent sequence. Usually, it is clear whether we are working with sequences or with sets of vectors, and this subtle difference is not problematic.

Exercises 4.2

Span

In Exercises 1–2, let p_1, p_2, p_3, and p_4 be in P_2, where

$$p_1 = 1 + 3x - x^2, \quad p_2 = -3x + 2x^2, \quad p_3 = x^2, \quad p_4 = 4 - x^2.$$

1. Answer the questions.
(a) Is p_4 in $\mathrm{Span}\{p_1, p_2, p_3\}$?
(b) Is p_4 in $\mathrm{Span}\{p_1, p_2\}$?
(c) Is p_4 in $\mathrm{Span}\{p_2, p_3\}$?

2. Prove that
(a) $\mathrm{Span}\{p_1, p_2, p_3\} = P_2$;
(b) $\mathrm{Span}\{p_1, p_2, p_4\} = P_2$;
(c) $\mathrm{Span}\{p_1, p_3, p_4\} = \mathrm{Span}\{p_2, p_3, p_4\}$.

In Exercises 3–7, determine whether the set spans P_2.

3. $\{1 + x + x^2,\ 1 + 2x + x^2,\ x\}$.

4. $\{1 - x + x^2,\ 1 + x - x^2,\ 1\}$.

5. $\{1 + x, -1 + x, 2 + x + x^2\}$.

6. $\{1 + x + x^2,\ 1 + x,\ 1\}$.

7. $\{-1+x+x^2,\ 1-x+x^2,\ 1+x-x^2\}$.

8. Does $\{4, 1+x, -1+x^2, -1+x^3\}$ span P_3?

In Exercises 9–13, determine whether or not the set spans M_{22}.

9. $\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 0 & 5 \end{bmatrix}\right\}$

10. $\left\{\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 4 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}\right\}$

11. $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ -1 & 1 \end{bmatrix}\right\}$.

12. $\{E_{11}, E_{12}, -E_{21}, E_{11}+E_{12}\}$.

13. $\{E_{11}, E_{11}+E_{12}, E_{11}+E_{12}+E_{21}, E_{11}+E_{12}+E_{21}+E_{22}\}$.

In Exercises 14–17, prove each statement.

14. $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ spans $\mathbf{R}^n$.

15. $\{1, x, x^2, \ldots, x^n\}$ spans P_n.

16. $\{E_{11}, E_{12}, \ldots, E_{ij}, \ldots, E_{mn}\}$ spans M_{mn}.

17. $\{E_{11}, E_{22}, \ldots, E_{ii}, \ldots, E_{nn}\}$ spans D_n.

18. Prove that in P, we have

$$\mathrm{Span}\,\{1+x, 1-x\} = \mathrm{Span}\,\{1-2x, 5x\}\,.$$

19. What restriction(s) on a and b ensure that the set $S = \{a+ax+ax^2, bx^2, 1\}$ spans P_2?

20. *True or False?* Explain.
(a) P_9 can be spanned by nine polynomials in it.
(b) P_9 can be spanned by ten polynomials from P_8.
(c) Any twenty polynomials of P_9 span P_9.
(d) Twenty polynomials of P_9 can span P_9.

21. Let V be a vector space. If $\mathbf{u}$, $\mathbf{v}$ are in V, then prove that

$$\mathrm{Span}\,\{\mathbf{u}, \mathbf{v}\} = \mathrm{Span}\,\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}\,.$$

22. Provide the details of the proof of Theorem 4.2.10.

Linear independence

In Exercises 23–26, determine whether or not the set is linearly independent.

23. $\{-2x+x^2, 1+x+x^2, 1-x\}$.

24. $\{1+ax+ax^2, 1+bx+bx^2, 1\}$.

25. $\{1+ax+ax^2, 1+bx+bx^2, x^2\}$ for unequal constants a and b.

26. The set of Exercise 11.

27. For which values of a is the set $\{1 + ax, a + (a + 2)x\} \subseteq P_1$ linearly dependent?

28. *True or False?* Explain.

(a) Any two distinct vectors of a vector space are linearly independent.

(b) Any two distinct polynomials of P_1 span P_1.

(c) Any two linearly independent polynomials of P_1 span P_1.

29. Prove that the subset

$$\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_1\}$$

of a vector space is linearly dependent.

30. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly independent subset of a vector space V. Prove that

$$\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}$$

is also linearly independent.

31. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent subset of a vector space V, then so is

$$\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 + \mathbf{v}_1\}.$$

32. In a vector space V, let

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + d_3\mathbf{v}_3$$

and $c_3 \neq d_3$. Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

33. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly independent subset of a vector space V. Find the scalars c_1, c_2, and c_3 if

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

and

$$\mathbf{v} = (2c_2 - c_1)\mathbf{v}_1 + (c_3 - c_2)\mathbf{v}_2 + (c_2 - 1)\mathbf{v}_3.$$

34. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a linearly dependent subset of a vector space V. Prove that for any scalar c, the set $\{c\mathbf{v}_1, \dots, c\mathbf{v}_k\}$ is also linearly dependent.

35. Let V be a vector space, and let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a linearly independent subset of V. Prove that any nonempty subset of S is also linearly independent.

36. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be a subset of a vector space V that contains a linearly dependent subset. Prove that S is also linearly dependent.

37. Provide the details of the proof of Theorem 4.2.15.

38. Provide the details of the proof of Theorem 4.2.22.

39. Let p, q, r be polynomials P_2. Suppose that $\{p, q\}$ and $\{q, r\}$ are linearly independent sets. Does this imply that $\{p, r\}$ is linearly independent? Explain.

40. Prove that

$$\{E_{11}, E_{12}, E_{13}, \dots, E_{mn}\}$$

is linearly independent in M_{mn}.

41. The sum $V_1 + V_2$ of two subspaces V_1 and V_2 of a vector space V was defined in Exercise 41, Section 4.2. If $V_1 = \mathrm{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ and $V_2 = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, then prove that

$$V_1 + V_2 = \mathrm{Span}\,\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_n\}\,.$$

42. The direct $V_1 \oplus V_2$ of two subspaces V_1 and V_2 of a vector space V was defined in Exercise 42, Section 4.2. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ be linearly independent in V_1, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be linearly independent in V_2. Prove the set

$$\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_n\}$$

is linearly independent in $V_1 \oplus V_2$. (*Hint:* Use Part (a) of Exercise 42, Section 4.2.)

43. Let f, g have the graphs in Figure 4.6. Explain why $\{f, g\}$ must be linearly independent in $F[0, 2\pi]$.

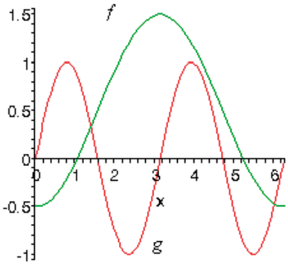

Figure 4.6: Linear independence of functions.

4.3 Basis, dimension

In this section, we discuss the fundamental concept of a *basis* of a vector space. Bases are spanning sets that are also linearly independent. Knowing a basis of a vector space can be useful in understanding the space and its properties. We also discuss the concept of *dimension*.

4.3.1 Basis of a vector space

Definition 4.3.1. A subset $\mathcal{B}$ of a nonzero vector space V is called a *basis* of V if

1. $\mathcal{B}$ is linearly independent, and
2. $\mathcal{B}$ spans V.

We also define as a basis of the zero vector space $\{\mathbf{0}\}$ the empty set $\emptyset$.

Let us express the definition of basis in concrete terms.

Linear independence For every finite subset $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of $\mathcal{B}$, if $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$, then $c_1 = 0, \dots, c_n = 0$.

Spanning For every vector $\mathbf{v}$ in V, we can choose scalars $c_1, \dots, c_n$ and vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in $\mathcal{B}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$.

A vector space that has a finite basis is called *finite dimensional*. In this case the finite subset can be taken to be $\mathcal{B}$ in the above definition. If a vector space does not have a finite basis, it is called *infinite dimensional*.

i Our main interest is on finite-dimensional vector spaces.

Here are some examples of bases. All the sets already seen were linearly independent and spanning.

Example 4.3.2. The standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ in $\mathbf{R}^n$ form a basis of $\mathbf{R}^n$.

Example 4.3.3. $\{1, x, x^2, \dots, x^n\}$ is a basis of P_n, called the *standard basis* of P_n (Figure 4.7).

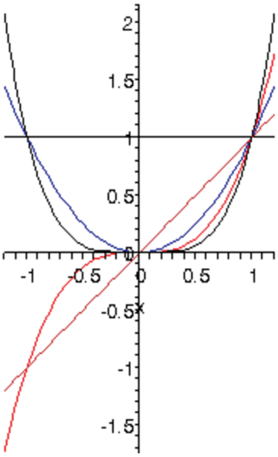

Figure 4.7: The standard basis $\{1, x, x^2, x^3, x^4\}$ of P_4.

Example 4.3.4. $\{E_{11}, E_{12}, \dots, E_{ij}, \dots, E_{mn}\}$ is a basis of M_{mn}, called the *standard basis* of M_{mn}.

Example 4.3.5. Prove that $\mathcal{B} = \{1 + x, -1 + x, x^2\}$ is a basis of P_2.

Solution.

(a) To prove that $\mathcal{B}$ spans P_2, we want every polynomial $p = a + bx + cx^2$ to be a linear combination in $\mathcal{B}$. So we look for scalars c_1, c_2, c_3 such that

$$c_1(1+x) + c_2(-1+x) + c_3x^2 = a + bx + cx^2 \Rightarrow$$
$$(c_1 - c_2) + (c_1 + c_2)x + c_3x^2 = a + bx + cx^2,$$

which leads to the system $c_1 - c_2 = a$, $c_1 + c_2 = b$, $c_3 = c$. We have

$$\begin{bmatrix} 1 & -1 & 0 & a \\ 1 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}a + \frac{1}{2}b \\ 0 & 1 & 0 & -\frac{1}{2}a + \frac{1}{2}b \\ 0 & 0 & 1 & c \end{bmatrix},$$

so the system is consistent for all choices of a, b, c. Thus $\mathcal{B}$ spans P_2.

(b) To prove that $\mathcal{B}$ is linearly independent, let c_1, c_2, c_3 be such that

$$c_1(1+x) + c_2(-1+x) + c_3x^2 = \mathbf{0} \Rightarrow$$
$$(c_1 - c_2) + (c_1 + c_2)x + c_3x^2 = \mathbf{0}.$$

Hence we have the system $c_1 - c_2 = 0$, $c_1 + c_2 = 0$, $c_3 = 0$. Now

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

So the system has only the trivial solution $c_1 = c_2 = c_3 = 0$. Thus $\mathcal{B}$ is linearly independent.

Example 4.3.6. Consider the set $W = \{c_1 \cos x + c_1 \sin x,\ c_1, c_2 \in \mathbf{R}\}$. Prove the following statements:

(a) W is a vector space;

(b) $\{\cos x, \sin x\}$ is a basis of W.

Solution. W is the span of $\{\cos x, \sin x\}$ in $F(\mathbf{R})$, and hence it is a subspace of $F(\mathbf{R})$ by Theorem 4.2.9. In particular, W is a vector space. In Example 4.2.20, we showed that $\{\cos x, \sin x\}$ is linearly independent in $F(\mathbf{R})$. Therefore it is linearly independent in the smaller vector space W. So it is a basis of W. The vectors $\cos x$, $\sin x$, $2\sin x + 3\cos x$, $3\cos x - 2\sin x$ of W are shown in Figure 4.8.

Example 4.3.7. $\mathcal{B} = \{1, x, x^2, \dots, x^n, \dots\}$ is a basis for P.

We have already proved that $\mathcal{B}$ in Example 4.3.7 is a spanning set. We let the reader prove linear independence. P is an example of an infinite-dimensional vector space. The vector space $F(R)$ is also infinite dimensional.

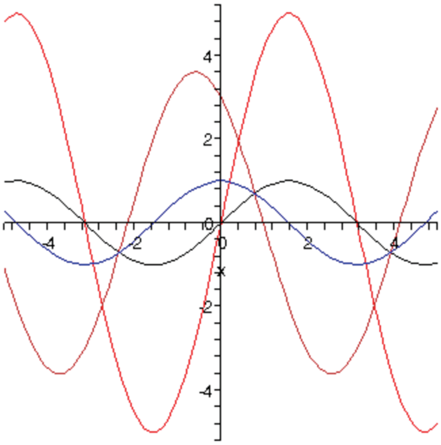

Figure 4.8: Some vectors of Span$\{\cos x, \sin x\}$.

One of the basic theorems of linear algebra, which we state without proof, is the following.

Theorem 4.3.8 (Existence of basis). *Every vector space has a basis.*

i A vector space can have many bases. For example, $\{\mathbf{e}_1, \mathbf{e}_2 - \mathbf{e}_1\}$ is a basis of $\mathbf{R}^2$, which is different from the standard basis.

One of the main characterizations of a basis is described in the following theorem.

Theorem 4.3.9. *A subset $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of a finite-dimensional vector space V is a basis of V if and only if for each vector $\mathbf{v}$ in V, there are unique scalars $c_1, \dots, c_n$ such that*

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

Proof.

$\Rightarrow$ Let $\mathcal{B}$ be a basis of V. Then each vector $\mathbf{v}$ in V is a linear combination $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, because $\mathcal{B}$ spans V. Also, the scalars c_i are unique by Theorem 4.2.22, because $\mathcal{B}$ is linearly independent.

$\Leftarrow$ Assume that each vector $\mathbf{v}$ of V can be written uniquely as a linear combination in $\mathcal{B}$. This implies that $\mathcal{B}$ spans V. To prove that $\mathcal{B}$ is linearly independent, we assume that

$$d_1\mathbf{v}_1 + \cdots + d_n\mathbf{v}_n = \mathbf{0}.$$

However, we also have

$$0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n = \mathbf{0}.$$

Therefore $d_1 = 0, \dots, d_n = 0$ by the uniqueness of the representation of $\mathbf{0}$. Hence $\mathcal{B}$ is linearly independent. Therefore $\mathcal{B}$ is a basis of V. □

4.3.2 Dimension

In this section, we make precise the intuitive notion of *dimension*. The following theorem is crucial in proving that the dimension of a finite-dimensional vector space is a well-defined number. It is due to the German mathematician *Ernst Steinitz* (1871–1928).

Theorem 4.3.10 (The exchange theorem). *If a vector space V is spanned by n vectors, then any subset of V with more than n vectors is linearly dependent. In other words, any linearly independent subset of V has at most n vectors.*

Proof. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be the spanning set, and let $T = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be a linearly independent set. It suffices to prove that $m \leq n$. The set

$$S' = \{\mathbf{u}_m, \mathbf{v}_1, \dots, \mathbf{v}_n\}$$

is linearly dependent by Theorem 4.2.15, because $\mathbf{u}_m$ is a linear combination in S. Now $\mathbf{u}_m \neq \mathbf{0}$, or else, T would be linearly dependent. Hence by the same theorem one of the $\mathbf{v}$s, say $\mathbf{v}_i$, is a linear combination of the preceding vectors. Thus the set

$$S'' = \{\mathbf{u}_m, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n\}$$

formed from S' by deleting $\mathbf{v}_i$ is still a spanning set by Theorem 4.2.10. We now add $\mathbf{u}_{m-1}$ to S'' to get

$$S''' = \{\mathbf{u}_{m-1}, \mathbf{u}_m, \mathbf{v}_1, \dots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \dots, \mathbf{v}_n\}$$

and use the same argument to prove that S''' is spanning. Also, one of the $\mathbf{v}$s is a linear combination of the preceding vectors, so it can be deleted. The $\mathbf{u}$s are not deleted, because none of them is a linear combination of the preceding ones due to their linear independence. We repeat this finite process and see that the $\mathbf{u}$s will be exhausted before the $\mathbf{v}$s. Otherwise, the remaining $\mathbf{u}$s would be linear combinations of the $\mathbf{u}$s that are already included in the set. Therefore $m \leq n$, as stated.[1] □

As a consequence of the exchange theorem, we have the following important theorem.

Theorem 4.3.11. *If a vector space V has a basis with n elements, then every basis of V has n elements (Figure 4.9).*

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[1] The name of the Exchange Theorem comes from its proof, where the spanning vectors were exchanged by linearly independent vectors.

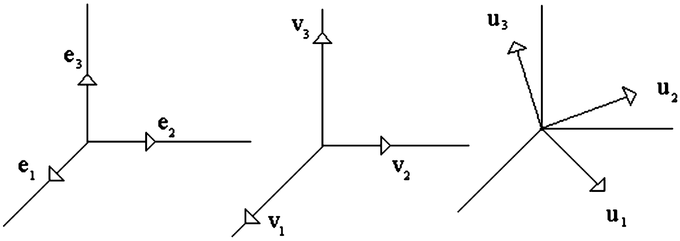
$\mathbf{e}_3$
$\mathbf{e}_2$
$\mathbf{e}_1$
$\mathbf{v}_3$
$\mathbf{v}_2$
$\mathbf{v}_1$
$\mathbf{u}_3$
$\mathbf{u}_2$
$\mathbf{u}_1$

Figure 4.9: All bases in a vector space have the same number of vectors.

Proof. Let $\mathcal{B}$ be a basis with n vectors, and let $\mathcal{B}'$ be another basis. If $\mathcal{B}'$ had more than n elements, then it would be a linearly dependent set by Theorem 4.3.10, because $\mathcal{B}$ is a spanning set. Hence $\mathcal{B}'$ is finite, and if m is its number of elements, then $m \leq n$. By the same argument, with $\mathcal{B}$ and $\mathcal{B}'$ interchanged, we see that $n \leq m$. Therefore $n = m$. □

Definition 4.3.12. If a vector space V has a basis with n elements, then n is called the *dimension* of V. We write

$$\dim(V) = n.$$

By Theorem 4.3.11 *the dimension is a well-defined number and does not depend on the choice of basis*. The dimension of the zero space $\{\mathbf{0}\}$ is *defined* to be zero.

For an infinite-dimensional vector space V, we write $\dim(V) = \infty$.

By counting the number of elements of the standard bases of $\mathbf{R}^n$, P_n, M_{mn} we see that (Figure 4.10)

1. $\dim(\mathbf{R}^n) = n$;
2. $\dim(P_n) = n + 1$;
3. $\dim(M_{mn}) = m \cdot n$.

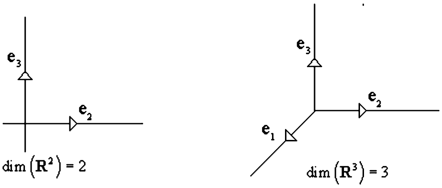
$\mathbf{e}_3$
$\mathbf{e}_2$
$\dim(\mathbf{R}^2) = 2$
$\mathbf{e}_3$
$\mathbf{e}_2$
$\mathbf{e}_1$
$\dim(\mathbf{R}^3) = 3$

Figure 4.10: The dimensions of $\mathbf{R}^2$ and $\mathbf{R}^3$.

Note that because a subspace of a vector space is itself a vector space, it makes sense to talk about the *dimension of a subspace*.

Example 4.3.13. Let

$$V = \{(2x + y + 5z, -x + 4y + 2z, 3x - 5y + z),\ x, y, z \in \mathbf{R}\}.$$

(a) Prove that V is a subspace of $\mathbf{R}^3$.
(b) Find a basis of V.
(c) Find the dimension of V.

Solution.

(a) We have

$$\begin{bmatrix} 2x + y + 5z \\ -x + 4y + 2z \\ 3x - 5y + z \end{bmatrix} = x \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix} + z \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} = x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3.$$

So $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Hence V is a subspace of $\mathbf{R}^3$ by Theorem 4.2.9.

(b) Now $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, because $[\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3]$ has only two pivot columns:

$$\begin{bmatrix} 2 & 1 & 5 \\ -1 & 4 & 2 \\ 3 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

In fact, the reduction shows that $\mathbf{v}_3 = 2\mathbf{v}_1 + \mathbf{v}_2$. So we may drop $\mathbf{v}_3$ and have $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. But now $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, and it spans V. So the set of two vectors

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix} \right\}$$

is a basis of V.

(c) The dimension of V is 2, because V has a basis with two elements.

The next theorem is labor saving. If a set S has as many vectors as the dimension of the vector space, then either assumption of linear independence or of a spanning set implies the other (Figure 4.11).

Theorem 4.3.14. *Let S be a subset of a finite-dimensional vector space V.*

1. *If S is linearly independent and has as many vectors as the dimension of V, then S is a basis of V.*
2. *If S spans V and has as many vectors as the dimension of V, then S is a basis of V.*

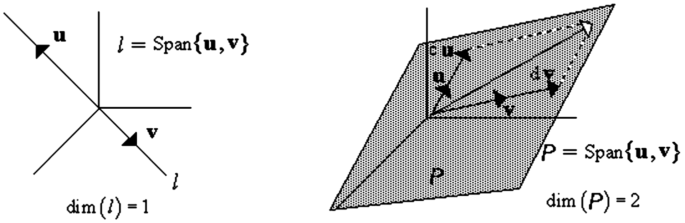

Figure 4.11: Dimensions of spans.

Proof. Let $\dim(V) = n$, and let $\mathcal{B}$ be a basis of V.

1. Suppose S is linearly independent. If S does not span V, then there is a vector $\mathbf{v}$ not in $\mathrm{Span}(S)$. Let S' be the set obtained by adding $\mathbf{v}$ to S. Then S' is linearly independent and has $n+1$ vectors. This contradicts the Exchange theorem, because $\mathcal{B}$ is a spanning set with fewer vectors than the linearly independent set S'. So S must span V. Thus S is a basis of V.
2. Let S be a spanning set. If S is linearly dependent, then one vector $\mathbf{v}$ in S is a linear combination of the remaining. Let S_1 be the set obtained from S by deleting $\mathbf{v}$. Then S_1 is a spanning set of V and has $n - 1$ elements. This contradicts the Exchange theorem, because $\mathcal{B}$ is a linearly independent set with more vectors than the spanning set S_1. So S is linearly dependent. Thus S is a basis of V. □

Example 4.3.15. Is $S = \{1 + x, -1 + x, 4 - x^2, 2 + x^3\}$ a basis of P_3?

Solution. S is linearly independent by Example 4.2.19. Furthermore, S has four elements, the same number as the dimension of P_3. Therefore S is a basis of P_3 by Part 1 of Theorem 4.3.14.

The next theorem claims that we can obtain a basis by either adding elements to a linearly independent set or by deleting elements from a spanning set.

Theorem 4.3.16. *Let V be an n-dimensional vector space, and let S be a set with m elements.*

1. *If S is linearly independent, then $m \le n$, and S can be enlarged to a basis.*
2. *If S spans V, then $m \ge n$, and S contains a basis.*

Proof of 1. If S is linearly independent, then $m \le n$ by the exchange theorem. If S is not already a spanning set, then we add a basis to it. The new set is spanning and linearly dependent. Hence one of its vectors that is not in S is a linear combination of the preceding. So we can drop this vector and still have a spanning set. We repeat this step until we have a linearly independent set. This set is a basis of V containing S. □

Example 4.3.17. If possible, find a basis of P_3 that contains the set

$$S = \{-1 + x^2, 3 - 2x\}.$$

Solution. It is possible to extend S to a basis, because it is linearly independent. We enlarge S to S' by adding the standard basis of P_3:

$$S' = \{-1 + x^2, 3 - 2x, 1, x, x^2, x^3\}.$$

Then S' spans P_3 and is linearly dependent. Hence one element after $3 - 2x$ is a linear combination of the preceding ones. The vector 1 is not a linear combination of the preceding (check), so we keep it. Both x and x^2 are linear combinations of $-1 + x^2, 3 - 2x, 1$ (check), so we drop them from S'. Lastly, x^3 is not a linear combination of $-1+x^2, 3-2x, 1$, so we keep it. Thus $\{-1+x^2, 3-2x, 1, x^3\}$ is linearly independent and still spans P_3. Hence it is a basis that contains S.

Example 4.3.18. Let S be a set of 10 vectors in $\mathbf{R}^k$. What can we say about k if S (a) is linearly independent? (b) spans $\mathbf{R}^k$? (c) is a basis of $\mathbf{R}^k$?

Solution. According to Theorem 4.3.16, we must have (a) $k \geq 10$, (b) $k \leq 10$, and (c) $k = 10$.

A *maximal* linearly independent set is a set such that if we add a vector to it, then the new set is no longer linearly independent. A *minimal* spanning set is a set such that if we take out one vector, then the new set is no longer a spanning set.

The Exchange theorem also implies the following:

1. *A basis is a maximal linearly independent set*;
2. *A basis is a minimal spanning set.*

Theorem 4.3.19. *Let W be a subspace of an n-dimensional vector space V. Then*

1. $\dim(W) \leq n$;
2. $\dim(W) = n$ *if and only if* $W = V$.

Proof. Exercise. ☐

4.3.3 Ordered bases

We often need the concept of an *ordered basis*, a basis of a vector space where the order of vectors matters. For example, although the bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{e}_2, \mathbf{e}_1\}$ of $\mathbf{R}^2$ define the same *set* and therefore the same basis, when we think of them as ordered bases, they are different. We may change the notation of a set, say $\{\mathbf{v}_1, \dots, \mathbf{v}_n, \dots\}$, to a different notation, say $\{\mathbf{v}_1, \dots, \mathbf{v}_n, \dots\}_O$, to avoid confusion. However, the context in which ordered bases are used will be clear, so additional notation is not really necessary.

The concept of ordered basis is useful in Section 4.4.

Exercises 4.3

In Exercises 1–3, determine whether the given sets are bases of M_{22}.

1. $\left\{\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 7 & 8 \\ 7 & 8 \end{bmatrix}\right\}$.

2. $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}$.

3. $\left\{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\right\}$.

4. Let $V \subseteq M_{22}$ denote the set of all matrices of the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$. Prove that $\mathcal{B} = \{E_{11} - E_{22}, E_{12}, E_{21}\}$ is a basis for V.

5. Prove that $\{x^2, (1 + x)^2, (-1 + x)^2\}$ is a basis of P_2.

6. Prove that for any real number c, the set $\{1, x + c, (x + c)^2\}$ is a basis of P_2.

7. Prove that $\{1, x, 2x^2, 3 - 3x + x^3\}$ is a basis of P_3. (These are the first four *Tchebysheff polynomials of the first kind*.)

In Exercises 8–13, let V be the *span* of the given set.
(a) Find a basis for V.
(b) Check whether or not $V = P_2$.

8. $\{1 + x + x^2, 1, -1 - x^2, x^2\}$.

9. $\{2 + x + 2x^2, x^2, 1 - x - x^2, 1\}$.

10. $\{x + x^2, 1 + x, -1 + x^2\}$

11. $\{x^2, 1 + x, -1 + x^2\}$.

12. $\{1 - x - 5x^2, 7 + x + 4x^2, 8 - x^2\}$.

13. $\{-x + x^2, -5 + x, -x^2, 3 + x^2\}$.

In Exercises 14–19, extend the given linearly independent set to a basis of P_2.

14. $\{1 + x + x^2, 1\}$.

15. $\{-x + x^2, x + x^2\}$.

16. $\{x + x^2, 1 + x\}$.

17. $\{1 + x, -1 + x^2\}$.

18. $\{1 - x + x^2, 2 - x^2\}$.

19. $\{-x + x^2, -5 + x\}$.

20. *True or False?* If

$$P_2 = \text{Span}\,\{p_1, p_2, p_3\},$$

then $\{p_1, p_2, p_3\}$ is a basis of P_2.

21. *True or False?* If

$$M_{22} = \text{Span}\{A_1, A_2, A_3, A_4\},$$

then $\{A_1, A_2, A_3, A_4\}$ is a basis of M_{22}.

22. Prove that the rows of an $n \times n$ invertible matrix form a basis of $\mathbf{R}^n$.

In Exercises 23–28, find the dimension of V.

23. $V = \left\{ \begin{bmatrix} a-b \\ 2a+b \end{bmatrix}, a, b \in \mathbf{R} \right\}$.

24. $V = \left\{ \begin{bmatrix} a \\ 0 \\ -2a \end{bmatrix}, a \in \mathbf{R} \right\}$.

25. $V = \left\{ \begin{bmatrix} a-c \\ b+c \\ 5c \end{bmatrix}, a, b, c \in \mathbf{R} \right\}$.

26. V is the set of all 3-vectors with first component zero.

27. V is the set of all 4-vectors with first and last components zero.

28. V is the set of all 4-vectors whose sum of components is zero.

In Exercises 29–31, determine the dimension of the span of the given sets in M_{22}.

29. $\left\{ \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \right\}$.

30. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$.

31. $\left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} \right\}$.

32. Find the dimension of the vector space V of all matrices of the form $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.

In Exercises 33–36, find the dimension of $V \subseteq P_2$, where V is the *span* of the given set.

33. $\{1 + x + x^2, 1, -1 - x^2, x^2\}$.

34. $\{x^2, 1 + x, -1 + x^2\}$.

35. $\{x + x^2, 1 + x, -1 + x^2\}$.

36. $\{1 - x - 5x^2, 7 + x + 4x^2, 8 - x^2\}$.

37. *True or False?* Explain. $\mathbf{R}^{10}$

(a) has a basis with 11 elements.
(b) has a basis with 10 elements.
(c) has a 9-dimensional subspace.
(d) has only one 9-dimensional subspace.
(e) is the only 10-dimensional subspace of $\mathbf{R}^{10}$.

38. *True or False?* Explain. Let V be a nonzero subspace of $\mathbf{R}^{10}$. Then V may have
(a) two distinct bases,
(b) two bases with different number of elements,
(c) a basis with 10 elements,
(d) exactly 10 elements,
(e) a basis with 11 elements,
(f) a basis with 9 elements.

39. Let f, g have the shown graphs in Figure 4.12. Explain why the dimension of $\operatorname{Span}\{f, g\}$ is 1.

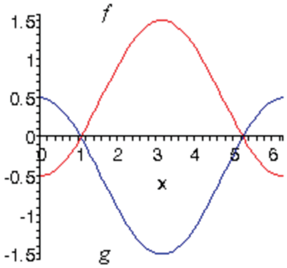

Figure 4.12: Dimension for the span.

In Exercises 40–42, find the dimension of $V \subseteq F(R)$.

40. $V = \operatorname{Span}\{e^x, e^{2x}, 2e^x\}$.

41. $V = \operatorname{Span}\{\cos(x)\sin(x), \sin(2x)\}$.

42. $V = \operatorname{Span}\{\cos^2(x), \sin^2(x), 1\}$.

43. Find the dimension of the solution space of $x_1 + x_2 + \cdots + x_n = 0$.

44. Prove Part 2 of Theorem 4.3.16.

45. Prove Theorem 4.3.19.

In Exercises 46–47, let V_1 and V_2 be vector subspaces of a finite-dimensional vector space V.

46. (*Grassmann's relation*) For the sum vector subspace $V_1 + V_2$, defined in Exercise 41, Section 4.1, prove that

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

47. For the direct sum vector subspace $V_1 \oplus V_2$, defined in Exercise 42, Section 4.1, prove that

$$\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2).$$

48. For the Cartesian product vector space $V_1 \times V_2$ of the finite-dimensional vector spaces V_1 and V_2, defined in Exercise 44, Section 4.1, prove that

$$\dim(V_1 \times V_2) = \dim(V_1)\dim(V_2).$$

49. Use Exercise 48 to find
(a) $\dim(\mathbf{R}^m \times \mathbf{R}^n)$,
(a) $\dim(P_m \times P_n)$,
(a) $\dim(M_{m,n} \times M_{k,r})$.

4.4 Coordinates, change of basis

Many physics and engineering problems can be greatly simplified by choosing the right coordinate system. Likewise, vector space problems can be simplified by choosing the right basis. Choosing the right basis is crucial in image compression and in data storage. Here we study how the components of vectors are affected when we change from one basis to another.

In this section, all bases are ordered.

4.4.1 Coordinate vectors

Definition 4.4.1. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis an n-dimensional vector space V. By Theorem 4.3.9, for each $\mathbf{v} \in V$, there exist unique scalars $c_1, \ldots, c_n$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

The vector with components the coefficients of $\mathbf{v}$, written as $[\mathbf{v}]_{\mathcal{B}}$, is called the *coordinate vector of* $\mathbf{v}$ *with respect to* $\mathcal{B}$:

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Note that $[\mathbf{v}]_{\mathcal{B}}$ changes as the basis $\mathcal{B}$ changes (Figure 4.13).

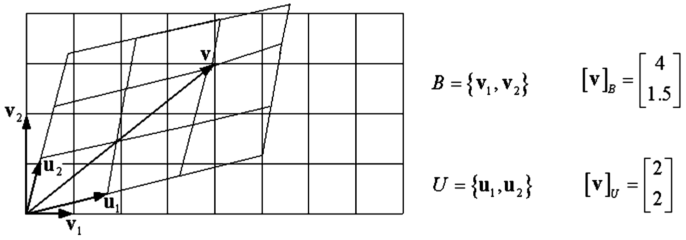

Figure 4.13: Coordinates with respect to different bases.

If $\mathbf{a} = [\, a_1 \cdots a_n \,]^T$ is any n-vector and $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis of $\mathbf{R}^n$, then $\mathbf{a} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$. Hence

$$[\mathbf{a}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{a}.$$

Example 4.4.2. Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $\mathcal{U} = \{\mathbf{e}_2, \mathbf{e}_1\}$ in $\mathbf{R}^2$, and let $\mathbf{a} = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$. Compute and compare $[\mathbf{a}]_{\mathcal{B}}$ and $[\mathbf{a}]_{\mathcal{U}}$.

Solution. We have $a = 1\mathbf{e}_1 + 2\mathbf{e}_2 = 2\mathbf{e}_2 + 1\mathbf{e}_1$. Therefore

$$[\mathbf{a}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [\mathbf{a}]_{\mathcal{U}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

So $[\mathbf{a}]_{\mathcal{B}} \neq [\mathbf{a}]_{\mathcal{U}}$.

Example 4.4.2 shows that $[\mathbf{v}]_{\mathcal{B}}$ also depends on the *order* of elements of $\mathcal{B}$. This is the reason we use *ordered bases*.

Example 4.4.3. Consider the following basis $\mathcal{B}$ on $\mathbf{R}^3$ and vector $\mathbf{v}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

(a) Find $[\mathbf{v}]_{\mathcal{B}}$.

(b) Find the vector $\mathbf{w}$ if $[\mathbf{w}]_{\mathcal{B}} = \left[\begin{smallmatrix} 6 \\ -3 \\ 2 \end{smallmatrix}\right]$.

Solution.

(a) We write $\mathbf{v}$ as a linear combination in $\mathcal{B}$:

$$\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

We solve this linear system to get $c_1 = -3$, $c_2 = -4$, $c_3 = 1$, i. e.,

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}.$$

(b) The components of $[\mathbf{w}]_{\mathcal{B}}$ are $6, -3, 2$, so $\mathbf{w}$ is given by

$$\mathbf{w} = 6\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \\ -4 \end{bmatrix}.$$

Example 4.4.4. Find $[-6 + x + 4x^2]_{\mathcal{U}}$, where $\mathcal{U}$ is the following basis of P_2:

$$\mathcal{U} = \{1 + x, 1 - x^2, 1 + x + x^2\}.$$

Solution. We seek scalars c_1, c_2, c_3 such that

$$\begin{aligned} -6 + x + 4x^2 &= c_1(1 + x) + c_2(1 - x^2) + c_3(1 + x + x^2) \Rightarrow \\ -6 + x + 4x^2 &= (c_1 + c_2 + c_3) + (c_1 + c_3)x + (-c_2 + c_3)x^2. \end{aligned}$$

So $-6 = c_1 + c_2 + c_3$, $1 = c_1 + c_3$, $4 = -c_2 + c_3$. We solve this system to get $c_1 = 4$, $c_2 = -7$, $c_3 = -3$. Therefore

$$[-6 + x + 4x^2]_{\mathcal{U}} = \begin{bmatrix} 4 \\ -7 \\ -3 \end{bmatrix}.$$

If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of a finite-dimensional vector space V, then

$$[\mathbf{v}_i]_{\mathcal{B}} = \mathbf{e}_i, \quad i = 1, \dots, n.$$

Theorem 4.4.5. *Let $\mathcal{B}$ be a basis of an n-dimensional vector space V. A vector $\mathbf{u}$ is a linear combination of vectors $\mathbf{u}_1, \dots, \mathbf{u}_m$ in V if and only if $[\mathbf{u}]_{\mathcal{B}}$ is a linear combination of $[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_m]_{\mathcal{B}}$ in $\mathbf{R}^n$. More precisely,*

$$\mathbf{u} = c_1\,\mathbf{u}_1 + \cdots + c_m\,\mathbf{u}_m \Leftrightarrow [\mathbf{u}]_{\mathcal{B}} = c_1\,[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_m\,[\mathbf{u}_m]_{\mathcal{B}}. \tag{4.2}$$

Proof. Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, and let

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad [\mathbf{u}_i]_{\mathcal{B}} = \begin{bmatrix} u_{i1} \\ \vdots \\ u_{in} \end{bmatrix}, \quad i = 1, \dots, m.$$

Assuming that $\mathbf{u} = c_1\,\mathbf{u}_1 + \cdots + c_m\,\mathbf{u}_m$, we have

$$\begin{aligned} \mathbf{u} &= c_1(u_{11}\mathbf{v}_1 + \cdots + u_{1n}\mathbf{v}_n) + \cdots + c_m(u_{m1}\mathbf{v}_1 + \cdots + u_{mn}\mathbf{v}_n) \\ &= (c_1u_{11} + \cdots + c_mu_{m1})\mathbf{v}_1 + \cdots + (c_1u_{1n} + \cdots + c_mu_{mn})\mathbf{v}_n. \end{aligned}$$

Hence

$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} c_1 u_{11} + \cdots + c_m u_{m1} \\ \vdots \\ c_1 u_{1n} + \cdots + c_m u_{mn} \end{bmatrix} = c_1 \begin{bmatrix} u_{11} \\ \vdots \\ u_{1n} \end{bmatrix} + \cdots + c_m \begin{bmatrix} u_{m1} \\ \vdots \\ u_{mn} \end{bmatrix}$$
$$= c_1 \, [\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_m \, [\mathbf{u}_m]_{\mathcal{B}} \, .$$

All steps can be reversed, so we have established equivalence (4.2). □

Theorem 4.4.5 has the following useful consequence.

Theorem 4.4.6. *Let $\mathcal{B}$ be a basis of an n-dimensional vector space V. Then the set $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is linearly independent in V if and only if the set $\{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_m]_{\mathcal{B}}\}$ is linearly independent in $\mathbf{R}^n$.*

Proof. Exercise. □

4.4.2 Change of basis

Let $\mathbf{v}$ be a vector of a n-dimensional vector space V, and let $\mathcal{B}$ and $\mathcal{U}$ be two bases. We want to find a relation between $[\mathbf{v}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{U}}$. The answer is given in the following theorem.

Theorem 4.4.7 (Change of basis). *Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\mathcal{U}$ be two bases of an n-dimensional vector space V. Let P be the $n \times n$ matrix with columns $[\mathbf{v}_i]_{\mathcal{U}}$,*

$$P = [\, [\mathbf{v}_1]_{\mathcal{U}} \, [\mathbf{v}_2]_{\mathcal{U}} \cdots [\mathbf{v}_n]_{\mathcal{U}} \,].$$

Then P is the only matrix such that for all $\mathbf{v} \in V$,

$$[\mathbf{v}]_{\mathcal{U}} = P \, [\mathbf{v}]_{\mathcal{B}} \, .$$

Proof. Because $\mathcal{B}$ is a basis, for each $\mathbf{v}$ in V, there are scalars c_i such that

$$\mathbf{v} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n.$$

Therefore by Theorem 4.4.5 applied to basis $\mathcal{U}$ we have

$$[\mathbf{v}]_{\mathcal{U}} = c_1 \, [\mathbf{v}_1]_{\mathcal{U}} + \cdots + c_n \, [\mathbf{v}_n]_{\mathcal{U}}$$
$$\Rightarrow [\mathbf{v}]_{\mathcal{U}} = P \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = P \, [\mathbf{v}]_{\mathcal{B}} \, .$$

Hence $[\mathbf{v}]_{\mathcal{U}} = P[\mathbf{v}]_{\mathcal{B}}$, as claimed. If P' also satisfies $[\mathbf{v}]_{\mathcal{U}} = P'[\mathbf{v}]_{\mathcal{B}}$ for all $\mathbf{v}$ in V, then for $\mathbf{v} = \mathbf{v}_i$, we have

$$[\mathbf{v}_i]_{\mathcal{U}} = P\,[\mathbf{v}_i]_{\mathcal{B}} = P\mathbf{e}_i \quad \text{and} \quad [\mathbf{v}_i]_{\mathcal{U}} = P'\,[\mathbf{v}_i]_{\mathcal{B}} = P'\mathbf{e}_i.$$

Hence $P\mathbf{e}_i = P'\mathbf{e}_i$, so P and P' have the same columns. Therefore $P = P'$. So P is the only matrix with the property $[\mathbf{v}]_{\mathcal{U}} = P[\mathbf{v}]_{\mathcal{B}}$ for all $\mathbf{v}$ in V. ☐

Definition 4.4.8. The matrix P of Theorem 4.4.7 is called the *transition matrix*, or *change of basis matrix* from $\mathcal{B}$ to $\mathcal{U}$.

Theorem 4.4.9. *Let $\mathcal{B}$ and $\mathcal{U}$ be two bases of an n-dimensional vector space V, and let P be the transition matrix from $\mathcal{B}$ to $\mathcal{U}$. Then*

1. *P is invertible;*
2. *P^{-1} is the transition matrix from $\mathcal{U}$ to $\mathcal{B}$.*

Proof. Let Q be transition matrix from $\mathcal{U}$ to $\mathcal{B}$. We prove that P^{-1} exists and equals Q. For each $\mathbf{v}$ in V, we have $Q[\mathbf{v}]_{\mathcal{U}} = [\mathbf{v}]_{\mathcal{B}}$. Thus

$$PQ\,[\mathbf{v}]_{\mathcal{U}} = P\,[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{U}} = I\,[\mathbf{v}]_{\mathcal{U}}\,.$$

Therefore $PQ = I$ by the uniqueness of the transition matrix stated in Theorem 4.4.7. Likewise, $QP = I$. So P is invertible, and $Q = P^{-1}$ (Figure 4.14). ☐

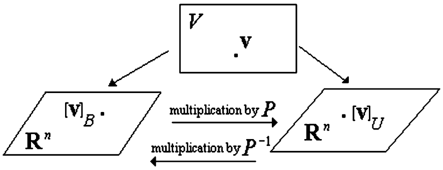

Figure 4.14: The action of the transition matrix and its inverse.

Example 4.4.10. Let $\mathcal{B}$ and $\mathcal{U}$ be the following bases of $\mathbf{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad \mathcal{U} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

(a) Find the transition matrix P from $\mathcal{B}$ to $\mathcal{U}$.

(b) Verify the relation $[\mathbf{v}]_{\mathcal{U}} = P[\mathbf{v}]_{\mathcal{B}}$ for $\mathbf{v} = [\,4\ -2\ 7\,]^T$.

Solution. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be vectors of $\mathcal{B}$, and let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ be vectors of $\mathcal{U}$.

(a) To find P, we need each $[\mathbf{v}_i]_{\mathcal{U}}$. So we must write each $\mathbf{v}_i$ as a linear combination in $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. We have to solve the following three systems for the coefficients c_{ij}:

$$\begin{aligned}\mathbf{v}_1 &= c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_1 + c_{31}\mathbf{u}_3,\\ \mathbf{v}_2 &= c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_1 + c_{32}\mathbf{u}_3,\\ \mathbf{v}_3 &= c_{13}\mathbf{u}_1 + c_{23}\mathbf{u}_1 + c_{33}\mathbf{u}_3.\end{aligned}$$

All three systems can be solved for c_{ij} in one step by reducing

$$\begin{bmatrix} 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{bmatrix}.$$

The coordinates of $[\mathbf{v}_i]_{\mathcal{U}}$ are $c_{ji}, j = 1, 2, 3$. These are also the columns of P. Therefore

$$P = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}.$$

(b) To find $[\mathbf{v}]_{\mathcal{U}}$, we write $\mathbf{v}$ as a linear combination $\mathbf{v} = k_1\mathbf{u}_1 + k_2\mathbf{u}_1 + k_3\mathbf{u}_3$ and solve the system for k_i to get

$$\begin{bmatrix} 0 & 0 & 1 & 4 \\ 0 & 1 & -1 & -2 \\ 1 & 0 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow [\mathbf{v}]_{\mathcal{U}} = \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}.$$

Similarly, for $[\mathbf{v}]_{\mathcal{B}}$, we have

$$\begin{bmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & 1 & -2 \\ -1 & 0 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \Rightarrow [\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ -5 \\ 3 \end{bmatrix}.$$

Now we check by using P from Part (a) to get

$$P\,[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix} = [\mathbf{v}]_{\mathcal{U}},$$

as expected.

Example 4.4.11. Let $\mathcal{B}$ be the standard basis of $\mathbf{R}^2$, and let $\mathcal{U}$ be the basis obtained by rotating $\mathcal{B}$ counterclockwise by $\pi/4$ radians bout the origin (Figure 4.15).

(a) Find the transition matrix from $\mathcal{B}$ to $\mathcal{U}$.

(b) Use P to find the new coordinates of the vector $\mathbf{v} = [\,1\;1\,]^T$.

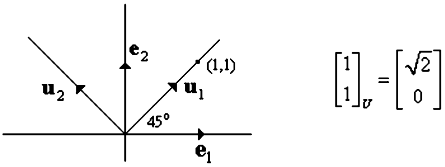
e2
(1,1)
u2
u1
45°
e1
$\begin{bmatrix}1\\1\end{bmatrix}_{\mathcal{U}} = \begin{bmatrix}\sqrt{2}\\0\end{bmatrix}$

Figure 4.15: The coordinates of (1, 1) with respect to 45°rotation.

Solution. Because $\sin(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$, we have

$$\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\} = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$$

(a) We write each $\mathbf{e}_i$ as a linear combination in $\mathbf{u}_1, \mathbf{u}_2$:

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}\mathbf{u}_1 - \frac{1}{\sqrt{2}}\mathbf{u}_2 \quad \text{and} \quad \mathbf{e}_2 = \frac{1}{\sqrt{2}}\mathbf{u}_1 + \frac{1}{\sqrt{2}}\mathbf{u}_2.$$

Thus the transition matrix is

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

(b) The new coordinates of $\mathbf{v}$ are computed by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{U}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}.$$

We see that $\mathbf{v}$ is now a vector along the new x-axis with respect to the new coordinate system.

Exercises 4.4

In Exercises 1–3, find the polynomial p, given a basis $\mathcal{B}$ of P_n and the coordinate vector $[p]_{\mathcal{B}}$.

1. $\mathcal{B} = \{1 + 2x, 5x\}$ and

$$[p]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}.$$

2. $\mathcal{B} = \{1 + x + 2x^2, -x^2, 1 + 2x\}$ and

$$[p]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}.$$

3. $\mathcal{B} = \{2 + 2x, -3 + 3x\}$ and

$$[p]_{\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

In Exercises 4–7, compute the coordinate vector $[p]_{\mathcal{B}}$ for the basis $\mathcal{B}$ of P_n and the polynomial p.

4. $p = 4 + 17x$ and $\mathcal{B} = \{1 + 2x, 1 - x\}$.

5. $p = 17 - 6x$ and

$$\mathcal{B} = \{-7 + 4x, 2 - 3x\}.$$

6. $p = -1 + 6x - 8x^2$ and

$$\mathcal{B} = \left\{1 + 2x + 2x^2, 2x - x^2, -1 - 2x\right\}.$$

7. $p = (a - b) + (7a + 3b)x$ and

$$\mathcal{B} = \{1 + 2x, 5x\}.$$

8. Let $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ be given in Figure 4.16, and let $\mathcal{B}$ be the standard basis of $\mathbf{R}^2$.

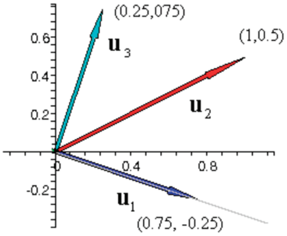

Figure 4.16: Coordinates with respect to a basis.

(a) Find $[\mathbf{u}_1]_{\mathcal{B}}$, $[\mathbf{u}_2]_{\mathcal{B}}$, and $[\mathbf{u}_3]_{\mathcal{B}}$.
(b) Find $[\mathbf{u}_1]_{\{\mathbf{u}_2,\mathbf{u}_3\}}$, $[\mathbf{u}_2]_{\{\mathbf{u}_1,\mathbf{u}_3\}}$, and $[\mathbf{u}_3]_{\{\mathbf{u}_1,\mathbf{u}_2\}}$.

In Exercises 9–10, let $\mathcal{B}$ be the following basis of M_{22}:

$$\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right\}.$$

9. Find the coordinate vector

$$\left[\begin{bmatrix} 4 & -1 \\ -4 & -4 \end{bmatrix}\right]_{\mathcal{B}}.$$

10. Find the matrix M if

$$[M]_{\mathcal{B}} = \begin{bmatrix} 4 & -3 & 8 & 10 \end{bmatrix}^T.$$

11. Prove Theorem 4.4.6.

12. For the 2-vector $\mathbf{v} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, find

(a) $[\mathbf{v}]_{\mathcal{B}}$, where $\mathcal{B}$ is the $\mathbf{R}^2$ basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}.$$

(b) $[\mathbf{v}]_{\mathcal{U}}$, where $\mathcal{U}$ is the $\mathbf{R}^2$ basis

$$\mathcal{U} = \left\{ \begin{bmatrix} 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix} \right\}.$$

In Exercises 13–14, find the transition matrix from basis $\mathcal{B}_1$ to basis $\mathcal{B}_2$.

13. $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}, \ \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}.$

14. $\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}, \ \mathcal{B}_2 = \left\{ \begin{bmatrix} 0 \\ -9 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\}.$

15. Find the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\begin{aligned} &\mathbf{v}_1 = \mathbf{e}_1, \quad \mathbf{v}_2 = \mathbf{e}_2, \quad \mathbf{v}_3 = \mathbf{e}_3, \\ &\mathbf{u}_1 = \mathbf{e}_3, \quad \mathbf{u}_2 = \mathbf{e}_1, \quad \mathbf{u}_3 = \mathbf{e}_2. \end{aligned}$$

In Exercises 16–21, find the transition matrix from the standard basis $\mathcal{B}_1$ of P_3 to the given basis $\mathcal{B}_2$.

16. (*Tchebysheff polynomials of the first kind*)

$$\mathcal{B}_2 = \left\{1, x, -1 + 2x^2, -3x + 4x^3\right\}.$$

17. (*Tchebysheff polynomials of the second kind*)

$$\mathcal{B}_2 = \left\{1, 2x, -1 + 4x^2, -4x + 8x^3\right\}.$$

18. (*Laguerre polynomials*)

$$\mathcal{B}_2 = \left\{1, 1 - x, 1 - 2x + (1/2)x^2, 1 - 3x + (3/2)x^2 - (1/6)x^3\right\}.$$

19. (*Hermite polynomials*)

$$\mathcal{B}_2 = \left\{1, 2x, -2 + 4x^2, -12x + 8x^3\right\}.$$

20. (*Legendre polynomials*)

$$\mathcal{B}_2 = \left\{1, x, -(1/2) + (3/2)x^2, -(3/2)x + (5/2)x^3\right\}.$$

21. (*Euler polynomials*)

$$\mathcal{B}_2 = \left\{1, -(1/2) + x, -x + x^2, 1/4 - (3/2)x^2 + x^3\right\}.$$

22. Find the transition matrix from the basis $\mathcal{B} = \{1 + x, 1 + x^2, 1 - x^2\}$ to the standard basis of P_2.

23. Let $\mathcal{A} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be a basis of a vector space V, and let $\mathcal{B} = \{3\mathbf{v}_1 + 5\mathbf{v}_2, \mathbf{v}_1 - 9\mathbf{v}_2\}$.
(a) Prove that $\mathcal{B}$ is also a basis of V.
(b) Find the transition matrix from $\mathcal{A}$ to $\mathcal{B}$.
(c) Find the transition matrix from $\mathcal{B}$ to $\mathcal{A}$.

24. Find the transition matrix from the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ to the basis $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2\}$, shown in Figure 4.17.

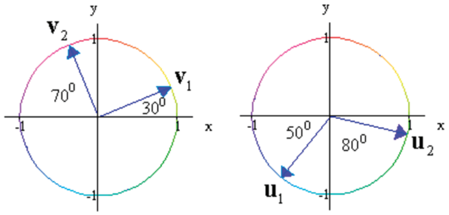

Figure 4.17: Find the transition matrix.

25. Find the transition matrix P from the standard basis $\mathcal{B}$ of $\mathbf{R}^2$ to the basis $\mathcal{B}'$ obtained by reflecting $\mathcal{B}$ about the line $y = -x$. Find the new coordinates of the vector $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

26. Let $\mathcal{B}$ be the standard basis of $\mathbf{R}^2$, and let $\mathcal{U}$ be the basis obtained by rotating $\mathcal{B}$ counterclockwise by θ radians about the origin.
(a) Find the transition matrix from $\mathcal{B}$ to $\mathcal{U}$.
(b) Use P to find the new coordinates of the vector $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$.
(c) Find the transition matrix from $\mathcal{U}$ to $\mathcal{B}$.

27. Find the transition matrix P from the standard basis $\mathcal{B}$ of $\mathbf{R}^3$ to the basis $\mathcal{B}'$ obtained by rotating $\mathcal{B}$ about the z-axis counterclockwise by 90°. Find the new coordinates of the vector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.

28. Let V be an n-dimensional vector space. A nonzero vector $\mathbf{v}$ of V has components $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}^T$ with respect to a basis $\mathcal{B}$ of V. Construct a basis $\mathcal{U}$ such that $[\mathbf{v}]_{\mathcal{U}} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$.

4.5 Null space

In this section, we study the null space, which is a vector space associated with any matrix, and then we examine its basic properties.

Definition 4.5.1. Let A be an $m \times n$ matrix. The *null space*, $\text{Null}(A)$ of A is the solution set of the homogeneous system $A\mathbf{x} = \mathbf{0}$:

$$\text{Null}(A) = \{\mathbf{x} \text{ in } \mathbf{R}^n \text{ such that } A\mathbf{x} = \mathbf{0}\}.$$

The dimension of $\text{Null}(A)$ is called the *nullity* of A.

Theorem 4.5.2. *The null space of an $m \times n$ matrix A is a subspace of* $\mathbf{R}^n$.

Proof. The null space is nonempty: It contains the zero vector of $\mathbf{R}^n$ (why?).

Let two vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ be in Null(A). Then $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. We then have

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Hence $\mathbf{x}_1 + \mathbf{x}_2$ is in Null(A). So the null space is closed under addition. The reader may also verify that the null space is closed under scalar multiplication. So the null space is a subspace of $\mathbf{R}^n$ (Figure 4.18). ☐

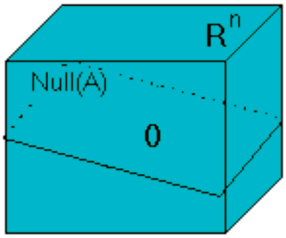

Figure 4.18: The null space is a subspace of $\mathbf{R}^n$.

We explain now how to compute the nullity and find a basis for Null(A).

Example 4.5.3. Let

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ -1 & 0 & -4 & 3 & -1 \\ 2 & -1 & 6 & 0 & 1 \\ -1 & 2 & 0 & -1 & 1 \end{bmatrix}.$$

Find

(a) a basis for the null space of A;

(b) the nullity of A.

Solution.

(a) The augmented matrix of the system $A\mathbf{x} = \mathbf{0}$ has reduced row echelon form

$$[A : \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore x_3 and x_5 are free variables. If we assign the parameters $x_3 = s$, $x_5 = r$ and solve for the leading variables, then we get $x_1 = -4s - r$, $x_2 = -2s - r$, $x_4 = 0$.

The solution in vector form is written as

$$\mathbf{x}=\begin{bmatrix}-4s-r\\-2s-r\\s\\0\\r\end{bmatrix}=r\begin{bmatrix}-1\\-1\\0\\0\\1\end{bmatrix}+s\begin{bmatrix}-4\\-2\\1\\0\\0\end{bmatrix}.$$

Hence the null space of A is spanned by the set

$$\mathcal{B}=\left\{\begin{bmatrix}-1\\-1\\0\\0\\1\end{bmatrix},\begin{bmatrix}-4\\-2\\1\\0\\0\end{bmatrix}\right\},$$

which is also linearly independent. So B is a basis of Null(A).

(b) Because the basis $\mathcal{B}$ in Part (a) has two elements, the nullity of A is 2.

As seen in Example 4.5.3, when we write the general solution of $A\mathbf{x}=\mathbf{0}$ as a linear combination with coefficients the parameters, the vectors of the linear combination not only span the null space, but they are also *linearly independent*. This is because the parameters occur at different components in the general solution vector $\mathbf{x}$. So the following algorithm finds a basis for the null space.

Algorithm 4.5.4 (Basis for null space). To find a basis for Null(A), write the general solution vector of $A\mathbf{x}=\mathbf{0}$ as a linear combination with coefficients the parameters. The vectors of the linear combination form a basis for Null(A).

Because the number of parameters determines the number of vectors in the basis of Null(A), we also have the following theorem.

Theorem 4.5.5. *The nullity of a matrix A equals the number of free variables in the general solution of the system $A\mathbf{x}=\mathbf{0}$.*

Exercises 4.5

In Exercises 1–7, find a basis for the null space and the nullity of the given matrix. (Recall that the zero subspace has dimension 0 and basis the empty set.)

1. (a) $\begin{bmatrix}-1&2\\2&-4\end{bmatrix}$, (b) $\begin{bmatrix}2&-2&2\\3&-3&3\\4&-4&5\end{bmatrix}$.

2. (a) $\begin{bmatrix}1&2\\2&4\\3&8\end{bmatrix}$, (b) $\begin{bmatrix}1&-1&1\\2&-2&2\\3&-3&3\\4&-4&4\end{bmatrix}$.

3. (a) $\begin{bmatrix} 1 & 2 & -1 & -3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$, (b) $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

4. (a) $\begin{bmatrix} -1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 4 \\ 0 & -3 & 3 & 9 \end{bmatrix}$, (b) $\begin{bmatrix} -1 & 1 & 1 & 2 \\ 2 & -2 & -2 & -4 \\ 0 & -3 & 3 & 9 \end{bmatrix}$.

5. (a) $\begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \\ 2 & -4 & 6 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -4 & 4 \\ 0 & 0 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \\ 2 & -4 & 6 & 0 \\ 3 & 3 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & -1 & 2 & 3 & 0 \\ 2 & -1 & 6 & 0 & 1 \\ -1 & 2 & 0 & -1 & 1 \end{bmatrix}$.

8. Find a matrix A whose null space consists of the points of the plane $2x - y + 2z = 0$ (Figure 4.19). What is the nullity of your matrix?

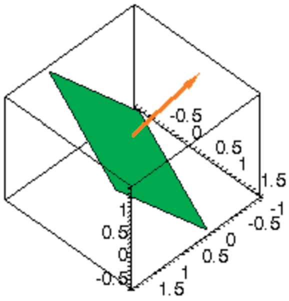

Figure 4.19: Find the nullity.

In Exercises 9–11, add the nullity to the number of pivot columns of the matrix. How does this sum relate to the size of the matrix?

9. (a) $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & -1 \\ 0 & 7 \end{bmatrix}$.

10. (a) $\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 2 & -2 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$.

11. $\begin{bmatrix} 1 & 2 & -1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

12. Prove that
(a) $\text{Null}(A) \subseteq \text{Null}(BA)$;
(b) $\text{Null}(A) \subseteq \text{Null}(A^k)$.

13. If B is invertible, then prove that $\text{Null}(BA) = \text{Null}(A)$.

4.6 Column space, row space, rank

In this section, we study two more vector spaces associated with any matrix, the *column space* and the *row space*. We also discuss the important rank theorem.

Recall the notational convention stated in Section 2.1 that in order to save space, we sometimes use the notation $(x_1, x_2, \ldots, x_n)$ for the vector $[\, x_1 \; x_2 \; \cdots \; x_n \,]^T$.

4.6.1 The column space

Definition 4.6.1. The *column space* $\text{Col}(A)$ of a matrix A is the span of its columns. The column space of an $m \times n$ matrix is a *subspace* of $\mathbf{R}^m$, because it is the span of m-vectors (Figure 4.20).

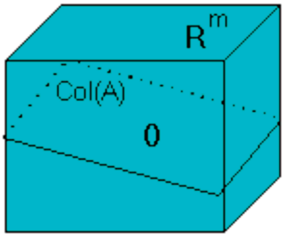

Figure 4.20: The column space is a subspace of $\mathbf{R}^m$.

As we know, a linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the span of the columns of A. We rephrase this in terms of the column space:

Theorem 4.6.2. *A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in $\text{Col}(A)$.*

Theorem 4.6.2 implies that the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if A and $[A : \mathbf{b}]$ have the same column space.

Next, we see how find a basis for the column space of A. It turns out that the pivot columns of A are such a basis.

Example 4.6.3. Find a basis for Col(B), where B is the echelon form matrix

$$B = \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution. Columns 1, 3, 5 are the pivot columns. They are also linearly independent. The nonpivot columns can be written as linear combinations of the pivot columns as $\mathbf{b}_2 = -2\mathbf{b}_1$ and $\mathbf{b}_4 = -\mathbf{b}_1 + \mathbf{b}_3$. So we may drop $\mathbf{b}_2$ and $\mathbf{b}_4$ from the span of the columns:

$$\mathrm{Col}(B) = \mathrm{Span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5\} = \mathrm{Span}\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}.$$

Therefore the pivot columns $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ form a basis for Col(B).

Theorem 4.6.4. *The columns of row equivalent matrices satisfy the same linear dependence relations. So if $A \sim B$, then*

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n = \mathbf{0} \Leftrightarrow c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = \mathbf{0}. \tag{4.3}$$

Proof. Because $A \sim B$, the systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solutions. Hence

$$A\mathbf{c} = \mathbf{0} \Leftrightarrow B\mathbf{c} = \mathbf{0}. \tag{4.4}$$

If $\mathbf{c}$ has components c_i, then we see that (4.4) and (4.3) are identical. ☐

Theorem 4.6.4 implies that any set of columns of A is linearly dependent if and only if the corresponding set of columns of B is linearly dependent. !

Example 4.6.5. Find a basis for Col(A), where

$$A = \begin{bmatrix} 1 & -2 & 2 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 2 & -4 & 6 & 4 & 0 \\ 3 & -6 & 8 & 5 & 1 \end{bmatrix}.$$

Solution. A reduces to matrix B of Example 4.6.3. Therefore the pivot columns of A are columns 1, 3, and 5. The pivot columns of A are linearly independent by Theorem 4.6.4. The nonpivot columns are linear combinations of the pivot columns: $\mathbf{a}_2 = -2\mathbf{a}_1$ and $\mathbf{a}_4 = -\mathbf{a}_1 + \mathbf{a}_3$. These are the same linear dependence relations as in Example 4.6.3. Hence the pivot columns $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ form a basis for Col(A).

Theorem 4.6.6. *The pivot columns of any matrix form a basis for its column space.*

Proof. Let A be an $m \times n$ matrix, and let B be its reduced row echelon form. We prove that the pivot columns of A are linearly independent and that the nonpivot columns are linear combinations of the pivot columns. By Theorem 4.6.4 it is sufficient to prove these claims for B.

Let B have k pivot columns $\mathbf{b}_{i_1}, \dots, \mathbf{b}_{i_k}$. B is in reduced row echelon form, so $\mathbf{b}_{i_1} = \mathbf{e}_1, \dots, \mathbf{b}_{i_k} = \mathbf{e}_k$ with $\mathbf{e}_i \in \mathbf{R}^m$. Therefore the pivot columns are linearly independent. Also, the last $m - k$ components of the columns of B are zero. So the span of pivot columns includes the nonpivot columns. Hence the pivot columns form a basis of the column space. ☐

> ⚡ A basis for Col(A) is, in general, not the same as a basis for Col(B), where B is a row echelon form of A. *Elementary row operations may change the column space of a matrix.* For a basis Col(A), the pivot columns of A are used, not the pivot columns of B.

Example 4.6.7 (Basis of spanning set). Find a basis from S for Span(S), where

$$S = \{(1, -1, 2, 3), (-2, 2, -4, -6), (2, -1, 6, 8), (1, 0, 4, 5), (0, 0, 0, 1)\}.$$

Solution. It suffices to find a basis for the column space of the matrix with columns the vectors of S. This matrix is the matrix A of Example 4.6.5, whose pivot columns were columns 1, 3, and 5. Therefore

$$\{(1, -1, 2, 3), (2, -1, 6, 8), (0, 0, 0, 1)\}$$

is a basis for Span(S) by Theorem 4.6.6.

Theorem 4.6.6 can be also used to extend a linearly independent set of $\mathbf{R}^n$ to a basis, as we see in the following example.

Example 4.6.8 (Extending independent set to basis). Extend the linearly independent set $S = \{(1, 0, -1, 0), (-1, 1, 0, 0)\}$ to a basis in $\mathbf{R}^4$.

Solution. Let $\mathbf{b}_1$ and $\mathbf{b}_2$ be vectors of S. We enlarge S to a spanning set S' by adding the standard basis of $\mathbf{R}^4$:

$$S' = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}.$$

We then row reduce the matrix with columns the elements of S' to get

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore the pivot columns 1, 2, 3, and 5 of S' form a basis for $\mathrm{Span}(S') = \mathbf{R}^4$. Hence

$$\{(1,0,-1,0),(-1,1,0,0),(1,0,0,0),(0,0,0,1)\}$$

is a basis of $\mathbf{R}^4$ that extends the linearly independent set S.

4.6.2 The row space

Definition 4.6.9. The *row space*, $\mathrm{Row}(A)$ of an $m \times n$ matrix A is the span of its rows. The row space is a *subspace* of $\mathbf{R}^n$, because it is the span of n-vectors (Figure 4.21).

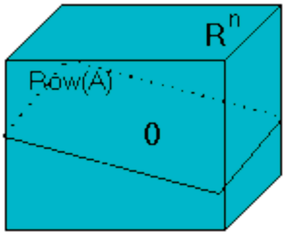

Figure 4.21: The row space is a subspace of $\mathbf{R}^n$.

Unlike column spaces, *row spaces are not affected by elementary row operations.* This is expressed in the following theorem.

Theorem 4.6.10. *If A and B are row equivalent, then* $\mathrm{Row}(A) = \mathrm{Row}(B)$.

Proof. B is obtained from A by a finite set of elementary row operations. Let B' be the outcome of the first operation, say, $\mathcal{O}$. If $\mathcal{O}$ is $R_i \leftrightarrow R_j$, then the set of rows remains the same. If $\mathcal{O}$ is either $cR_i \to R_i$ or $R_i + cR_j \to R_i$, then the new ith row is a linear combination of old ones. Hence $\mathrm{Row}(B') \subseteq \mathrm{Row}(A)$. $\mathcal{O}$ is reversible, so we also have $\mathrm{Row}(A) \subseteq \mathrm{Row}(B')$. Therefore $\mathrm{Row}(A) = \mathrm{Row}(B')$. We repeat this process with the remaining operations until we reach B. In the end, we have $\mathrm{Row}(A) = \mathrm{Row}(B)$. ☐

Theorem 4.6.11. *The nonzero rows of a row echelon form of any matrix A*

1. *are linearly independent and*
2. *form a basis for* $\mathrm{Row}(A)$.

Proof.

1. Let B be an echelon form of A, and let $\mathbf{r}_{i_1}, \dots, \mathbf{r}_{i_k}$ be the nonzero rows of B. If $c_1\mathbf{r}_{i_1} + \cdots + c_k\mathbf{r}_{i_k} = \mathbf{0}$, then $c_1 = 0$, because B is echelon form, and hence all entries below the leading entry of $\mathbf{r}_{i_1}$ are 0. So we can drop the term $c_1\mathbf{r}_{i_1}$ and repeat the argument. Eventually, all c_i are zero. Hence $\{\mathbf{r}_{i_1}, \dots, \mathbf{r}_{i_k}\}$ is linearly independent.

2. The nonzero rows $\mathbf{r}_{i_1},\dots,\mathbf{r}_{i_k}$ in Part 1 are linearly independent and span Row(B). So they are a basis of Row(B). But Row(A) = Row(B) by Theorem 4.6.10. So $\{\mathbf{r}_{i_1},\dots,\mathbf{r}_{i_k}\}$ is also a basis for Row(A). ☐

Example 4.6.12. Find a basis for Row(A), where

$$A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 1 & 3 & 1 & -2 \\ 1 & 1 & 3 & 0 \\ 0 & 1 & -1 & -1 \\ 1 & 2 & 2 & -1 \end{bmatrix}.$$

Solution. A reduces to the row echelon form matrix

$$B = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

According to Theorem 4.6.11, $\{(1,2,2,-1),\ (0,1,-1,-1)\}$ is a basis for Row(A).

The method of Example 4.6.12 offers an alternative way of finding a basis for the span of a finite set of n-vectors. We form the matrix with rows the given vectors and compute a basis for its row space. This time the basis may *not* consist entirely of the given vectors.

Example 4.6.13 (Basis for the span). Find a basis for Span(S), where

$$S = \{(1,-1,2,3), (-2,2,-4,-6), (2,-1,6,8), (1,0,4,5), (0,0,0,1)\}.$$

Solution. We answered this question in Example 4.6.7. Here is another way by computing the row space of the matrix with *rows* the elements of S. We have

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 \\ -2 & 2 & -4 & -6 \\ 2 & -1 & 6 & 8 \\ 1 & 0 & 4 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So $\{(1,-1,2,3), (0,1,2,2), (0,0,0,1)\}$ is a basis for Span(S). This basis is *different* from the one found in Example 4.6.7. Note that $(0,1,2,2)$ is *not* in S.

Elementary row operations do not preserve linear dependence relations among rows. For example, consider

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

We see that $\mathbf{r}_2 = 2\mathbf{r}_1$ for the first matrix, but $\mathbf{r}_2 \neq 2\mathbf{r}_1$ for the second matrix.

4.6.3 Rank

Theorem 4.6.14. *For any matrix A, we have*

$$\dim \mathrm{Col}(A) = \dim \mathrm{Row}(A).$$

Proof. The dimension of $\mathrm{Col}(A)$ is the number of pivots of A, which is also the same as the number of nonzero rows of an echelon form of A. The latter is the dimension of $\mathrm{Row}(A)$. ☐

Definition 4.6.15. The common dimension of the column and row spaces of A is called the *rank* of A and is denoted by $\mathrm{Rank}(A)$ (Figure 4.22).

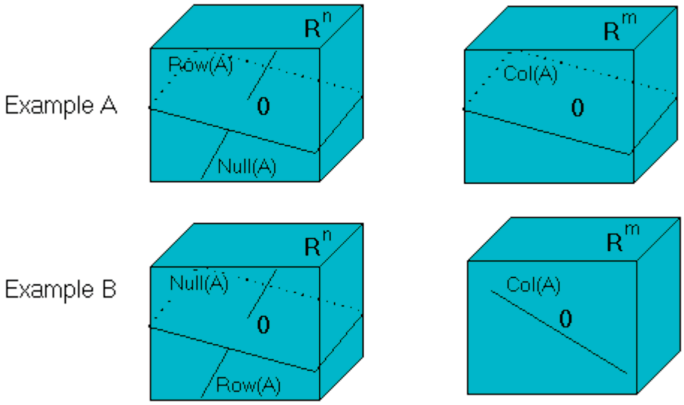

Figure 4.22: The row and column spaces have the same dimension.

The rank is the number of the pivots of A. To compute it, we reduce A to echelon form and count the number of nonzero rows or the number of pivot columns.

Example 4.6.16. The rank of A of Example 4.6.12 is 2, because the row echelon form B has two nonzero rows.

The rank of an $m \times n$ matrix is a nonnegative integer k such that $k \leq m, n$.

Example 4.6.17. Can a 5×9 matrix have rank 6?

Answer. No, because the rank cannot exceed 5.

An important consequence of Theorem 4.6.14 is the following theorem.

Theorem 4.6.18. *A and A^T have the same rank.*

Proof. The column space of A is the same as the row space of A^T. □

The next result is one of the *most important theorems* of linear algebra.

Theorem 4.6.19 (The rank theorem). *For any matrix A, we have*

$$\text{Rank}(A) + \text{Nullity}(A) = \text{number of columns of } A.$$

Proof. The rank of A is the number of pivot columns of A. On the other hand, by Theorem 4.5.5 the nullity of A is the number of free variables of $A\mathbf{x} = \mathbf{0}$. There are as many free variables as nonpivot columns, so the nullity equals the number of nonpivot columns. The theorem now follows from

$$\#\text{ pivot columns } + \#\text{ nonpivot columns } = \#\text{ columns}.$$ □

Example 4.6.20. Suppose the system $A\mathbf{x} = \mathbf{0}$ has 20 unknowns and its solution space is spanned by 6 linearly independent vectors.
(a) What is the rank of A?
(b) Can A have size 13×20?

Solution.
(a) The number of columns of A is 20, and the nullity is 6. Hence the rank of A is $20 - 6 = 14$ by the rank theorem (Theorem 4.6.19).
(b) No, the rank cannot the exceed the number of rows, so A should have at least 14 rows.

4.6.4 Rank and linear systems

The theory and methods developed in this section are strongly related to linear systems. Because a linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in $\text{Col}(A)$ by Theorem 4.6.2, we have the following theorem.

Theorem 4.6.21 (Kronecker–Capelli). *The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if*

$$\text{Rank}(A) = \text{Rank}([A : \mathbf{b}]).$$

Next, we summarize the particular cases, where the rank of an $m \times n$ matrix is either m or n.

Theorem 4.6.22. *Let A be an $m \times n$ matrix. The following are equivalent.*

1. *A has rank m.*
2. *Each row of A has a pivot.*
3. *The system $A\mathbf{x} = \mathbf{b}$ is consistent for all m-vectors $\mathbf{b}$.*
4. *The columns of A span $\mathbf{R}^m$.*
5. $\text{Nullity}(A) = n - m$.

Theorem 4.6.23. *Let A be an $m \times n$ matrix. The following are equivalent.*

1. *A has rank n.*
2. *Each column of A is a pivot column.*
3. *The columns of A are linearly independent.*
4. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. $\text{Nullity}(A) = 0$.

Exercises 4.6

Let

$$\mathbf{a} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -4 \\ -8 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}.$$

In Exercises 1–5, determine which of **a**, **b**, **c**, **u**, **v**, and **w** are in the column space of the given matrix.

1. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

2. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$.

4. $\begin{bmatrix} -2 & 1 & 4 & -6 \\ 0 & 4 & -5 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

5. $\begin{bmatrix} -2 & 1 & 4 & -6 & 1 \\ 0 & 4 & -5 & 0 & 2 \\ 0 & -8 & 10 & 0 & -4 \end{bmatrix}$.

In Exercises 6–12, find a basis for the column space of the matrix.

6. $\begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

7. $\begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

8. $\begin{bmatrix} 5 & 0 & 2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$.

9. $\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 2 & 4 & 6 & -5 \\ 0 & 2 & -4 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 1 & 2 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & -2 & 4 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 8 & 7 \\ 1 & -2 & -8 & -7 \end{bmatrix}$.

11. $\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 2 & 2 & 4 & 6 & -5 \\ 0 & 0 & 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 2 & 1 & 2 \end{bmatrix}$.

12. $\begin{bmatrix} 4 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & -3 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

In Exercises 13–14, draw Col(A).

13. $A = \begin{bmatrix} 2 & 1 & 0 & -1 \\ -2 & 1 & 1 & 1 \end{bmatrix}$.

14. $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$.

15. Sketch Col(A) and Nul(A) for $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{bmatrix}$.

16. Sketch Col(A) and Nul(A) for $A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \\ 6 & -2 \end{bmatrix}$.

17. Find matrices A and B such that $A \sim B$ and $\mathrm{Col}(A) \neq \mathrm{Col}(B)$.

18. For the graph in Figure 4.23, suppose that the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are in the column space C of a matrix, whereas the vector $\mathbf{u}$ is not in C. Draw and describe C.

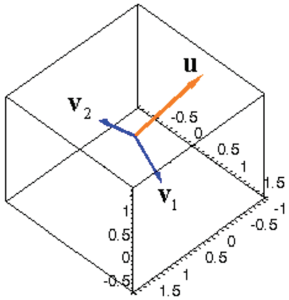

Figure 4.23: Draw C.

In Exercises 19–21, find a basis for the span of the given set of vectors.

19. $\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ -8 \end{bmatrix} \right\}.$

20. $\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$

21. $\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -1 \\ 3 \end{bmatrix} \right\}.$

In Exercises 22–25, enlarge the given linearly independent set of n-vectors to a basis of $\mathbf{R}^n$.

22. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$

23. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$

24. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}.$

25. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$

26. Prove that
(a) $\operatorname{Col}(AB) \subseteq \operatorname{Col}(A)$;
(b) $\operatorname{Col}(A^k) \subseteq \operatorname{Col}(A)$;
(c) $\operatorname{Null}(I - A) \subseteq \operatorname{Col}(A)$;
(d) $\operatorname{Col}(AB) = \operatorname{Col}(A)$, if B is invertible.

The Row Space

In Exercises 27–30, find a basis for Row(A) and compute Rank(A).

27. $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 0 & -1 & 2 & 3 \\ 1 & 1 & 4 & 2 \end{bmatrix}$.

28. $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

29. $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 2 \\ 1 & 1 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$.

30. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \\ 1 & 4 & -8 \end{bmatrix}$.

In Exercises 31–33, find a basis for the span of the given set of vectors by working with the row space of some matrix.

31. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\}$.

32. $\left\{ \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right\}$.

33. $\left\{ \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$.

Rank

In Exercises 34–35, let

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

34. Verify

(a) Theorem 4.6.18 for A and B.

(b) The Rank theorem (Theorem 4.6.19) for A, B, and B^T.

35. Use Theorem 4.6.21 to prove that the system $[B : \mathbf{b}]$, with $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, is consistent.

36. Suppose the system $A\mathbf{x} = \mathbf{0}$ has 250 unknowns and its solution space is spanned by 50 linearly independent vectors.
(a) What is the rank of A?
(b) Can A have size 150×250?
(c) Can A have size 200×200?
(d) Can A have size 250×150?
(e) Can A have size 200×250?
(f) Can A have size 250×250?

37. Let $A\mathbf{x} = \mathbf{b}$ be a system with 400 equations and 450 unknowns. Suppose that the null space of A is spanned by 50 linearly independent vectors. Is the system consistent for all 400-vectors $\mathbf{b}$?

38. Prove Theorem 4.6.22.

39. Prove Theorem 4.6.23.

40. Prove that
(a) $\text{Rank}(AB) \leq \text{Rank}(A)$;
(b) $\text{Rank}(AB) \leq \text{Rank}(B)$;
(c) $\text{Rank}(A) = \text{Rank}(AB)$ if B is invertible.

4.7 Application to coding theory

Nearly all transmitted messages, from human speech to receiving data from a satellite, are subject to noise. It is important, therefore, to be able to encode a message in a way that after it gets scrambled by noise, it can be decoded to its original form (Figure 4.24). This is done sometimes by repeating the message two or three times, something very common in human speech. However, repetition is not always very efficient: copying once or twice the data stored on a hard disk requires extra storage.

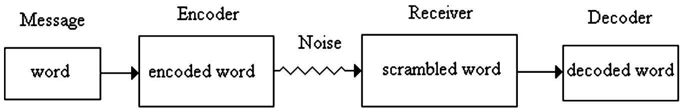

Figure 4.24: The coding–decoding process.

We examine ways of encoding and decoding a message after it gets distorted by noise. This process is called *coding*. A code that detects errors in a scrambled message is called *error-detecting*. If, in addition, it can correct the error, then it is called *error-correcting*. It is harder to write error-correcting codes.

Most messages are *digital*, i. e., sequences of 0s and 1s such as 10101 or 1010011. Suppose we want to sent the message 1011. This binary "word" may stand for a real word, such as "buy", or a sentence, such as "buy stock on Beatles' songs" and so on. Encoding 1011 means to attach a binary tail to it so that if the message gets distorted to, say, 0011, we can detect the error. A simple thing to do is attach a 1 or 0, depending on whether we have an odd or even number of 1s in the word. This way all encoded words will have an even number of 1s. So 1011 will be encoded as 10111. Now if this is distorted to 00111, then we know that an error has occurred because we only received an odd number of ones. This error-detecting code is call *parity check* (Figure 4.25). Parity check is too simple to be very useful. For example, if 2 digits were changed, then our scheme will not detect the error. Also, in the case of a single error, we would do not where it is to fix it. This is not an error-correcting code. Another approach would be to encode the message by repeating it twice, for example, 10111011. Then if 00111011 is received, then we know that one of the two equal halves was distorted. If only one error occurred, then it is clearly at position 1. This coding scheme also gives poor results and is not often used. We could get better results by repeating the message several times, but that takes space.

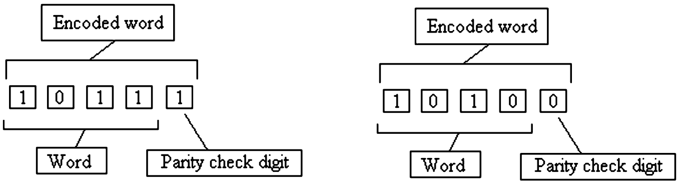

Figure 4.25: Coding with one parity check.

4.7.1 Vector spaces over Z_2

We examine an interesting single error-correcting code introduced by R. H. Hamming in the 1950s.[2]

Before we discuss details, we extend the definition of a vector space so that scalars other than real numbers or complex numbers can be used. We are interested in the set of scalars $Z_2 = \{0, 1\}$, the *integers mod* 2. Addition and multiplication in Z_2 is defined by

$$\begin{gathered} 0+0=0, \quad 1+0=1, \quad 0+1=1, \quad 1+1=0; \\ 0\cdot 0=0, \quad 1\cdot 0=0, \quad 0\cdot 1=0, \quad 1\cdot 1=1 \end{gathered}$$

---

[2] *Richard Wesley Hamming* (1915–1998) was born in Chicago, Illinois, and died in Monterey, California. In 1946, he joined the Bell Telephone Laboratories. In 1950, he published a fundamental paper on error-detecting and error-correcting codes. This started a new area within information theory. Hamming worked on the early computer and the development of computer languages. He was awarded the Turing Prize from the Association for Computing Machinery.

Since $1 + 1 = 0$, the opposite of 1 is again 1, so $-1 = 1$. Thus *subtraction is identical to addition*! These operations satisfy the usual properties of addition and multiplication. For example,

$$(1 + 1) + (1 \cdot 0 + 1) + 1 \cdot (0 + 1) = 0 + 1 + 1 = 0.$$

Let Z_2^n be the set of n-vectors with components the elements of Z_2. If $n = 3$, then Z_2^3 consists of the eight vectors

$$Z_2^3 = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)\}.$$

In general, Z_2^n has 2^n elements.

Just as we did with $\mathbf{R}^n$, we equip Z_2^n with componentwise addition and scalar multiplication with Z_2-operations. In Z_2^4, we have

$$\begin{aligned}(1,1,0,1) + (0,1,1,0) &= (1,0,1,1),\\ 1(1,0,0,1) &= (1,0,0,1),\\ 0(1,1,1,0) &= (0,0,0,0).\end{aligned}$$

Under these operations, Z_2^n satisfies all the axioms of a vector space, except that the scalars are from Z_2. We say that Z_2^n *is a vector space over* Z_2. All the basic concepts and properties, such as subspaces, bases, linearly independent vectors, spanning sets, row reduction of matrices, column space, row space, rank, nullity, etc., apply to vector spaces over Z_2 and to matrices with entries from Z_2.

Theorem 4.7.1. *If V is a vector subspace over Z_2 with dimension n, then V has 2^n elements.*

Proof. If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of V, then V consists of all different linear combinations

$$c_1\mathbf{u}_1 + \cdots + c_n\mathbf{v}_n \quad \text{with } c_1, \dots, c_n \text{ either 0 or 1.}$$

For each coefficient, there are two choices, so we have a total of 2^n different combinations. ☐

Example 4.7.2. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Find

(a) bases for Col(A) and Null(A) over Z_2,
(b) the rank and nullity of A over Z_2, and
(c) the rank and nullity of A over $\mathbf{R}$.

Solution.

(a) Over Z_2, A reduces as (the reduction is done with Z_2-arithmetic)

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \mathrm{Col}\,(A)\,.$$

The null space is obtained by setting the reduced row echelon equal to $(0, 0, 0)$ and solving for the leading variables. If $x_4 = r$ and $x_3 = s$, then $x_1 = -r = r$ and $x_2 = -r - s = r + s$, where $r, s \in \{0, 1\}$. So we have

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for } \mathrm{Null}\,(A)\,.$$

(b) Over Z_2, the rank is 2, and the nullity is 2. By Theorem 4.7.1, Col(A) has $2^2 = 4$ elements, and Null(A) has $2^2 = 4$ elements.

(c) Over **R**, we row reduce

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

So, the rank is 3, and the nullity is 1 by the rank theorem.

4.7.2 The Hamming (7, 4)-code

We are now ready to define Hamming's interesting single error correcting code.[3] An (n, k) *linear code* is a subspace of Z_2^n of dimension k. All vectors of a linear code are called *codewords*, or *encoded words*.

---

[3] See: 1. *Error Correcting Codes* by Welsey Peterson, 1961, published by The M. I. T., Second Printing. 2. *Introduction to the Theory of Error Correcting Codes* by Vera Pless, 1982, published by John Wiley and Sons.

Consider the matrix H over Z_2,

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

Note that the columns $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_7$ of H are all the nonzero vectors of Z_2^3.

The null space of H is called a *Hamming* $(7,4)$-*code*. Let Null(H) be abbreviated to N_H. Matrix H is called a *parity check matrix* for the code N_H. Just as in Example 4.7.2, we may easily compute a basis B for N_H to get

$$B = \{(1,0,0,0,0,1,1), (0,1,0,0,1,0,1), (0,0,1,0,1,1,0), (0,0,0,1,1,1,1)\}.$$

So N_H is a linear $(7,4)$ code and it has $2^4 = 16$ vectors. Because, $H(\mathbf{e}_i) = \mathbf{h}_i$ for $i = 1, \ldots, 7$, we see that none of the standard basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_7$ of Z_2^7 is in N_H.

The matrix G whose rows are the elements of B,

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

is called a *generator matrix* of the Hamming $(7,4)$-code.

The following theorem is A key to encoding and decoding by the Hamming $(7,4)$-code method. It states that if any coordinate of a vector in N_H is changed, then the new vector is no longer in N_H. Also, if $H\mathbf{v}$ is the jth column of H, then changing the jth coordinate of $\mathbf{v}$ *only* will put the new vector in N_H.

Theorem 4.7.3. *Let* $\mathbf{v} = (v_1, \ldots, v_7)$ *in* Z_2^7.

1. *If* $\mathbf{v} \in \mathrm{N}_H$, *then* $\mathbf{v} + \mathbf{e}_i$ *is not in* N_H *for* $i = 1, \ldots, 7$.
2. *If* $H\mathbf{v} = \mathbf{h}_j$, *then* $\mathbf{v} + \mathbf{e}_j \in \mathrm{N}_H$. *Furthermore,* $\mathbf{v} + \mathbf{e}_i$ *is not in* N_H *for* $i \neq j$.

Proof.

1. Because, $H\mathbf{v} = \mathbf{0}$, we have

$$H(\mathbf{v} + \mathbf{e}_i) = H(\mathbf{v}) + H(\mathbf{e}_i) = \mathbf{0} + \mathbf{h}_i = \mathbf{h}_i \neq \mathbf{0}.$$

2. We have

$$H(\mathbf{v} + \mathbf{e}_j) = H(\mathbf{v}) + H(\mathbf{e}_j) = \mathbf{h}_j + \mathbf{h}_j = 2\mathbf{h}_j = \mathbf{0},$$

so $\mathbf{v} + \mathbf{e}_i \in \mathrm{N}_H$. In addition,

$$H(\mathbf{v} + \mathbf{e}_i) = H(\mathbf{v}) + H(\mathbf{e}_i) = \mathbf{h}_j + \mathbf{h}_i \neq \mathbf{0}\ , \quad i \neq j.$$

□

4.7.3 Encoding and decoding

Let us see now how to encode a message and decode the distorted reception of it. We are assuming that the word to be coded is binary of length 4, say 1011, and that noise altered only one binary digit of the encoded word.

To encode 1011, we form the linear combination $\mathbf{v}$ in the basis B of the Hamming $(7,4)$-code with coefficients the digits $1,0,1,1$ of our message:

$$\begin{aligned}\mathbf{v} &= 1(1,0,0,0,0,1,1)+0(0,1,0,0,1,0,1)\\ &\quad +1(0,0,1,0,1,1,0)+1(0,0,0,1,1,1,1)=(1,0,1,1,0,1,0)\,.\end{aligned}$$

This is the same as right matrix multiplication by G over Z_2:

$$\mathbf{v}^T G=\begin{bmatrix}1 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 1 & 1\\ 0 & 1 & 0 & 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 1 & 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 1 & 1 & 0 & 1 & 0\end{bmatrix}.$$

The encoded word $\mathbf{v}$ is in N_H by construction. It contains the original message in the first four components and adds a sort of parity check $0,1,0$ in the end. Suppose that the string 1011010 gets transmitted and received as 0011010. Let $\mathbf{u}=(0,0,1,1,0,1,0)$. To correct the received message, we compute the product $H\mathbf{u}$:

$$H\mathbf{u}=\begin{bmatrix}0 & 0 & 0 & 1 & 1 & 1 & 1\\ 0 & 1 & 1 & 0 & 0 & 1 & 1\\ 1 & 0 & 1 & 0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\\1\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Because $H\mathbf{u}$ is the first column of H, Theorem 4.7.3, Part 2, implies that $\mathbf{u}+\mathbf{e}_1$ is in N_H and none of $\mathbf{u}+\mathbf{e}_i$, $i\neq 1$, is in N_H. Hence $\mathbf{u}+\mathbf{e}_1=\mathbf{v}$ is the only corrected coded message, and the original message 1011 is recovered.

Example 4.7.4. Suppose we received the Hamming encoded messages 1010101 and 1100111. If there was at most one error in each transmission, what were the original messages?

Solution. Let $\mathbf{v}_1=(1,0,1,0,1,0,1)$ and $\mathbf{v}_2=(1,1,0,0,1,1,1)$.

(a) $H\mathbf{v}_1=(0,0,0)$. Hence $\mathbf{v}_1\in\mathrm{N}_H$. Because the original encoded message was already in N_H, a single error would throw $\mathbf{u}_1$ out of N_H by Theorem 4.7.3, Part 1. So there was no error in the transmission of the first message, which was 1010.

(b) $H\mathbf{v}_2 = (1, 1, 1)$. So the seventh component of $\mathbf{v}_2$ needs to be corrected to 0 by Theorem 4.7.3, Part 2. Therefore the original message was 1100. This time the noise affected the parity check part, and the original message was never altered.

Algorithm for error correction with the Hamming (7, 4)-code

Suppose that a four-digit binary word $\mathbf{w}$ is coded as $\mathbf{u}$, so that $\mathbf{u}$ is in N_H. If $\mathbf{u}$ is distorted to $\mathbf{v} = (v_1, \dots, v_7)$ by at most one component change, to recover the original message $\mathbf{w}$m do the following.

```
Input:   $\mathbf{v}$
1.  Compute $H\mathbf{v}$.
2.  If $H\mathbf{v} = \mathbf{0}$, then let $\mathbf{w} = v_1v_2v_3v_4$. Stop.
3.  If $H\mathbf{v} = \mathbf{h}_i$, then change the $i$th component of $\mathbf{v}$ to get a new vector
    $\mathbf{u}_1 = (v_1', \dots, v_7')$.
4.  Let $\mathbf{w} = v_1'v_2'v_3'v_4'$.
Output:   $\mathbf{w}$
```

4.7.4 Other types of codes

Our study of the Hamming (7, 4)-code was intended as an illustration of some of the many fruitful ideas of C. E. Shannon, R. H. Hamming, and others in the late 1940s and early 1950s in the areas of electrical engineering and information theory. We did not attempt to be thorough. The Hamming code is only good for encoding binary words of length 4, of which there are only $2^4 = 16$. If we want a larger "alphabet", or if we want to correct at least two errors in a scrambled message, then we need other types of codes.

In practice a wide variety coding techniques are used that allow more words and thus longer messages to be coded. In addition, some codes allow more noise errors than the Hamming code. Many of the interesting codes are *nonlinear*. Definitions and examples are included in standard texts on the subject.

In the study of codes, mathematics is the main contributor with linear algebra, number theory, and field theory in the front line.

Exercises 4.7

Z_2^n-arithmetic

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

1. Perform the indicated operations in Z_2^3.

$$\text{(a)}\quad \mathbf{u}+\mathbf{v},\qquad \text{(b)}\quad -\mathbf{v},\qquad \text{(c)}\quad 1\mathbf{u}+0\mathbf{v}-1\mathbf{w},\qquad \text{(d)}\quad \mathbf{u}+\mathbf{v}+\mathbf{w}.$$

2. Solve the following equation for $\mathbf{x}$ over Z_2:

$$\mathbf{x}-\mathbf{u}+\mathbf{v}=\mathbf{w}+\mathbf{u}.$$

3. Compute A^2 and A^3 over Z_2.

4. Is $\{\mathbf{u},\mathbf{v}\}$ linearly independent over Z_2? What about $\{\mathbf{u},\mathbf{w}\}$?

5. Is $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$ linearly independent over Z_2?

6. Add a vector to $\{\mathbf{u},\mathbf{v}\}$ so that the resulting set is a basis of Z_2^3.

7. Find a basis and the vectors of the null space of A over Z_2. Repeat over $\mathbf{R}$.

8. Find the inverse of

$$A=\begin{bmatrix}1&0&0\\0&1&1\\1&1&0\end{bmatrix}$$

over Z_2 and verify your answer.

9. Let A and B be 2×2 binary commuting matrices. Prove that over Z_2, we have

$$(A+B)^2=A^2+B^2.$$

Codes

10. Encode the message 1110.

11. Encode the message 0101.

12. Encode the message 0010.

In Exercises 13–17, suppose that a message word was encoded by the Hamming coding method. During transmission, at most one coordinate was altered. Recover the original message from the received binary vector shown.

13.
(a) $(1,1,1,1,0,1,1)$,
(b) $(1,1,1,1,1,0,0)$.

14.
(a) $(0,1,1,1,1,0,1)$,
(b) $(0,1,1,1,1,0,0)$.

15.
(a) $(0,1,1,0,0,0,1)$,
(b) $(0,1,0,0,0,1,1)$.

16.
(a) $(0, 1, 1, 0, 0, 1, 1)$,
(b) $(1, 1, 1, 1, 0, 0, 0)$.

17.
(a) $(1, 1, 1, 0, 0, 1, 0)$,
(b) $(1, 1, 1, 0, 0, 0, 0)$.

18. Write the 16 elements of the Hamming $(7, 4)$-code N_H.

The *weight* $w(\mathbf{v})$ of a vector $\mathbf{v}$ in Z_2^n is the number of its nonzero entries. For example,

$$w(0, 1, 1, 0) = 2 \quad \text{and} \quad w(1, 0, 1, 1, 1) = 4.$$

The *distance* $d(\mathbf{u}, \mathbf{v})$ between two vectors $\mathbf{u}$ and $\mathbf{v}$ in Z_2^n is the number of entries at which $\mathbf{u}$ and $\mathbf{v}$ differ. Hence

$$d(\mathbf{u}, \mathbf{v}) = w(\mathbf{u} - \mathbf{v}) = w(\mathbf{u} + \mathbf{v}).$$

19. Prove that

$$d(\mathbf{u}, \mathbf{v}) = d(\mathbf{0}, \mathbf{u} - \mathbf{v}).$$

20. Prove that $w(\mathbf{v}) \geq 3$ for all nonzero vectors $\mathbf{v}$ in N_H. (*Hint:* Use Exercise 18.)

21. Prove that $d(\mathbf{u} - \mathbf{v}) \geq 3$ for all distinct vectors $\mathbf{u}$ and $\mathbf{v}$ in N_H. (*Hint:* Use Exercise 20.)

Error detecting codes can also be defined in terms of the distance function d. A linear code $V \subseteq Z_2^n$ is *single error detecting* if for any codeword $\mathbf{v} \in V$ and any vector $\mathbf{u}$ in Z_2^n, the relation $d(\mathbf{u}, \mathbf{v}) \leq 1$ implies that $\mathbf{u}$ is not a codeword, unless $\mathbf{v} = \mathbf{u}$.

22. Use Exercise 21 to prove that N_H is single error detecting according to the above definition. Moreover, prove that the statement "the relation $d(\mathbf{u}, \mathbf{v}) \leq 1$ implies that $\mathbf{u}$ is not a codeword, unless $\mathbf{v} = \mathbf{u}$" is equivalent to Part 1 of Theorem 4.7.3.

4.8 Miniprojects

The focus of this project section is to discuss an even further generalization of vector space. In Section 4.7, we defined vector spaces over Z_2. Now we allow more general types of scalars that are elements of a *field*.

Fields

Definition 4.8.1. A *field* F is a set of elements called *scalars*, equipped with two operations, addition $(a + b)$ and multiplication (ab), that satisfy the following properties.

Addition

(A1) $a + b$ belongs to F for all $a, b \in F$.

(A2) $a + b = a + b$ for all $a, b \in F$.

(A3) $(a + b) + c = a + (b + c)$ for all $a, b, c \in F$.

(A4) There exists a unique scalar $0 \in F$, called the *zero* of F, such that for all a in F,

$$a + 0 = a.$$

(A5) For each $a \in F$, there exists a unique scalar $-a$, called the *negative* or *opposite* of a, such that

$$a + (-a) = 0.$$

Multiplication

(M1) ab belongs to F for all $a, b \in F$.

(M2) $(a + b)c = ab + bc$ for all $a, b, c \in F$.

(M3) $ab = ba$ for all $a, b \in F$.

(M4) $(ab)c = a(bc)$ for all $a, b, c \in F$.

(M5) There exists a unique *nonzero* scalar $1 \in F$, called *one*, such that for all a in F,

$$a1 = a.$$

(M6) For each $a \in F$, $a \neq 0$, there exists a unique scalar a^{-1} (or $\frac{1}{a}$), called the *inverse* or *reciprocal* of a, such that

$$aa^{-1} = 1.$$

We usually write $a - b$ for the sum $a + (-b)$:

$$a - b = a + (-b).$$

Problem A. Prove that in a field F,

$$\text{if} \quad ab = 0, \quad \text{then} \quad a = 0 \quad \text{or} \quad b = 0.$$

Problem B. Prove that the following are fields. In each case, use the usual addition, multiplication, and reciprocation.

1. The set of real numbers **R**.
2. The set of rational numbers **Q**.
3. The set of complex numbers **C**.
4. The set of integers mod 2, Z_2.
5. The set $\mathbf{Q}(\sqrt{2})$ of all numbers of the form $a + b\sqrt{2}$, where a and b are rational numbers.

(*Hint:* For $\mathbf{Q}(\sqrt{2})$, the reciprocal of $a+b\sqrt{2}$ can written in the form $A+B\sqrt{2}$ by multiplying and dividing $\frac{1}{a+b\sqrt{2}}$ by the *conjugate* $a-b\sqrt{2}$. For example, the inverse of $1-3\sqrt{2}$ is $\frac{1}{1-3\sqrt{2}} = \frac{1+3\sqrt{2}}{(1-3\sqrt{2})(1+3\sqrt{2})} = \frac{1+3\sqrt{2}}{-17} = -\frac{1}{17} - \frac{3}{17}\sqrt{2}$.)

Problem C. Explain why the following sets are *not* fields. In each case, use the usual addition and multiplication.

1. The set of integers $\mathbf{Z}$.
2. The set of positive integers $\mathbf{N}$.
3. The set $\mathbf{R}^2$ with the usual componentwise addition $(a,b)+(a',b')=(a+a',b+b')$ and componentwise "multiplication" $(a,b)(a',b')=(aa',bb')$.

Vector spaces over any field F

A vector space V over a field F is a nonempty set equipped with two operations, addition and scalar multiplication, that satisfy all axioms of a vector space as defined in Section 4.1, except that all scalars come from the field F, instead from the real numbers $\mathbf{R}$. The elements are called *vectors* just as before. If the field F is $\mathbf{R}$, then we say that V is a *real* vector space. If $F=\mathbf{Q}$, then the set of rationals, then we say that V is a *rational* vector space. If $F=\mathbf{C}$, the set of complex numbers, then we have a *complex* vector space, which was defined in Section 4.1.

We denote by F^2 the set of all ordered pairs (a,b) of elements a and b of F. In general, we denote by F^n the set of all ordered n-tuples $(a_1,\dots,a_n)$, where $a_1,\dots,a_n$ are any elements of F. F^n is equipped with componentwise addition and scalar multiplication:

$$(a_1,\dots,a_n)+(b_1,\dots,b_n)=(a_1+b_1,\dots,a_n+b_n),$$
$$c(a_1,\dots,a_n)=(ca_1,\dots,ca_n).$$

Problem A. Prove that the following sets are vector spaces over the specified field F.

1. $\mathbf{Q}$ over $\mathbf{Q}$.
2. $\mathbf{Q}^n$ over $\mathbf{Q}$.
3. $\mathbf{C}$ over $\mathbf{C}$.
4. $\mathbf{C}^n$ over $\mathbf{C}$.
5. Any field F over F.
6. F^n over F.

Problem B. Prove that the following sets are vector spaces over the specified field F.

1. The real numbers $\mathbf{R}$ over the set of rational numbers $\mathbf{Q}$. Addition is the usual r_1+r_2, $r_1,r_2\in\mathbf{R}$. Scalar multiplication is of the form qr, where q is a rational number, and r is real.

2. The complex numbers **C** over the set of real numbers **R**. Addition is the usual $z_1 + z_2$, $z_1, z_2 \in \mathbf{C}$. Scalar multiplication is of the form rz, where r is a real number, and z is complex.
3. The set $\mathbf{Q}(\sqrt{2})$ of all numbers of the form $a + b\sqrt{2}$, $a, b \in \mathbf{Q}$, over **Q**.

Problem C. Find the *dimension* of the given vector spaces over the specified field F.

1. **C** over **C**.
2. $\mathbf{C}^2$ over **C**.
3. **C** over **R**.
4. F^n over F.
5. $\mathbf{Q}(\sqrt{2})$ over **Q**.

Vector spaces over finite fields

In this section, we define some interesting fields that consist of finitely many elements and some vector spaces defined over them.

The integers modulo *p*

A prime number p is a positive integer whose only divisors are 1 and p. For example, $2, 3, 5, 7, 11, 13, 17, 19, 23, 29$ are prime numbers.

Let p be a prime number. The set Z_p of integers mod p consists of the p elements $\{0, 1, \ldots, p-1\}$. In Z_p, we define addition and multiplication as follows:

If a and b are in Z_p, then $a + b$ is the smallest positive remainder that we get if we divide the integer $a + b$ by p. For example, if $p = 5$ and $Z_5 = \{0, 1, 2, 3, 4\}$, then $2 + 3$ yields the remainder 0 when divided by 3, so $2 + 3 = 0$ in Z_5. Also, $3 + 4 = 2$ in Z_5, because $7 = 5 \cdot 1 + 2$ has the remainder 2 when divided by 5.

If a and b are in Z_p, then ab is defined in the same way, i. e., the smallest positive remainder that we get if we divide the integer ab by p. For example, in Z_5, $2 \cdot 3 = 1$, because $6 = 5 \cdot 1 + 1$. Likewise, $3 \cdot 3 \cdot 3 \cdot 3 = 1$, because $81 = 20 \cdot 5 + 1$. In Z_5 the reciprocal of 2 is 3, because $2 \cdot 3 = 1$ in Z_5. We may write $\frac{1}{2} = 3$. Likewise, $\frac{1}{3} = 2$ and $\frac{1}{4} = 4$. The operations we just defined are called “mod p” operations of the integers.

Problem A. Find

$$-1 \in Z_7, \quad -10 \in Z_{17}, \quad \frac{1}{3} \in Z_{17}, \quad \frac{1}{6} \in Z_7, \quad \frac{1}{10} \in Z_{11}, \quad \frac{1}{p-1} \in Z_p \quad (p \text{ prime}).$$

If m is any positive integer, then $Z_m = \{0, \ldots, m-1\}$, the integers mod m, is defined just as Z_p and is given the same mod m operations.

Problem B. With respect to the mod p operations, prove that

1. Z_3, Z_7, and Z_p (p prime) are fields.
2. Z_4, Z_m, m not prime, are *not* fields.

Because Z_p is a field for any prime p, we may talk about vector spaces over Z_p. Because F^n is a vector space over F by Project 2, Z_p^n, the set of n-tuples $(a_1, \dots, a_n)$ with $a_i \in Z_p$, is a vector space over Z_p of dimension n.

Problem C. Consider the vector space Z_p^n over Z_p.
1. Prove that Z_p^n has p^n elements.
2. If V is a subspace of Z_p^n of dimension m, then V has p^m elements.
3. Find a basis for Z_3^2.
4. Let
$$A = \begin{bmatrix} 4 & 1 & 0 \\ 4 & 3 & 2 \\ 3 & 3 & 1 \end{bmatrix}$$
be a matrix with entries in Z_5. Row reduce A using only mod 5 elementary row operations.
5. Find bases for the null and column spaces of A over Z_5. Verify the rank theorem (Theorem 4.6.19).

4.9 Technology-aided problems and answers

Let

$$M = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_5\ \mathbf{v}_6] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 8 \end{bmatrix}$$

and

$$N = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3\ \mathbf{u}_4\ \mathbf{u}_5] = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 4 & 3 & 4 & 5 \\ 3 & 6 & 5 & 7 & 9 \\ 4 & 8 & 7 & 10 & 13 \end{bmatrix},$$

and let

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}, \quad T = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}, \quad \mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{u}_3, \mathbf{u}_4\}.$$

1. Is S a basis of $\mathbf{R}^3$?

2. Is T a basis of $\mathbf{R}^4$?

3. Prove that $\mathcal{B}$ is a basis of $\mathbf{R}^4$.

4. Find $[\mathbf{u}_1]_{\mathcal{B}}$ and $[\mathbf{u}_2]_{\mathcal{B}}$.

5. If $[\mathbf{x}]_{\mathcal{B}} = \mathbf{u}_5$, then what is $\mathbf{x}$?

6. Find a basis for $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

7. Find a basis for the span of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

8. Find bases for Col(M), Row(M), and Null(M).

9. Compute the rank and nullity of M. Verify the rank theorem (Theorem 4.6.19).

10. Verify that M and M^T have the same rank.

11. Enlarge the linearly independent set $\{\mathbf{u}_2, \mathbf{u}_3\}$ to a basis of $\mathbf{R}^4$.

12. Let $a = -7$, $b = 2$, and

$$\mathbf{u} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 & 1 \\ 0 & 8 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 & 1 \\ 0 & -9 \end{bmatrix}.$$

Verify Axioms (A2), (A3), (M2), (M3), and (M4) for these a, b, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

13. Define the three-variable function $f(a,b,x) = a\cos(3x) + b\sin(2x)$, which represents the linear combinations of $\cos(3x)$ and $\sin(2x)$ in $F(\mathbf{R})$. Use f to plot in one graph the linear combinations with $\{a = 1, b = 1\}$, $\{a = 3, b = 0\}$, $\{a = 0, b = -3\}$, $\{a = 3, b = -4\}$, and $\{a = -3, b = 4\}$.

14. Prove that the set V of polynomials of the form $ax^3 + bx^2$, $a, b \in \mathbf{R}$, *is* a subspace of P_3.

15. Prove that the following sets of polynomials form bases of P_3:

$$\mathcal{B}_1 = \{1, -x + 1, x^2 - x, -x^3 + x^2 - 1\}, \quad \mathcal{B}_2 = \{x + 2, 4x^2 - x, x^3 - x, x^2 + 1\}.$$

16. Find the transition matrix P from $\mathcal{B}_1$ to $\mathcal{B}_2$.

17. Find the transition matrix Q from $\mathcal{B}_2$ to $\mathcal{B}_1$.

18. Verify that $P = Q^{-1}$.

19. Verify that $[-x^3 + 5x + 1]_{\mathcal{B}_2} = P[-x^3 + 5x + 1]_{\mathcal{B}_1}$.

4.9.1 Selected solutions with Mathematica

```
(* Data. *)
v1 = {{1},{2},{3}}; v2 = {{2},{3},{4}}; v3 = {{3},{4},{5}};
v4 = {{4},{5},{6}}; v5 = {{5},{6},{7}}; v6 = {{6},{7},{8}};
u1 = {{1},{2},{3},{4}}; u2 = {{2},{4},{6},{8}};
u3 = {{1},{3},{5},{7}}; u4 = {{1},{4},{7},{10}};
u5 = {{1},{5},{9},{13}};
e1 = {{1},{0},{0},{0}}; e2 = {{0},{1},{0},{0}};(* Standard basis vectors.*)
e3 = {{0},{0},{1},{0}}; e4 = {{0},{0},{0},{1}};
M = Join[v1, v2, v3, v4, v5, v6, 2]
n =Join[u1,u2,u3,u4,u5,2]        (* N is already used by the program. *)
B = Join[e1,e2,u3,u4,2]    (* The matrix with columns the vectors of B. *)
(* Exercises 1-7. *)
RowReduce[M]       (* 2 pivots, 3 rows: does not span R^3. Not a basis.  *)
RowReduce[n]       (* 2 pivots, 4 rows: does not span R^4. Not a basis.  *)
RowReduce[B]       (* 4 pivots, 4 rows, 4 columns: spanning and          *)
(* linearly independent: Basis of R^4.                 *)
RowReduce[Join[B, u1, 2]]   (* We solve the system {B:u1} by reduction.  *)
%[[All, 5]]    (* The last col. gives the coords. of u1. Repeat with u2. *)
```

```
B . u5                (* Exercise 5: x is just Bu5. *)
RowReduce[Join[v1,v2,v3,2]]    (* Pivots at (1,1),(2,2). {v1,v2} basis. *)
RowReduce[Join[u1,u2,u3,u4,2]] (* Pivots at (1,1),(2,3). {u1,u3} basis. *)
(* Exercises 8-9. *)
RowReduce[M]            (* Pivot cols. 1,2 so basis {v1,v2}. Rank=2.  *)
RowReduce[Transpose[M]] (* The first two rows form a basis for row space. *)
NullSpace[M]            (* Basis for null space. 4 vectors so nullity=4. *)
(* Nulity+rank= 4+2 equals the number of columns, 6. Rank Theorem is OK. *)
(* Exercise 10. *)
MatrixRank[M]                    (* rank of M is 2 *)
MatrixRank[Transpose[M]]         (* rank of M^T is 2 *)
(* Exercise 11 *)
m = Join[u1,u2,e1,e2,e3,e4,2]       (* Pivot columns 1,3,4,5. So basis: *)
RowReduce[m]                            (* {u1,e1,e2,e3}.                  *)
(* Exercise 12. *)
a=-7; b=2; u={{1,3},{0,2}};                   (* Entering scalars           *)
v={{-1,1},{0,8}}; w={{2,1},{0,-9}};           (* and matrices.              *)
(u+v) - (v+u)                                 (* A(2),                      *)
(u+v)+w - (u+(v+w))                           (* A(3),                      *)
a (u+v) - (a u+a v)                           (* M(2),                      *)
(a+b) u - (a u+b u)                           (* M(3), and                  *)
(a b) u - a (b u)                             (* M(4)  hold.                *)
(* Exercise 13. *)
Clear[a,b]                                   (* Clear the values of a and b *)
f[a_,b_,x_] = a Cos[3 x] + b Sin[2 x]        (* Defining f.                 *)
Plot[{f[1,1,x],f[3,0,x],f[0,-3,x],f[3,-4,x],f[4,-3,x]},{x,0,Pi}](* Plotting.*)
(* Exercise 14. *)
p1 = a1 x^3 + b1 x^2;   p2 = a2 x^3 + b2 x^2;  (* General polynomials of V. *)
Collect[p1+p2,x]        (* The sum p1 + p2 is in V.                         *)
Collect[Expand[c p1],x] (*  c*p1 is in V. So V is a subspace.               *)
(* Exercise 15. *)
B1 = {{0,0,0,1},{0,0,-1,1},{0,1,-1,0},{-1,1,0,-1}}
B2 = {{0,0,1,2},{0,4,-1,0},{1,0,-1,0},{0,1,0,1}}
RowReduce[B1]   (* All matrices have 4 pivot rows, so they are linearly *)
RowReduce[B2] (* independent, hence they form a basis, since dim(P_3)=4.*)
(* Exercises 16-18. *)
P1=RowReduce[Join[B2,B1,2]]         (* Reduce [B1:B2] and keep the *)
P=P1[[All, 5 ;; 8]]                 (* last 4 columns to get P.    *)
Q1=RowReduce[Join[B1,B2,2]]         (* Repeat with Q.              *)
Q=Q1[[All, 5 ;; 8]]
P . Q                               (* PQ=I and same size, so P^(-1)=Q. *)
(* Exercise 19. *)
p={{-1},{0},{5},{1}}                     (* x^3+5x+1 in vector form.   *)
pb1 = RowReduce[Join[B1, p, 2]][[All, 5]]     (* [p]_B1.        *)
pb2 = RowReduce[Join[B2, p, 2]][[All, 5]]     (* [p]_B2.        *)
P . pb1                     (* P[p}_B2 yields [p]_B1 as expected.      *)
```

4.9.2 Selected solutions with MATLAB

```
% Data.
v1 = [1; 2; 3]; v2 = [2; 3; 4]; v3 = [3; 4; 5];
v4 = [4; 5; 6]; v5 = [5; 6; 7]; v6 = [6; 7; 8];
u1 = [1; 2; 3; 4]; u2 = [2; 4; 6; 8];
u3 = [1; 3; 5; 7]; u4 = [1; 4; 7; 10];
u5 = [1; 5; 9; 13];
e1 = [1; 0; 0; 0]; e2 = [0; 1; 0; 0];      % Standard basis vectors.
e3 = [0; 0; 1; 0]; e4 = [0; 0; 0; 1];
M = [v1 v2 v3 v4 v5 v6]
N = [u1 u2 u3 u4 u5]
B = [e1 e2 u3 u4]    % The matrix with columns the vectors of B.
% Exercises 1-7.
rref(M)                 % 2 pivots, 3 rows: does not span R^3. Not a basis.
rref(N)                 % 2 pivots, 4 rows: does not span R^4. Not a basis.
rref(B)                 % 4 pivots, 4 rows, 4 columns: spanning and
% linearly independent: Basis of R^4.
rref([B u1])            % We solve the system [B:u1] by reduction. The last
ans(:,5)                % column gives the coordinates of u1. Repeat with u2.
B * u5                  % Exercise 5: x is just Bu5.
rref([v1 v2 v3])        % Pivots at (1,1),(2,2). So {v1,v2} is a basis.
rref([u1 u2 u3 u4])     % Pivots at (1,1),(2,3). So {u1,u3} is a basis.
% Exercises 8-9.
rref(M)                 % Pivot cols. 1,2 so basis {v1,v2}. Rank=2.
rank(M)                 % The rank is 2.
rref(M')                % The first two rows form a basis for row space.
null(M)                 % Basis for null space. 4 vectors so nullity=4.
% Nulity + rank = 4 +2 equals the number of columns, 6. Rank Theorem is OK.
% Exercise 10.
rank(M')                % The rank of the transpose is also 2.
% Exercise 11.
m = [u1 u2 e1 e2 e3 e4]        % Pivot columns 1,3,4,5. So basis:
rref(m)                        % {u1,e1,e2,e3}.
% Exercise 12.
a=-7; b=2; u=[1 3; 0 2];                       % Entering scalars
v=[-1 1; 0 8]; w=[2 1; 0 -9];                  % and matrices.
(u+v) - (v+u)                                  % A(2),
(u+v)+w - (u+(v+w))                            % A(3),
a*(u+v) - (a*u+a*v)                            % M(2),
(a+b)*u - (a*u+b*u)                            % M(3), and
(a*b)*u - a*(b*u)                              % M(4)  hold.
% Exercise 13.
function [A] = f(a,b,x)            % Defining f in an m-file. Type the
A=a*cos(3*x)+b*sin(2*x);       % code on the left in a file named f.m .
end                            % Then in MATLAB session type:
fplot(@(x)[f(1,1,x),f(3,0,x),f(0,-3,x),f(4,-3,x),f(-3,4,x)]) % Plotting.
% Exercise 15.
B1 = [0 0 0 1; 0 0 -1 1; 0 1 -1 0; -1 1 0 -1]
```

```
B2 = [0 0 1 2; 0 4 -1 0; 1 0 -1 0; 0 1 0 1]
rref(B1)     % All matrices have 4 pivot rows, so the polynomials are linearly
rref(B2)     % independent, hence they form a basis, since dim(P_3)=4.
% Exercises 16-18.
P1=rref([B2 B1])            % Reduce [B1:B2] and keep the last 4
P=P1(:,5:8)                 % columns to get P.
Q1=rref([B1 B2])            % Repeat with Q.
Q=Q1(:,5:8)
P * Q                       % PQ=I and same size, so P^(-1)=Q
% Exercise 19.
p=[-1;0;5;1]                % x^3+5x+1 in vector form.
rref([B1,p])
pb1=ans(:,5)                % [p]_B1.
rref([B2,p])
pb2=ans(:,5)                % [p]_B2.
P * pb1                     % P[p]_B2 yields [p]_B1 as expected.
```

4.9.3 Selected solutions with Maple

```
# Data.
with(LinearAlgebra);
v1 := Vector([1,2,3]); v2 := Vector([2,3,4]); v3 := Vector([3,4,5]);
v4 := Vector([4,5,6]); v5 := Vector([5,6,7]); v6 := Vector([6,7,8]);
u1 := Vector([1,2,3,4]); u2 := Vector([2,4,6,8]);
u3 := Vector([1,3,5,7]); u4 := Vector([1,4,7,10]);
u5 := Vector([1,5,9,13]);
e1 := Vector([1,0,0,0]);e2:=Vector([0,1,0,0]);#Standard basis vectors.
e3 := Vector([0,0,1,0]); e4:=Vector([0,0,0,1]);
M := <v1|v2|v3|v4|v5|v6>;
N := <u1|u2|u3|u4|u5>;
B := <e1|e2|u3|u4>;   # The matrix with columns the vectors of B.
# Exercises 1-7.
ReducedRowEchelonForm(M);  # 2 pivots, 3 rows:
# does not span R^3. Not a basis.
ReducedRowEchelonForm(N);  # 2 pivots, 4 rows:
# does not span R^4. Not a basis.
ReducedRowEchelonForm(B);  # 4 pivots, 4 rows, 4 columns:
# spanning and linearly independent: Basis of R^4.
ReducedRowEchelonForm(<B|u1>);# Solve system [B:u1] by reduction.
Column(%,5);   #The last  column gives the coordinates of u1.
#  Repeat with u2.
B . u5;                 # Exercise 5: x is just Bu5.
ReducedRowEchelonForm(<v1|v2|v3>); # Pivots at (1,1),(2,2).
# So {v1,v2} is a basis.
ReducedRowEchelonForm(<u1|u2|u3|u4>);# Pivots at (1,1),(2,3).
# So {u1,u3} is a basis.
# Exercises 8-9.
```

```
ColumnSpace(M);     # Basis for column space. 2 vectors, rank=2.
Rank(M);              # Finding the rank directly.
RowSpace(M);          # Basis for row space.
NullSpace(M);      # Basis for null space. 4 vectors, nullity=4.
# rank=2 + nullity=4 equals the number of columns 6. Rank Theorem.
# Exercise 10.
Rank(Transpose(M));      #  The rank of the transpose is also 2.
# Exercise 11.
m := <u1|u2|e1|e2|e3|e4>;    # Pivot columns 1,3,4,5. So basis:
ReducedRowEchelonForm(m);                    # {u1,e1,e2,e3}.
# Exercise 12.
a:=-7: b:=2: u:=Matrix(2,2,[1,3,0,2]):  # Entering scalars and
v:=Matrix(2,2,[-1,1,0,8]): w:=Matrix(2,2,[2,1,0,-9]): #matrices.
(u+v) - (v+u);                                 # A(2),
(u+v)+w - (u+(v+w));                           # A(3),
a*(u+v) - (a*u+a*v);                           # M(2),
(a+b)*u - (a*u+b*u);                           # M(3), and
(a*b)*u - a*(b*u);                             # M(4)  hold.
# Exercise 13.
f := proc(a,b,x) a*cos(3*x)+b*sin(2*x) end:       # Defining f.
plot({f(1,1,x),f(3,0,x),f(0,-3,x),f(3,-4,x),f(4,-3,x)},x=0..Pi);
# Exercise 14.
p1 := a1*x^3 + b1*x^2:  p2 := a2*x^3 + b2*x^2:    # General p
collect(p1+p2,x);         # The sum p1 + p2 is in V.
collect(expand(c*p1),x);  # c*p1 is in V. So V is a subspace.
# Exercise 15.
B1 := Matrix([[0,0,0,1],[0,0,-1,1],[0,1,-1,0],[-1,1,0,-1]]);
B2 := Matrix([[0,0,1,2],[0,4,-1,0],[1,0,-1,0],[0,1,0,1]]);
ReducedRowEchelonForm(B1);   # 4 pivot rows, so linearly
ReducedRowEchelonForm(B2); # Independent. A basis since dim(P_3)=4.
# Exercises 16-18.
P1:=ReducedRowEchelonForm(<B2|B1>); # Reduce [B1:B2] and keep
P:=SubMatrix(P1,1..4,5..8);    # the last 4 columns to get P.
Q1:=ReducedRowEchelonForm(<B1|B2>);        # Repeat with Q.
Q:=SubMatrix(Q1,1..4,5..8);
P.Q;                     # PQ=I and same size so P^(-1)=Q.
# Exercise 19.
p:=Vector([-1,0,5,1]);           # x^3+5x+1 in vector form.
pb1:=Column(ReducedRowEchelonForm(<B1|p>),5);     # [p]_B1.
pb2:=Column(ReducedRowEchelonForm(<B2|p>),5);     # [p]_B2.
P.pb1;                # P[p]_B2 yields [p]_B1 as expected.
```

5 Linear transformations

> My methods are really methods of working and thinking; this is why they have crept in everywhere anonymously.
>
> Emmy Noether, German mathematician (Figure 5.1).

Figure 5.1: Emmy Noether.
Unknown author Publisher: Mathematical Association of America, Brooklyn Museum, Agnes Scott College, Public domain, via Wikimedia Commons.
Emmy Noether (1882–1935) was a remarkable German mathematician renowned for her pioneering contributions to abstract algebra, invariant theory, Galois theory, and theoretical physics. She held positions at the University of Göttingen in Germany, Bryn Mawr College PA, and the Institute for Advanced Study in Princeton.

Introduction

In this chapter, we study the main transformations between vector spaces, called *linear transformations*, introduced by *Peano* in 1888 (Figure 5.2).[1] These generalize our familiar matrix transformations.

Figure 5.2: Giuseppe Peano.
(Image by Unknown author, Un. of St Andrews, Public Domain, https://commons.wikimedia.org/w/index.php?curid=2633677.)
Giuseppe Peano (1858–1932) studied mathematics at the University of Turin, Italy, and became a professor there. In 1888, he published the book Geometrical Calculus, which contains the first modern definition of a vector space (based on Grassmann's idea). He also published the so-called Peano axioms, which define the natural numbers and discovered a space-filling curve named after him.

We start with the definition, examples, and main properties of linear transformations. Then we introduce the kernel and the range, which generalize the familiar con-

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[1] Linear transformations are introduced in the final chapter of Peano's book “Giuseppe Peano, Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann, preceduto dalle operazioni della Iogica deduttiva”, Turin: Bocca, 1888.

https://doi.org/10.1515/9783111331850-005

cepts of the null space and the column space of a matrix. The concept of isomorphism is discussed at this point. It is fundamental in distinguishing between two vector spaces, whether or not they are essentially the same. Then the useful concept of the matrix of a linear transformation is discussed. It gives us ways to use matrix arithmetic to answer questions about linear transformations. Then we discuss addition and scalar multiplication of linear transformations that make them a vector space in its own right. We conclude the chapter by examining compositions of linear transformations.

5.1 Linear transformations

A linear transformation is a transformation between two vector spaces that preserves addition and scalar multiplication. This new concept generalizes the concept of a matrix transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$.

Definition 5.1.1. A *linear transformation* or *linear map* from a vector space V to a vector space W is a transformation $T : V \to W$ such that for all vectors $\mathbf{u}$ and $\mathbf{v}$ of V and any scalar c, we have (Figure 5.3)

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$;
2. $T(c\mathbf{u}) = cT(\mathbf{u})$.

The addition in $\mathbf{u}+\mathbf{v}$ is addition in V, whereas the addition in $T(\mathbf{u})+T(\mathbf{v})$ is addition in W. Likewise, scalar multiplications $c\mathbf{u}$ and $cT(\mathbf{u})$ occur in V and W, respectively. In the particular case where $V = W$, the linear transformation $T : V \to V$ is called a *linear operator* of V.

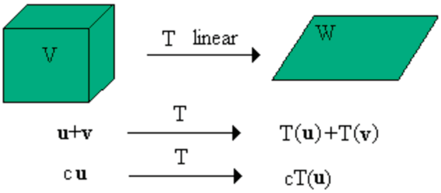

Figure 5.3: Linear transformations preserve the vector space operations.

The most important examples of linear transformations are matrix transformations. As we will see, in a very precise sense a linear transformation can be viewed as a matrix transformation.

Example 5.1.2 (Matrix transformations). A matrix transformation is a linear transformation. (See Section 2.2.)

Example 5.1.3. Prove that $T : M_{22} \to P_3$ defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = d + cx + (b - a)\,x^3$$

is linear.

Solution. We have

$$\begin{aligned}
T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= T\begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix} \\
&= (d_1 + d_2) + (c_1 + c_2)\,x + \{(b_1 + b_2) - (a_1 + a_2)\}x^3 \\
&= \{d_1 + c_1x + (b_1 - a_1)\,x^3\} + \{d_2 + c_2x + (b_2 - a_2)\,x^3\} \\
&= T\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + T\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
T\left(c\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) &= T\begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix} \\
&= cd_1 + cc_1x + (cb_1 - ca_1)\,x^3 \\
&= c[d_1 + c_1x + (b_1 - a_1)\,x^3] \\
&= cT\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}.
\end{aligned}$$

Both parts of the definition are satisfied, so T is linear.

Theorem 5.1.4. *$T : V \to W$ is a linear transformation if and only if for all vectors $\mathbf{v}_1$ and $\mathbf{v}_2 \in V$ and all scalars c_1 and c_2,*

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2).$$

Proof. Exercise. ☐

More generally, if $\mathbf{v}_i$ are vectors in V and c_i are scalars ($i = 1, \dots, n$), then

$$T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n).$$

In other words, *a linear transformation maps a linear combination of vectors to the linear combination of the images with the same coefficients.*

Theorem 5.1.5. *Let $T : V \to W$ be a linear transformation. Then*

1. $T(\mathbf{0}) = \mathbf{0}$;
2. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$.

Proof of 2. By Theorem 5.1.4 with $c_1 = 1$ and $c_2 = -1$, we have

$$T(\mathbf{u} - \mathbf{v}) = 1T(\mathbf{u}) + (-1)T(\mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}). \qquad \square$$

The transformation $\mathbf{0} : V \to W$ that maps all vectors of V to $\mathbf{0}$ in W, i. e.,

$$\mathbf{0}(\mathbf{v}) = \mathbf{0} \quad \text{for all } \mathbf{v} \in V,$$

is a linear transformation. It is called the *zero transformation*.

The transformation $I : V \to V$ that maps each vector of V to itself, i. e.,

$$I(\mathbf{v}) = \mathbf{v} \quad \text{for all } \mathbf{v} \in V,$$

is a linear transformation. It is called the *identity transformation of* V.

Example 5.1.6 (Homothety). Prove that for a fixed scalar c, the transformation $T : V \to V$ defined by

$$T(\mathbf{v}) = c\,\mathbf{v}$$

is linear.

Solution. Let $\mathbf{u}, \mathbf{w} \in V$ and $r \in \mathbf{R}$. T is linear, because

$$\begin{aligned} T(\mathbf{u} + \mathbf{w}) &= c(\mathbf{u} + \mathbf{w}) = c\mathbf{u} + c\mathbf{w} = T(\mathbf{u}) + T(\mathbf{w}), \\ T(r\mathbf{u}) &= c(r\mathbf{u}) = r(c\mathbf{u}) = rT(\mathbf{u}). \end{aligned}$$

The transformation defined in Example 5.1.6 is called a *homothety*. If $c > 1$, then the homothety is a *dilation*, and its effect on $\mathbf{v}$ is “stretching” $\mathbf{v}$ by a factor of c (Figure 5.4). If $0 < c < 1$, then the homothety is a *contraction*, and its effect on $\mathbf{v}$ is “shrinking” $\mathbf{v}$ by a factor of c. If $c < 0$, then this transformation reverses the direction of $\mathbf{v}$.

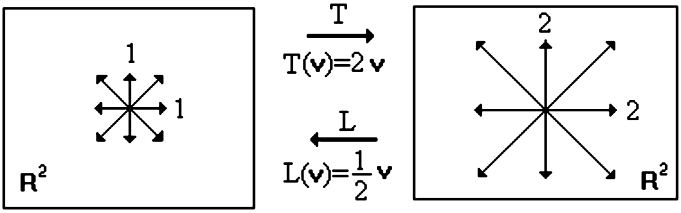

Figure 5.4: Dilation and contraction by a factor of 2.

Example 5.1.7 (Multiplication by fixed matrix). Let A be a fixed $m \times n$ matrix. The transformation $T : M_{nk} \to M_{mk}$ defined by

$$T(B) = AB$$

is linear.

Solution. We have

$$\begin{aligned} T(B + C) &= A(B + C) = AB + AC = T(B) + T(C), \\ T(cB) &= A(cB) = c(AB) = cT(B). \end{aligned}$$

Example 5.1.8. The transformation $T : P_2 \to P_1$ defined by

$$T(a + bx + cx^2) = b + 2cx$$

is linear.

Solution. Let $p_1 = a_1 + b_1x + c_1x^2$ and $p_2 = a_2 + b_2x + c_2x^2$. Then

$$\begin{aligned} T(p_1 + p_2) &= T\big((a_1 + a_2) + (b_1 + b_2)x + (c_1 + c_2)x^2\big) \\ &= (b_1 + b_2) + 2(c_1 + c_2)x \\ &= (b_1 + 2c_1x) + (b_2 + 2c_2x) \\ &= T(p_1) + T(p_2). \end{aligned}$$

The verification of Part 2 of the definition is left as an exercise.

Example 5.1.9 (Dotting by fixed vector). Let $\mathbf{u}$ be a fixed vector in $\mathbf{R}^n$. The transformation $T : \mathbf{R}^n \to \mathbf{R}$ defined by

$$T(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$$

is linear.

Solution. The dot product may be viewed as the matrix multiplication $T(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$. So this is a particular case of Example 5.1.7.

Example 5.1.10 (Requires calculus). Let V be the vector space of all differentiable real-valued functions defined on $\mathbf{R}$. Then the transformation $T : V \to V$ defined by differentiating each $f \in V$,

$$T(f) = f',$$

is linear.

Solution. If $f, g \in V$ and $c \in \mathbf{R}$, then the basic properties of derivatives yield

$$T(f+g) = (f+g)' = f' + g' = T(f) + T(g),$$
$$= T(cf) = (cf)' = cf' = cT(f).$$

So T is linear.

5.1.1 Evaluation of linear transformation from a basis

One of the most important properties of a linear transformation is that all its images can be determined *uniquely* given only its values on a basis. This is expressed by the following theorem. First, recall that the *range* of any map is the set of all its images.

Theorem 5.1.11. *Let $T : V \to W$ be a linear transformation with V being n-dimensional. Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V. Then the set $T(B) = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ spans the range of T.*

Proof. Let $\mathbf{w}$ be in the range of T. Then there is $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. Since B spans V, there are scalars c_i such that $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$. So

$$\mathbf{w} = T(\mathbf{v}) = T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n).$$

Hence $\mathbf{w}$ is a linear combination of $T(B)$. □

Example 5.1.12. Let $T : P_1 \to \mathbf{R}^3$ be a linear transformation such that

$$T(-1+x) = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}, \quad T(1+x) = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

Find $T(a+bx)$.

Solution. $\mathcal{B} = \{-1+x, 1+x\}$ is a basis of P_2 (check). Every polynomial $a+bx$ is a linear combination

$$a + bx = c_1(-1+x) + c_2(1+x)$$
$$= (-c_1 + c_2) + (c_1 + c_2)x.$$

Hence $-c_1 + c_2 = a$, $c_1 + c_2 = b$. Solving for c_i, we get $c_1 = \frac{1}{2}b - \frac{1}{2}a$, $c_2 = \frac{1}{2}a + \frac{1}{2}b$. By linearity

$$T(a+by) = T\big(c_1(-1+x) + c_2(1+x)\big) = c_1T(-1+x) + c_2T(1+x)$$
$$= \left(\frac{1}{2}b - \frac{1}{2}a\right)T(-1+x) + \left(\frac{1}{2}a + \frac{1}{2}b\right)T(1+x)$$

$$= \left(\frac{1}{2}b - \frac{1}{2}a\right)\begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix} + \left(\frac{1}{2}a + \frac{1}{2}b\right)\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} a+b \\ 3b \\ 2a-b \end{bmatrix}.$$

Exercises 5.1

In Exercises 1–6, determine whether or not $T : P_1 \longrightarrow P_1$ is linear.

1. $T(a + bx) = (3a + b) + (a - 2b)x$.

2. $T(a + bx) = (a - b) + (a + b + 1)x$.

3. $T(a + bx) = (a - 5b) + abx$.

4. $T(p) = 15p$.

5. $T(p) = 6p - 2$.

6. $T(p) = 5p^3$.

7. Provide the details of the proof of Theorem 5.1.4.

8. *True or False?*

(a) $T : M_{22} \longrightarrow P_2$ defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a + b) + x + (c - d)x^2$$

is linear.

(b) $T : P_2 \longrightarrow P_1$ defined by

$$T\left(a + bx + cx^2\right) = b + 2cx$$

is linear.

(c) $T : P_1 \longrightarrow P_2$ defined by

$$T(a + bx) = ax + \frac{b}{2}x^2$$

is linear.

9. Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ be a linear transformation such that

(a) $T(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$,

(b) $T(-\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, and

(c) $T(\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Find $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $T\begin{bmatrix} -10 \\ 15 \\ -25 \end{bmatrix}$.

10. Let $T : P_1 \to P_1$ be the linear transformation such that

$$T(-1+x) = 1+x, \quad T(1+x) = 2-x.$$

Find $T(a+bx)$ and $T(b+ax)$.

11. Let $T : P_2 \to P_1$ be the linear transformation such that

$$T\left(1+x+x^2\right) = -1+3x, \quad T\left(1+x-x^2\right) = -3+2x, \quad T\left(1-x+x^2\right) = 1+2x.$$

Find $T(a+bx+cx^2)$.

12. Let $T : P \to P$ be the linear transformation that satisfies

$$T\left(x^n\right) = \frac{1}{n+1}x^{n+1}, \quad n \geq 0.$$

Find
(a) $T(x+x^2)$,
(b) $T(-1+x^3)$,
(c) $T((1+x^2)^2)$.

13. *True or False?* Let q_1 and q_2 be two fixed polynomials in P_n. Then the transformation $T : P_n \to P_n$ such that $T(p) = q_1 p q_2 - 7p$ is linear.

14. Let C be an invertible $n \times n$ matrix. Prove that the following transformations T, L, and R from M_{nn} to itself are linear:
(a) $T(X) = CXC^{-1}$;
(b) $L(X) = C^{-1}XC$;
(c) $R(X) = C^{-1}XC - X$.

15. Explain why a linear transformation $T : P_1 \to \mathbf{R}^2$ such that

$$T(1+x) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad T(3+3x) = \begin{bmatrix} 12 \\ 3 \end{bmatrix}$$

cannot be uniquely determined. Find at least two such linear transformations.

16. Give an example of a nonlinear transformation $T : \mathbf{R}^2 \to P_1$ with the property that $T(\mathbf{0})$ is the zero polynomial.

17. Give an example of a linear transformation $T : P_1 \to \mathbf{R}^2$ and linearly independent polynomials p and q such that $\{T(p), T(q)\}$ is linearly dependent.

18. For any linear transformation $T : V \to W$, prove that if a subset $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subseteq V$ is linearly dependent, then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_k)\} \subseteq W$ is also linearly dependent.

19. Find all linear transformations from $\mathbf{R}$ to $\mathbf{R}$.

In Example 5.1.9, we saw that dotting by a fixed vector in $\mathbf{R}^n$ is a linear transformation. In Exercise 20, we see that this is the *only* type of linear transformations from $\mathbf{R}^n$ to $\mathbf{R}$.

20. Let $T : \mathbf{R}^n \to \mathbf{R}$ be linear. Prove that there exists an n-vector $\mathbf{u}$ such that

$$T(\mathbf{v}) = \mathbf{u} \cdot \mathbf{v} \quad \text{for all } \mathbf{v} \in \mathbf{R}^n.$$

5.2 Kernel and range

In this section, we study two subspaces are associated with any linear transformation, the *kernel* and the *range*. We also discuss when we consider two vector spaces to be the “same”, or *isomorphic*.

Definition 5.2.1. The *kernel* $\operatorname{Ker}(T)$ of a linear transformation $T : V \to W$ is the set of all vectors in V that map to zero in W:

$$\operatorname{Ker}(T) = \{\mathbf{v} \in V,\ T(\mathbf{v}) = \mathbf{0} \in W\}.$$

The *range* $\operatorname{Range}(T)$ of T is the set of all images of T in W:

$$\operatorname{Range}(T) = \{\mathbf{w} \in W,\ \mathbf{w} = T(\mathbf{v}) \text{ for some } \mathbf{v} \in V\}.$$

Example 5.2.2. Compute the kernel and range of
(a) the zero linear transformation $\mathbf{0} : V \to W$,
(b) the identity linear transformation $I : V \to V$,
(c) the projection linear transformation $p : \mathbf{R}^2 \to \mathbf{R}^2$, $p(x,y) = (x,0)$.

Solution.

(a) $\mathbf{0}(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$, so the kernel is V. Since $\mathbf{0}$ is the only image, the range is $\{\mathbf{0}\}$. Thus

$$\operatorname{Ker}(\mathbf{0}) = V, \quad \operatorname{Range}(\mathbf{0}) = \{\mathbf{0}\}.$$

(b) Since $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$, every nonzero vector is mapped to a nonzero one. Hence the kernel is $\{\mathbf{0}\}$. Every $\mathbf{v}$ is its own image, so the range is V. Thus

$$\operatorname{Ker}(I) = \{\mathbf{0}\}, \quad \operatorname{Range}(I) = V.$$

(c) Vector (x,y) is in $\operatorname{Ker}(p)$ if and only if $p(x,y) = (x,0) = (0,0)$. So $x = 0$. So the kernel consists of the points $(0,y)$. Also, (z,w) is in the range if and only if there is (x,y) such that $p(x,y) = (x,0) = (z,w)$. Hence $w = 0$. So the range consists of the points $(x,0)$ (Figure 5.5). Thus

$$\operatorname{Ker}(p) = \{(0,y),\ y \in \mathbf{R}\}, \quad \operatorname{Range}(p) = \{(x,0),\ x \in \mathbf{R}\}.$$

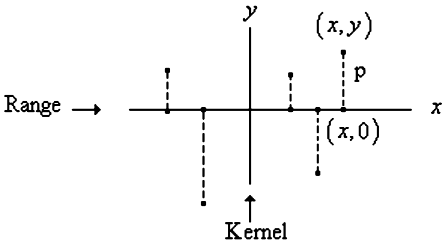

Figure 5.5: The kernel and range of the projection onto the x-axis.

Example 5.2.3. Find the kernel and range of $T : P_2 \to \mathbf{R}^2$ defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a - c \\ b + c \end{bmatrix}.$$

Solution. Ker(T) is the set of polynomials $a + bx + cx^2$ such that

$$T(a + bx + cx^2) = \begin{bmatrix} a - c \\ b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the homogeneous system $a - c = 0$, $b + c = 0$, we get $a = r$, $b = -r$, $c = r$, $r \in \mathbf{R}$. Hence

$$\operatorname{Ker}(T) = \{r - rx + rx^2,\ r \in \mathbf{R}\} = \operatorname{Span}\{1 - x + x^2\}.$$

Let $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$ be a vector in Range(T). Then there are a, b, c such that

$$T(a + bx + cx^2) = \begin{bmatrix} a - c \\ b + c \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}.$$

The resulting system

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

is solvable for all s, t, because each row of the coefficient matrix has a pivot. Therefore Range$(T) = \mathbf{R}^2$ (Figure 5.6).

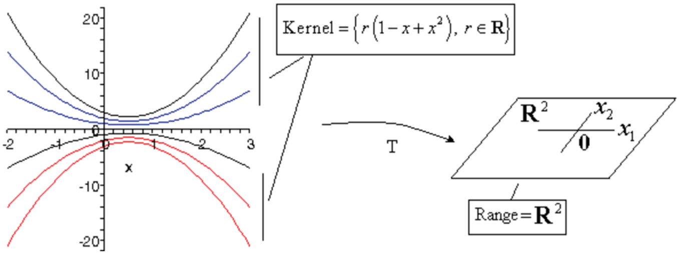

Figure 5.6: The kernel and range of T.

Theorem 5.2.4. *If $T : V \to W$ is a linear transformation, then*

1. $\mathrm{Ker}(T)$ *is a subspace of* V*;*
2. $\mathrm{Range}(T)$ *is a subspace of* W*.*

Proof of 2. $\mathrm{Range}(T)$ contains $\mathbf{0}$ (why?). If $\mathbf{w}_1, \mathbf{w}_2 \in \mathrm{Range}(T)$, then there are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of V such that $\mathbf{w}_1 = T(\mathbf{v}_1)$ and $\mathbf{w}_2 = T(\mathbf{v}_2)$. T is linear, so we have

$$\begin{aligned} \mathbf{w}_1 + \mathbf{w}_2 &= T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2), \\ c\,\mathbf{w}_1 &= c\,T(\mathbf{v}_1) = T(c\,\mathbf{v}_1), \qquad c \in \mathbf{R}. \end{aligned}$$

We found vectors $\mathbf{v}_1 + \mathbf{v}_2$ and $c\mathbf{v}_1$ that are mapped to $\mathbf{w}_1 + \mathbf{w}_2$ and $c\mathbf{w}_1$, respectively. Hence $\mathbf{w}_1 + \mathbf{w}_2$, $c\mathbf{w}_1$ are in $\mathrm{Range}(T)$. So $\mathrm{Range}(T)$ is a subspace of W. ☐

Example 5.2.5. Find the kernel and range of T of Example 5.1.9 (Figure 5.7).

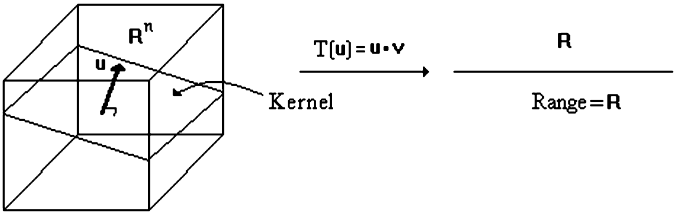

Figure 5.7: Kernel and image of “dotting” by a fixed vector.

Solution. The kernel consists of all vectors $\mathbf{v}$ such that $\mathbf{u} \cdot \mathbf{v} = 0$, i. e., all n-vectors *orthogonal* to $\mathbf{u}$. This is the hyperplane through the origin with normal $\mathbf{u}$.

To find the range, we observe that since $\mathbf{u}$ is nonzero,

$$T(\mathbf{u}) = \mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2 > 0.$$

Hence the nonzero number $\|\mathbf{u}\|^2$ is in the range of T. So the range contains the span of $\|\mathbf{u}\|^2$, which is $\mathbf{R}$. Therefore $\mathrm{Range}(T) = \mathbf{R}$.

Definition 5.2.6. The dimension of the kernel is called the *nullity* of T. The dimension of the range is called the *rank* of T.

Theorem 5.2.7. *Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a matrix transformation with standard matrix A. Then*

1. $\operatorname{Ker}(T) = \operatorname{Null}(A)$;
2. $\operatorname{Range}(T) = \operatorname{Col}(A)$;
3. $\operatorname{Nullity}(T) = \operatorname{Nullity}(A)$;
4. $\operatorname{Rank}(T) = \operatorname{Rank}(A)$.

Proof. Exercise. ☐

The next theorem is one of the cornerstones of linear algebra. It generalizes the Rank theorem (Theorem 4.6.19) and it is proved in Section 5.3.

Theorem 5.2.8 (The dimension theorem). *If $T : V \to W$ is a linear transformation from a finite-dimensional vector space V into any vector space W, then*

$$\operatorname{Nullity}(T) + \operatorname{Rank}(T) = \dim(V).$$

Example 5.2.9. Determine the range of the linear transformation $T : \mathbf{R}^4 \to P_2$ defined by

$$T\begin{bmatrix} a & b & c & d \end{bmatrix}^T = (a - b) + (c + d)x + (2a + b)x^2.$$

Solution. The null space of T is spanned by $\begin{bmatrix} 0 & 0 & -1 & 1 \end{bmatrix}^T$, which is obtained immediately from solving the system $a - b = 0$, $c + d = 0$, $2a + b = 0$. Therefore the nullity of T is 1. Hence by the dimension theorem

$$\operatorname{Rank}(T) = \dim(\mathbf{R}^4) - \operatorname{Nullity}(T) = 4 - 1 = 3.$$

Thus the range is a three-dimensional subspace of P_2, so it must be all of P_2.

5.2.1 Isomorphisms

There is a wide variety of useful vector spaces. However, setting the different notations of the individual examples aside, we find that many of these spaces are essentially the same. We analyze the notion of two vector spaces being the same. Such spaces are called isomorphic.

Recall that a transformation $T : A \to B$ between two sets allows for

(a) two or more elements of A to have the same image,
(b) the range of T to be strictly contained in the codomain B.

If either (a) or (b) does not occur, then we have two interesting particular cases.

Definition 5.2.10. Let $T : A \to B$.

1. The transformation T is called *one-to-one* if for each element b in the range, there is *exactly one* element a with image $b = T(a)$ (Figure 5.8). This can be rephrased as

   $$T(a_1) = T(a_2) \Rightarrow a_1 = a_2 \tag{5.1}$$

   or, equivalently, as

   $$a_1 \neq a_2 \Rightarrow T(a_1) \neq T(a_2), \tag{5.2}$$

2. The transformation T is called *onto* if its range equals its codomain (Figure 5.8), i. e., if

   $$\text{Range}(T) = B. \tag{5.3}$$

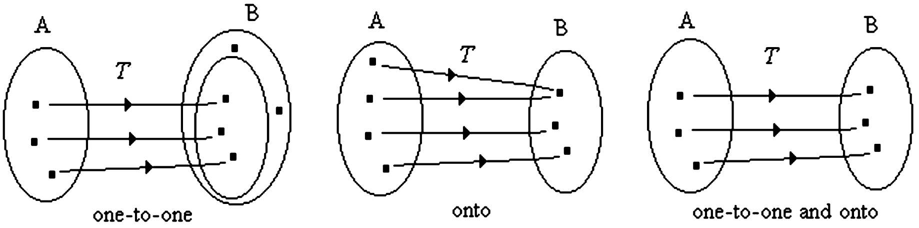

Figure 5.8: One-to-one and onto.

Theorem 5.2.11. *Let $T : V \to W$ be a linear transformation. Then the following are equivalent.*

1. *T is one-to-one.*
2. $\text{Ker}(T) = \{\mathbf{0}\}$.

Proof.

⇒ Let T be one-to-one. If $\mathbf{v}$ is in the kernel, then $T(\mathbf{v}) = \mathbf{0} = T(\mathbf{0})$. Hence $\mathbf{v} = \mathbf{0}$, because T is one-to-one. Therefore, $\text{Ker}(T) = \{\mathbf{0}\}$ (Figure 5.9).

⇐ Conversely, suppose $\text{Ker}(T) = \{\mathbf{0}\}$. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors of V such that $T(\mathbf{u}) = T(\mathbf{v})$. Since T is linear,

$$T(\mathbf{u}) = T(\mathbf{v}) \Rightarrow T(\mathbf{u}) - T(\mathbf{v}) = \mathbf{0} \Rightarrow T(\mathbf{u} - \mathbf{v}) = \mathbf{0}.$$

So $\mathbf{u} - \mathbf{v}$ is in the kernel of T, which was assumed to be $\{\mathbf{0}\}$. Hence $\mathbf{u} - \mathbf{v} = \mathbf{0}$, i. e., $\mathbf{u} = \mathbf{v}$. Therefore T is one-to-one. ☐

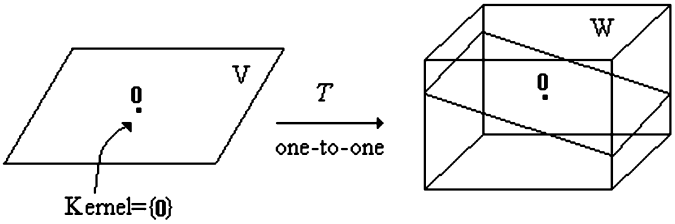

Figure 5.9: T linear is one-to-one if and only if Kernel$(T) = \{\mathbf{0}\}$.

Example 5.2.12. Prove that $T : P_2 \to M_{22}$ defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a+b & b+c \\ a+c & 0 \end{bmatrix}$$

is linear and one-to-one. Is T onto?

Solution. The linearity of T is left as an exercise. Let $p = a + bx + cx^2$ be in the kernel of T. Then

$$T(p) = \begin{bmatrix} a+b & b+c \\ a+c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $a+b = 0, b+c = 0, a+c = 0$. This homogeneous system has only the trivial solution. So $p = \mathbf{0}$. Therefore T is one-to-one by Theorem 5.2.11. T is *not* onto. For example, there is no polynomial that maps to I_2 (why?).

Theorem 5.2.13. *Let $T : V \to W$ be a one-to-one linear transformation. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent subset of V, then $T(S) = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)\}$ is a linearly independent subset of W.*

Proof. Assuming that $c_1T(\mathbf{u}_1)+\cdots+c_kT(\mathbf{v}_k) = \mathbf{0}$, we have, by linearity and Theorem 5.2.11,

$$\begin{aligned} c_1T(\mathbf{u}_1) + \cdots + c_kT(\mathbf{v}_k) &= \mathbf{0} \\ \Rightarrow T(c_1\mathbf{u}_1 + \cdots + c_k\mathbf{v}_k) &= \mathbf{0} \\ \Rightarrow c_1\mathbf{u}_1 + \cdots + c_k\mathbf{v}_k &= \mathbf{0}. \end{aligned}$$

Hence $c_1 = \cdots = c_k = 0$, because S is linearly independent. Therefore $T(S)$ is linearly dependent. □

Definition 5.2.14. A linear transformation between two vector spaces that is one-to-one and onto is called an *isomorphism*. Two vector spaces are called *isomorphic* if there is an isomorphism between them. We consider isomorphic spaces to be the same, because their elements correspond one for one and the structure of the vector space operations is preserved through linearity.

In the next three examples, we ask the reader to verify that T is an isomorphism by showing that T is linear, one-to-one, and onto.

Example 5.2.15. $\mathbf{R}^6$ and $M_{2,3}$ are isomorphic with isomorphism $T : \mathbf{R}^6 \to M_{2,3}$ defined by

$$T(a_1, \dots, a_6) = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}.$$

Example 5.2.16. $M_{3,2}$ and $M_{2,3}$ are isomorphic with isomorphism $T : M_{3,2} \to M_{2,3}$ defined by

$$T \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix}.$$

Example 5.2.17. $\mathbf{R}^n$ and P_{n-1} are isomorphic with isomorphism $T : \mathbf{R}^n \to P_{n-1}$ defined by

$$T(a_0, \dots, a_{n-1}) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}.$$

(Figure 5.10.)

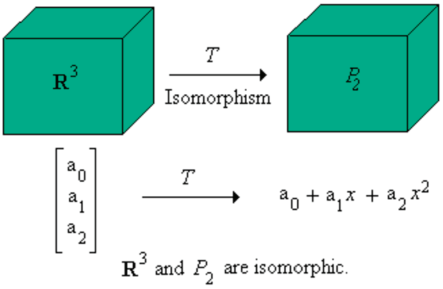
$\mathbf{R}^3$
T
Isomorphism
P_2
$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$
T
$a_0 + a_1 x + a_2 x^2$
$\mathbf{R}^3$ and P_2 are isomorphic.

Figure 5.10: Isomorphism of $\mathbb{R}^3$ and P_2.

Example 5.2.18. Prove that $\mathbf{R}^{mn}$ and $M_{m,n}$ are isomorphic.

Solution. Exercise (find a linear transformation that is one-to-one and onto).

Theorem 5.2.19. *Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces V and W with* $\dim(V) = \dim(W)$.

1. *If T is one-to-one, then it is an isomorphism.*
2. *If T is onto, then it is an isomorphism.*

Proof of 1. Let T be one-to-one. Then its nullity is zero. So, by the dimension theorem $\dim \operatorname{Range}(T) = \dim(V) = \dim(W)$. Therefore $\operatorname{Range}(T) = W$ by Theorem 4.3.19, Section 4.3. Hence T is onto. Therefore T is an isomorphism. ☐

! Theorem 5.2.19 is labor saving: One-to-one is equivalent to onto, but only if the dimensions of V and W are the same. (See Example 5.2.12 for one-to-one but not onto!)

Theorem 5.2.20. *Let V and W be finite-dimensional vector spaces. Then the following are equivalent:*

1. *V and W are isomorphic.*
2. *V and W have the same dimension.*

Proof of $\Leftarrow$. Let V and W have the same dimension n. If $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\mathcal{U} = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ are bases of V and W, then we define an isomorphism $T : V \to W$ as follows: For each $\mathbf{v}$ in V, there are unique scalars c_i such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

We define T by

$$T(\mathbf{v}) = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n.$$

It is well defined, because the c_i are uniquely determined, since $\mathcal{B}$ is a basis. It is left as an exercise to prove that T is linear. Also, T is one-to-one, because if $\mathbf{v}$ is in $\operatorname{Ker}(T)$, then

$$\mathbf{0} = T(\mathbf{v}) = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$$

implies $c_1 = \cdots = c_n = 0$, since $\mathcal{U}$ is linearly independent. Therefore $\mathbf{v} = \mathbf{0}$ by Theorem 5.2.11. So T is isomorphism by Theorem 5.2.19. ☐

Example 5.2.21. Prove that $\mathbf{R}^2$ and $\mathbf{R}^3$ are *not* isomorphic (Figure 5.11).

Solution. They do not have the same dimension.

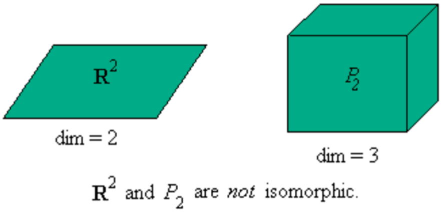

Figure 5.11: $\mathbf{R}^2$ and P_2 are not isomorphic.

Example 5.2.22. $\mathbf{R}^n$ and P_n are *not* isomorphic. Why?

Exercises 5.2

Kernel and range

In Exercises 1–3, find bases for the kernel and range and compute the nullity and rank of T. In each case, verify the dimension theorem.

1. $T : P_2 \to P_2$ is defined by

$$T\left(a + bx + cx^2\right) = (a - b) + (b - c)x + (-a + c)x^2.$$

2. $T : P_3 \to P_2$ is defined by

$$T\left(a + bx + cx^2 + dx^3\right) = (a - b) + (b - d)x + (c + d)x^2.$$

3. $T : P_1 \to P_3$ is defined by

$$T\,(a + bx) = (a - 2b)x + (2a - 4b)x^3.$$

4. Prove Part 1 of Theorem 5.2.4.

5. Prove Theorem 5.2.7.

In Exercises 6–8, find a basis for the kernel and the range of $T : P_2 \to P_2$ if T satisfies the given equations.

6. $T(p(x)) = p(1 + x)$.

7. $T(1) = x, T(x) = x^2, T(x^2) = -1$.

8. $T(1) = 0, T(x) = 0, T(x^2) = 1$.

In Exercises 9–10, for the given A, find the dimension of the kernel of the linear transformation $T : M_{33} \to M_{33}$ defined by

$$T(X) = AX.$$

9. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

10. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$

In Exercises 11–12, for given n and A, find the nullity and rank of the linear transformation $T : M_{22} \to M_{22}$ defined by

$$T(X) = AX - XA.$$

11. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$

12. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$

One-to-one, onto, isomorphisms

In Exercises 13–14, use Theorem 5.2.11 to prove that the linear transformations are one-to-one.

13. $T : P_1 \to M_{2,2}$ is defined by

$$T\left(a+bx\right)=\begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}.$$

14. $T : P_1 \to P_4$ is defined by

$$T\left(a+bx\right)=(-a+b)x^2+(a-2b)x^3.$$

In Exercises 15–16, prove that the linear transformations are onto.

15. $T : P_2 \to P_1$ is defined by

$$T\left(a+bx+cx^2\right)=(a+b)+(a+c)x.$$

16. $T : M_{22} \to \mathbf{R}^2$ is defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} a+b \\ c-d \end{bmatrix}.$$

In Exercises 17–18, prove that the linear transformations are isomorphisms.

17. $T : P_1 \to P_1$ is defined by

$$T\left(a+bx\right)=(a-2b)+(-2a+b)x.$$

18. $T : P_2 \to \mathbf{R}^3$ is defined by

$$T\left(a+bx+cx^2\right)=\begin{bmatrix} b-a \\ c-b \\ a+c \end{bmatrix}.$$

In Exercises 19–23, check T for isomorphism.

19. $T\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} x \\ 0 \\ y \end{bmatrix}.$

20. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} x \\ y \end{bmatrix}.$

21. $I : \mathbf{R}^n \to \mathbf{R}^n, I(\mathbf{x})=\mathbf{x}.$

22. $T : \mathbf{R}^n \to \mathbf{R}^n, T(\mathbf{x})=2\mathbf{x}.$

23. $T : \mathbf{R}^n \to \mathbf{R}^m, T(\mathbf{x})=A\mathbf{x}, m \neq n.$

24. Prove that any isomorphism $T : \mathbf{R}^2 \to \mathbf{R}^2$ maps straight lines to straight lines.

25. Prove that any isomorphism $T : \mathbf{R}^3 \to \mathbf{R}^3$ maps planes to planes.

26. Explain geometrically why a rotation of the plane is an isomorphism.

27. Let A be an $n \times n$ matrix of rank n, and let $\mathbf{b}$ be a *nonzero* n-vector. Prove that the affine transformation

$$T : \mathbf{R}^n \to \mathbf{R}^n, \quad T(\mathbf{x}) = A\mathbf{x} + \mathbf{b},$$

is one-to-one and onto. Is it an isomorphism? Explain.

28. Let V and W be finite-dimensional vector spaces and let $T : V \to W$ be a linear transformation. Prove the following statements:

(a) If T is one-to-one, then

$$\dim(V) \leq \dim(W);$$

(b) If T is onto, then

$$\dim(V) \geq \dim(W);$$

(c) If T is onto and a set S spans V, then $T(S)$ spans W;
(d) If T is isomorphism and a set $\mathcal{B}$ is a basis of V, then $T(\mathcal{B})$ is a basis of W.

5.3 Matrix of linear transformation

In this section, we show that a linear transformation between finite-dimensional vector spaces can be represented by a matrix transformation. This allows us to evaluate linear transformations by using only matrix multiplication.

In this section, we work with *ordered bases* of vector spaces.

Theorem 5.3.1. *Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces V and W. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V, and let $\mathcal{U}$ be a basis of W. Then the $m \times n$ matrix A with columns*

$$[T(\mathbf{v}_1)]_{\mathcal{U}}, \ \ldots, \ [T(\mathbf{v}_n)]_{\mathcal{U}}$$

is the only matrix that satisfies

$$[T(\mathbf{v})]_{\mathcal{U}} = A\,[\mathbf{v}]_{\mathcal{B}}$$

for all $\mathbf{v} \in V$.

Proof. Since $\mathcal{B}$ spans V, there are scalars c_i such that $\mathbf{v} = c_1\mathbf{u}_1 + \cdots + c_n\,\mathbf{v}_n$. By linearity we have

$$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + \cdots + c_n T(\mathbf{v}_n).$$

We use Theorem 4.4.5, Section 4.4, to get

$$\begin{aligned}[T(\mathbf{v})]_{\mathcal{U}} &= c_1[T(\mathbf{v}_1)]_{\mathcal{U}} + \cdots + c_n[T(\mathbf{v}_n)]_{\mathcal{U}} \\ &= A\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = A\,[\mathbf{v}]_{\mathcal{B}}\,.\end{aligned}$$

The proof that A is the *only* matrix with this property is left as an exercise. ☐

Definition 5.3.2. The matrix A of Theorem 5.3.1 is called the *matrix of T with respect to $\mathcal{B}$ and $\mathcal{U}$*. If $V = W$ and $\mathcal{B} = \mathcal{U}$, then A is called the *matrix of T with respect to $\mathcal{B}$* (Figure 5.12).

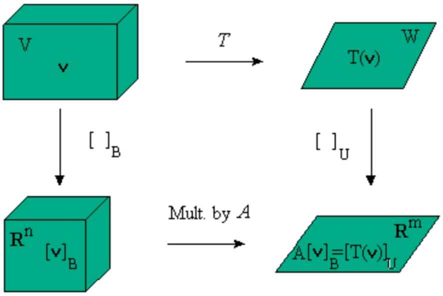

Figure 5.12: Matrix of linear transformation.

Example 5.3.3. Let $T : P_2 \to M_{22}$ be the linear transformation defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a+b & b+c \\ a+c & 0 \end{bmatrix},$$

and let $\mathcal{B}$ and $\mathcal{U}$ be the following bases of P_2 and M_{22}, respectively:

$$\mathcal{B} = \{x^2, 1-x, 1+x\}, \quad \mathcal{U} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}.$$

(a) Find the matrix A of T with respect to the bases $\mathcal{B}$ and $\mathcal{U}$.
(b) Evaluate $T(1 + x + x^2)$ (i) directly and (ii) by using A of Part (a).

Solution.
(a) First, we evaluate T on the basis $\mathcal{B}$:

$$T(x^2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad T(1-x) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad T(1+x) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then we write the images as linear combinations in $\mathcal{U}$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-2)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

as seen by the reduction

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

According to Theorem 5.3.1, the columns of A are the coefficients in the linear combinations. Hence

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) (i) Direct evaluation yields $T(1 + x + x^2) = \left[\begin{smallmatrix} 2 & 2 \\ 2 & 0 \end{smallmatrix}\right]$.

(ii) To use A, we first need to find $[1 + x + x^2]_{\mathcal{B}}$ by writing $1 + x + x^2$ as a linear combination in $\mathcal{B}$. We easily see that

$$1 + x + x^2 = 1x^2 + 0\,(1 - x) + 1\,(1 + x)\,.$$

Hence

$$[1 + x + x^2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then we calculate $[T(1 + x + x^2)]_{\mathcal{U}}$ by computing the product $A[1 + x + x^2]_{\mathcal{B}}$,

$$A[1 + x + x^2]_{\mathcal{B}} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = [T(1 + x + x^2)]_{\mathcal{U}},$$

and find $T(1 + x + x^2)$ from the components of $[T(1 + x + x^2)]_{\mathcal{U}}$:

$$\begin{aligned} T(1+x+x^2) &= 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}. \end{aligned}$$

We obtained the same answer as by direct evaluation.

! The most important conclusion from Theorem 5.3.1 and Example 5.3.3 is that general linear transformations can be easily manipulated by matrices, matrix row reduction, and matrix multiplication. Linear transformations are in essence matrix transformations. The matrix of a linear transformation reduces it into a matrix transformation.

Theorem 5.3.4. *Let $T : V \to W$ be a linear transformation with* $\dim V = n$ *and* $\dim W = m$. *Let A be the matrix of T with respect to bases $\mathcal{B}$ and $\mathcal{U}$ of V and W. Then*

1. *$\mathbf{v}$ is in the kernel of T if and only if $[\mathbf{v}]_{\mathcal{B}}$ is in the null space of A;*
2. *$\mathbf{w}$ is in the range of T if and only if $[\mathbf{w}]_{\mathcal{U}}$ is in the column space of A;*
3. *T is one-to-one if and only if A has n pivots;*
4. *T is onto if and only if A has m pivots;*
5. *T is isomorphism if and only if A is invertible.*

Proof. Exercise. (*Hint:* Use Theorems 5.3.1 and 4.4.5.) □

5.3.1 Change of basis and the matrix of a linear transformation

The next theorem tells us how the matrix of a linear transformation $T : V \to V$ changes when we change bases in V. Sometimes, there are bases that yield a very simple matrix for T, for instance, a *diagonal matrix*. Evaluation of T then becomes very easy.

Theorem 5.3.5. *Let $T : V \to V$ be a linear operator from a finite-dimensional vector space V into itself. Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases of V, and let P be the transition matrix from $\mathcal{B}'$ to $\mathcal{B}$. If A is the matrix of T with respect to $\mathcal{B}$ and A' is the matrix of T with respect to $\mathcal{B}'$, then*

$$A' = P^{-1}AP.$$

Proof. By Theorem 4.4.9, P^{-1} is the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$. Hence $[\mathbf{w}]_{\mathcal{B}'} = P^{-1}[\mathbf{w}]_{\mathcal{B}}$ for all $\mathbf{w}$ in V. In particular, $[T(\mathbf{v})]_{\mathcal{B}'} = P^{-1}[T(\mathbf{v})]_{\mathcal{B}}$ for all $\mathbf{v}$ in V. Therefore

$$\begin{aligned} [T(\mathbf{v})]_{\mathcal{B}'} &= P^{-1}[T(\mathbf{v})]_{\mathcal{B}} = P^{-1}(A[\mathbf{v}]_{\mathcal{B}}) = (P^{-1}A)[\mathbf{v}]_{\mathcal{B}} \\ &= P^{-1}A(P[\mathbf{v}]_{\mathcal{B}'}) = (P^{-1}AP)[\mathbf{v}]_{\mathcal{B}'}. \end{aligned}$$

So the matrix $P^{-1}AP$ satisfies $[T(\mathbf{v})]_{\mathcal{B}'} = (P^{-1}AP)[\mathbf{v}]_{\mathcal{B}'}$ for all $\mathbf{v}$ in V. Hence by the uniqueness of the transition matrix it must equal the transition matrix A' of T with respect to $\mathcal{B}'$ (Figure 5.13). $\square$

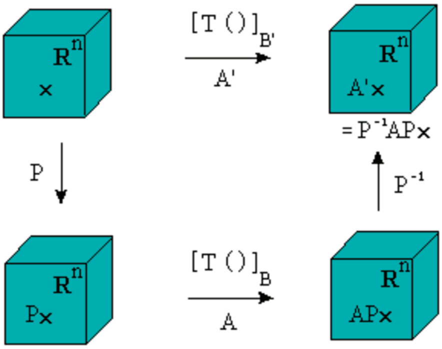

Figure 5.13: The effect of change of basis on $[T]_{\mathcal{B}}$.

Example 5.3.6. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation given by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5x + 6y \\ -3x + 4y \end{bmatrix},$$

and let $\mathcal{B}$ and $\mathcal{B}'$ be the bases

$$\mathcal{B} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}, \quad \mathcal{B}' = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}.$$

(a) Compute the matrix A of T with respect to $\mathcal{B}$.
(b) Compute the transition matrix P from $\mathcal{B}'$ to $\mathcal{B}$.
(c) Use Theorem 5.3.5 to find the matrix of T with respect to $\mathcal{B}'$.
(d) Compute the matrix A' of T with respect to $\mathcal{B}'$ directly from $\mathcal{B}'$.

Solution.

(a) Since $\mathcal{B}$ is the standard basis,

$$A = \begin{bmatrix} -5 & 6 \\ -3 & 4 \end{bmatrix}$$

is the standard matrix of T.

(b) The coordinates of the vectors of $\mathcal{B}'$ with respect to $\mathcal{B}$ are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Hence $P = \left[\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right]$.

(c) By Theorem 5.3.5 we have

$$A' = P^{-1}AP = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -5 & 6 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

(d) Now we compute A' directly from $\mathcal{B}'$ and T. We evaluate T at $\mathcal{B}'$ to get

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ -2 \end{bmatrix}_{\mathcal{B}'} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}.$$

Therefore $A' = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$. We get the same *diagonal* matrix as in Part (c).

Definition 5.3.7. Let A and B be $n \times n$ matrices. We say that B *is similar to* A if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

Theorem 5.3.5 can be rephrased by saying that *The matrices of a linear transformation with respect to two different bases are similar*.

The following basic facts about similar matrices are left as exercises:

1. A is similar to itself;
2. If B is similar to A, then A is similar to B;
3. If B is similar to A and C is similar to B, then C is similar to A.

In the exercises, we show that two similar matrices give rise to the *same* linear transformation with respect to different bases.

5.3.2 Proof of dimension theorem

Let us now prove the dimension theorem discussed in Section 5.2.

If $T : V \to W$ is a linear transformation from a finite-dimensional vector space V into a vector space W, then

$$\text{Nullity}(T) + \text{Rank}(T) = \dim(V).$$

Proof. Since V is finite dimensional, Range(T) is spanned by a finite set by Theorem 5.1.11. Thus Range(T) is finite dimensional, and T may be viewed as a linear transformation between the two finite-dimensional vector spaces V and Range(T).

Let A be the matrix of T with respect to bases $\mathcal{B}$ and $\mathcal{U}$ of V and Range(T), respectively. The number of columns of A equals $\dim(V)$, the dimension of V. Theorem 5.3.4 implies that

$$\text{Nullity}(T) = \text{Nullity}(A) \quad \text{and} \quad \text{Rank}(T) = \text{Rank}(A).$$

The dimension theorem now follows from the rank theorem (Theorem 4.6.19). ☐

Exercises 5.3

Let $\mathcal{B}$, $\mathcal{B}'$, and $\mathcal{B}''$ be the following bases of P_1, P_2, and P_3, respectively:

$$\mathcal{B} = \{1 + x, -1 + x\}, \quad \mathcal{B}' = \left\{-x + x^2, 1 + x, x\right\}, \quad \mathcal{B}'' = \left\{-x + x^3, 1 + x^2, x, -1 + x\right\}.$$

Let p, p', p'' be any polynomials of P_1, P_2, and P_3, respectively:

$$p = a + bx, \quad p' = a + bx + cx^2, \quad p'' = a + bx + cx^2 + dx^3.$$

In Exercises 1–3, find the matrix of T with respect to:

1. $\mathcal{B}$ and $\mathcal{B}'$, if $T(p) = (a - b) + bx + ax^2$.

2. $\mathcal{B}'$ and $\mathcal{B}$, if $T(p') = (a - b) + (b - 4c)x$.

3. $\mathcal{B}'$, if $T(p') = (a - b) + (b - c)x + (-a + 3c)x^2$.

4. Let $T : P_1 \to P_1$, $T(p) = (-a + b) + (2a - 3b)x$, and let $\mathcal{S} = \{2 + x, 1\}$.
(a) Compute the matrix A of T with respect to $\mathcal{B}$.
(b) Compute the transition matrix P from $\mathcal{S}$ to $\mathcal{B}$.
(c) Use (a) and (b) to find the matrix of T with respect to $\mathcal{S}$.
(d) Compute the matrix A' of T with respect to $\mathcal{S}$ directly from $\mathcal{S}$.

5. Let $T : P_2 \to P_2$ be the linear transformation defined by $T(p') = -2c + bx$.
(a) Find the matrix A of T with respect to the standard basis $\{1, x, x^2\}$.
(b) Find the matrix A' of T with respect to the basis $\mathcal{B}'$.
(c) Evaluate $T(6x - 2x^2)$ (i) directly, (ii) by using A, and (iii) by using A'.

Let $T : P_1 \to P_2$ be a linear transformation. In Exercises 6–7, for the given T,
(a) find the matrix A of T with respect to $\mathcal{B}$ and $\mathcal{B}'$;
(b) evaluate $T(5 - 2x)$ directly and then by using A, $\mathcal{B}$, $\mathcal{B}'$.

6. $T(p) = b + ax - ax^2$.

7. $T(p) = a - ax - bx^2$.

8. Let $T : M_{22} \to P_3$ be the linear transformation defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + (b - c)\,x + (c + d)\,x^2 + dx^3,$$

and let $\mathcal{B}$ and $\mathcal{U}$ be the following bases of M_{22} and P_3, respectively:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}, \quad \mathcal{U} = \{x^3, x^2, x, 1\}.$$

(a) Find the matrix A of T with respect to the bases $\mathcal{B}$ and $\mathcal{U}$.
(b) Use A from Part (a) to prove that T is an isomorphism.

9. Prove the uniqueness of the transition matrix as claimed by Theorem 5.3.1.

10. Prove Theorem 5.3.4.

In Exercises 11–13, A is the matrix of a linear transformation $T : P_n \to P_m$. Find n and m and a formula for $T(q)$, $q \in P_n$, with respect to:

11. $\mathcal{B}'$, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

12. $\mathcal{B}$ and $\mathcal{B}''$, if

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -4 & 1 & 2 & -8 \end{bmatrix}.$$

13. $\mathcal{B}''$ and $\mathcal{B}$, if $A = \begin{bmatrix} -4 & 2 \\ 0 & 9 \\ 1 & -1 \\ 2 & -3 \end{bmatrix}$.

14. Consider the linear transformation T of the plane that projects all of $\mathbf{R}^2$ onto the line $y = x$ as shown in Figure 5.14.

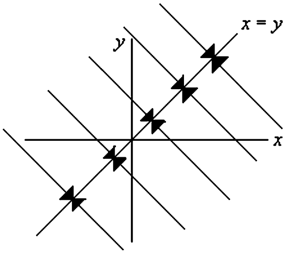

Figure 5.14: Linear transformation of projection.

(a) Find the matrix A of T with respect to the standard basis.
(b) Find the matrix B of T with respect to the basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.
(c) Find the image of $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ by using A of Part (a).
(d) Find the image of $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$ by using B of Part (b).

15. Find the matrix A of $T : M_{22} \to M_{22}$ defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -b & d \\ c & -a \end{bmatrix}$$

with respect to the basis

$$\{E_{11} - E_{12}, E_{12} - E_{21}, E_{21} - E_{22}, E_{22} + E_{11}\}.$$

Prove that T is an isomorphism by row reducing A.

In Exercises 16–18, prove the statements.

16. A is similar to itself.

17. If B is similar to A, then A is similar to B.

18. If B is similar to A and C is similar to B, then C is similar to A.

19. For two $n \times n$ matrices A and B with at least one of them invertible, prove that AB is similar to BA.

20. Let A and B be similar $n \times n$ matrices. Prove that there are a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ and bases $\mathcal{B}$ and $\mathcal{U}$ such that the matrix of T with respect to $\mathcal{B}$ is A and the matrix of T with respect to $\mathcal{U}$ is B.

5.4 The algebra of linear transformations

We study the basic operations of linear transformations and relate them to the corresponding matrix operations. Throughout this section, V, W, and U are finite-dimensional vector spaces.

5.4.1 Sums and scalar Products

Let $T, L : V \to W$ be linear transformations. The *sum*, $T + L$ of T and L is the transformation $T + L : V \to W$ defined by

$$(T + L)(\mathbf{v}) = T(\mathbf{v}) + L(\mathbf{v})$$

for all $\mathbf{v} \in V$. Let c be any scalar. The *scalar multiple* cT of T by c is the transformation $cT : V \to W$ defined by

$$(cT)(\mathbf{v}) = cT(\mathbf{v})$$

for all $\mathbf{v} \in V$. Just as with vectors, we may form *linear combinations*.

$$c_1T_1 + \cdots + c_nT_n.$$

Theorem 5.4.1. *$T + L$ and cL are linear transformations.*

Proof. Let $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $c_1, c_2 \in \mathbf{R}$. Then

$$\begin{aligned}(T+L)(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) &= T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + L(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)\\ &= c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + c_1 L(\mathbf{v}_1) + c_2 L(\mathbf{v}_2)\\ &= c_1\big(T(\mathbf{v}_1) + L(\mathbf{v}_1)\big) + c_2\big(T(\mathbf{v}_2) + L(\mathbf{v}_2)\big)\\ &= c_1(T+L)(\mathbf{v}_1) + c_2(T+L)(\mathbf{v}_2).\end{aligned}$$

Hence $T + L$ is linear. The verification that cT is linear is left as an exercise. □

Theorem 5.4.2 (Laws for addition and scalar multiplication). *Let T, L, and K be linear transformations between vector spaces, $V \to W$. Let c be any scalar. Then*

1. $(T + L) + K = T + (L + K)$;
2. $T + L = L + T$;
3. $T + \mathbf{0} = \mathbf{0} + T = T$;
4. $T + (-T) = (-T) + T = \mathbf{0}$;
5. $c(T + L) = cT + cL$;
6. $(a + b)T = aT + bT$;
7. $(ab)T = a(bT) = b(aT)$;
8. $1T = T$;
9. $0T = \mathbf{0}$.

Proof. Exercise. □

Theorems 5.4.1 and 5.4.2 imply the next basic result.

Theorem 5.4.3 (Vector space of linear transformations). *The set of all linear transformations $T : V \to W$ is a vector space under the above addition and scalar multiplication.*

5.4.2 Composition of linear transformations

Definition 5.4.4. Let $L : U \to V$ and $T : V \to W$ be linear transformations. The *composition of T with L* is the transformation $T \circ L : U \to W$ defined for $\mathbf{v} \in U$ by

$$T \circ L(\mathbf{v}) = T\big(L(\mathbf{v})\big).$$

Theorem 5.4.5. *$T \circ L : U \to W$ is a linear transformation.*

Proof. Exercise. □

Example 5.4.6. Let $L : P_1 \to P_1$ and $T : P_1 \to P_2$ be the linear transformations

$$L(a + bx) = b - ax \quad \text{and} \quad T(a + bx) = a + ax - bx^2.$$

To find $T \circ L(a+bx)$, we evaluate

$$T \circ L(a+bx) = T\big(L(a+bx)\big) = T(b-ax) = b + bx + ax^2.$$

Composition of linear transformations satisfy the following properties which parallel to the properties of matrix multiplication.

Theorem 5.4.7 (Laws of composition). *Let T, L, and K be linear transformations such that the operations below can be performed. Let c be any scalar. Then*

1. $(T \circ L) \circ K = T \circ (L \circ K)$;
2. $T \circ (L + K) = T \circ L + T \circ K$;
3. $(L + K) \circ T = L \circ T + K \circ T$;
4. $c(T \circ L) = (cT) \circ L = T \circ (cL)$;
5. $I \circ T = T \circ I = T$;
6. $\mathbf{0} \circ T = \mathbf{0}$, $T \circ \mathbf{0} = \mathbf{0}$.

Proof. Exercise. ☐

Just like matrix multiplication, composition is not commutative,

$$T \circ L \neq L \circ T.$$

Like a square matrix, *powers* of a linear operator $T : V \to V$ are defined by

$$T^0 = I, \quad T^1 = T, \quad T^2 = T \circ T, \quad \dots, \quad T^k = T \circ T \cdots \circ T \quad (k \text{ “factors”}).$$

5.4.3 Projections

A linear operator $P : V \to V$ is called a *projection* if $P^2 = P \circ P = P$.

Example 5.4.8. The linear operator $T : P_1 \to P_1$ defined by

$$T(a+bx) = (3a-6b) + (a-2b)x$$

is a projection. (Verify.)

If a projection is a matrix transformation $T : \mathbf{R}^n \to R^n$, then its standard matrix, say A, is called a *projection matrix* or *idempotent matrix* and has the property $A^2 = A$.

Example 5.4.9. The following are projection matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ c & 1 \end{bmatrix}.$$

Interesting cases of projection matrices occur when the matrix is *orthogonal*. Orhtogonal matrices are studied in Chapter 8. Special cases of orthogonal projections were encountered in Section 2.6.

5.4.4 Linear transformation and matrix operations

Theorems 5.4.2 and 5.4.7 show a similarity between matrix operations and operations of linear transformations. From Section 5.3 we know that a linear transformation $T : V \to W$ is represented by a matrix transformation via

$$[T(\mathbf{v})]_{\mathcal{B}'} = A\,[\mathbf{v}]_{\mathcal{B}} \quad \text{for all } \mathbf{v} \in V, \tag{5.4}$$

where $\mathcal{B}$ and $\mathcal{B}'$ are fixed bases of V and W, and A is the matrix of T with respect to $\mathcal{B}$ and $\mathcal{B}'$. Recall that A is the only matrix that satisfies (5.4).

The next theorem tells us exactly how the linear transformation operations correspond to matrix operations.

Theorem 5.4.10. *Let T and L be linear transformations between finite-dimensional vector spaces with matrices A and B with respect to fixed bases. Then the matrix of the linear transformation*

1. $T + L$ *is* $A + B$,
2. $T - L$ *is* $A - B$,
3. $-T$ *is* $-A$,
4. cT *is* cA,
5. $T \circ L$ *is* AB.

Proof of 1. For each $\mathbf{v} \in V$, we have

$$\begin{aligned}[(T+L)(\mathbf{v})]_{\mathcal{B}'} &= [T(\mathbf{v}) + L(\mathbf{v})]_{\mathcal{B}'} \\ &= [T(\mathbf{v})]_{\mathcal{B}'} + [L(\mathbf{v})]_{\mathcal{B}'} \\ &= A\,[\mathbf{v}]_{\mathcal{B}} + B\,[\mathbf{v}]_{\mathcal{B}} \\ &= (A+B)\,[\mathbf{v}]_{\mathcal{B}}\,.\end{aligned}$$

Hence $A + B$ is the matrix of $T + L$ with respect to any bases $\mathcal{B}$ and $\mathcal{B}'$. □

5.4.5 Invertible linear transformations

Definition 5.4.11. A linear transformation $T : V \to V$ is *invertible* if there is a transformation $L : V \to V$ such that

$$T \circ L = I \quad \text{and} \quad L \circ T = I.$$

The transformation L is called an *inverse* of T. If an inverse exists, then it is unique. The proof is identical to that of the uniqueness of the inverse of a matrix. This unique inverse is denoted by T^{-1}. So

$$T \circ T^{-1} = I \quad \text{and} \quad T^{-1} \circ T = I.$$

Note that if T is invertible, then

$$T(\mathbf{v}) = \mathbf{w} \Leftrightarrow T^{-1}(\mathbf{w}) = \mathbf{v}$$

The inverse of a transformation, if it exists, reverses the effect of the transformation.

The next theorem identifies the invertible linear transformations with isomorphisms studied in Section 5.2.

Theorem 5.4.12. *Let $T : V \to V$ be a linear transformation.*
1. *T is invertible if and only if it is an isomorphism.*
2. *If T is invertible, then T^{-1} is linear.*

Proof.
1. Let T be invertible, and let L be its inverse. We prove that T is one-to-one and onto. For $\mathbf{v}_1, \mathbf{v}_2 \in V$, let $T(\mathbf{v}_1) = T(\mathbf{v}_2)$. Then

$$\begin{aligned} T(\mathbf{v}_1) &= T(\mathbf{v}_2) \\ &\Rightarrow L(T(\mathbf{v}_1)) = L(T(\mathbf{v}_2)) \\ &\Rightarrow L \circ T(\mathbf{v}_1) = L \circ T(\mathbf{v}_2) \\ &\Rightarrow \mathbf{v}_1 = \mathbf{v}_2. \end{aligned}$$

Therefore T is one-to-one. Now let $\mathbf{w} \in V$ and $\mathbf{v} = L(\mathbf{w})$. Then

$$T(\mathbf{v}) = T(L(\mathbf{w})) = T \circ L(\mathbf{w}) = \mathbf{w}.$$

So for each $\mathbf{w}$, there is some $\mathbf{v}$ that is mapped to $\mathbf{w}$. Hence T is onto.

The proof of the converse in Part 1 is left as an exercise.

2. Let T be invertible. We prove that T^{-1} is linear. If $\mathbf{v}_1, \mathbf{v}_2 \in V$, then there are unique vectors $\mathbf{w}_1, \mathbf{w}_2 \in V$ such that $T(\mathbf{w}_1) = \mathbf{v}_1$ and $T(\mathbf{w}_2) = \mathbf{v}_2$, because T is one-to-one and onto by Part 1. Therefore $\mathbf{w}_1 = T^{-1}(\mathbf{v}_1)$ and $\mathbf{w}_2 = T^{-1}(\mathbf{v}_2)$. We have

$$\begin{aligned} T^{-1}(\mathbf{v}_1 + \mathbf{v}_2) &= T^{-1}(T(\mathbf{w}_1) + T(\mathbf{w}_2)) \\ &= T^{-1}(T(\mathbf{w}_1 + \mathbf{w}_2)) \\ &= T^{-1} \circ T(\mathbf{w}_1 + \mathbf{w}_2) \\ &= I(\mathbf{w}_1 + \mathbf{w}_2) \\ &= \mathbf{w}_1 + \mathbf{w}_2 \\ &= T^{-1}(\mathbf{v}_1) + T^{-1}(\mathbf{v}_2). \end{aligned}$$

We leave it to the reader to prove that for each $\mathbf{v} \in V$ and each scalar c,

$$T^{-1}(c\,\mathbf{v}) = c\,T^{-1}(\mathbf{v}).$$

Hence T^{-1} is a linear transformation. □

Finally, we relate the invertibility of a linear transformation to that of its matrix with respect to two bases.

Theorem 5.4.13. *Let $T : V \to V$ be a linear transformation with matrix A with respect to bases $\mathcal{B}$ and $\mathcal{B}'$ of V. Then:*

1. *T is invertible if and only if A is invertible;*
2. *If T is invertible, then A^{-1} is the matrix of T^{-1} with respect to $\mathcal{B}'$ and $\mathcal{B}$.*

Proof. Exercise. □

Exercises 5.4

1. Evaluate $T + L$ and $-3T$ at $a + bx$ and at $-6 + 7x$, if

$$T(a + bx) = b - ax, \quad L(a + bx) = (3a + b) - bx.$$

2. Evaluate $T \circ L$ and $L \circ T$ at any vector of their domains, if

$$T\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -x - 2y + z \\ x - w \end{bmatrix}, \quad L\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x - 3y \\ x - y \\ x \end{bmatrix}.$$

3. Find two linear transformations T and L such that $T \circ L \neq L \circ T$.

4. Prove that T and L are inverse to each other, if

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + y \\ x - z \\ x + y - z \end{bmatrix}, \quad L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x - y + z \\ -y + z \\ -x - 2y + z \end{bmatrix}.$$

5. Prove that T is invertible (do not compute the inverse), if

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -x + y + z \\ x - 2z \\ 2x - y \end{bmatrix}.$$

In Exercises 7–6, prove that the transformation is invertible and compute its inverse.

6. $T : \mathbf{R}^4 \to \mathbf{R}^4$ given by

$$T(\mathbf{x}) = \begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \mathbf{x}.$$

7. $T(a + bx) = (b - a) + (a + b)x$.

8. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ and $L : \mathbf{R}^2 \to \mathbf{R}^3$ be linear transformations. Prove that the composition $L \circ T : \mathbf{R}^3 \to \mathbf{R}^3$ is *not* an isomorphism.

9. Complete the proof of Theorem 5.4.1.

10. Prove Theorem 5.4.2.

11. Prove Theorem 5.4.5.

12. Prove Theorem 5.4.7.

13. Complete the proof of Theorem 5.4.10.

14. Complete the proof of Theorem 5.4.12.

15. Prove Theorem 5.4.13.

16. Find an example of a projection transformation $T : M_{22} \to M_{22}$.

17. If A is invertible and projection, then prove that $A = I$.

18. Let A be a 2×2 projection matrix.
(a) If A is diagonal, then prove that its diagonal entries are either 0 or 1.
(b) If A is not diagonal, then prove that its trace must be 1.

Right and left inverses

In this paragraph, we introduce right and left inverses of linear transformations as we did for matrices in the exercises of Section 3.2.

Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. We say that the linear transformation L is *right inverse* of T if $T \circ L = I$. Likewise, K is a *left inverse* of T if $K \circ T = I$.

Let A be the matrix of T with respect to fixed bases of V and W. By recalling the definitions of left and right inverses of matrices in the exercises of Section 3.2, solve the following three exercises.

19. Prove that the following statements are equivalent.
(a) T has a right inverse.
(b) T is onto.
(c) A has a right inverse.

20. Prove that the following statements are equivalent.
(a) T has a left inverse.
(b) T is one-to-one.
(c) A has a left inverse.

21. Prove that if T has both a right inverse L and a left inverse K, then
(a) $L = K$;
(b) T is one-to-one and onto;
(c) T is an isomorphism;
(d) A has a left and a right inverse that coincide;
(e) A is invertible.

Negative powers of invertible transformations
Just as we did with invertible matrices, we may define negative powers of invertible transformations. If T is invertible and n is a positive integer, then we define

$$T^{-n} = \left(T^{-1}\right)^n = \underbrace{T^{-1} \circ T^{-1} \circ \cdots \circ T^{-1}}_{n \text{ factors}}.$$

22. Find $T^{-3}(a + bx)$ if $T(a + bx) = -b + ax$.

23. Find $T^{-2}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ if $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$.

5.5 Special topic: Fractals

5.5.1 Fractals

In recent years a new area of mathematics, called *Fractal Geometry*, has emerged. Although fractal geometry has its roots in important works by Cantor, Sierpinski, von Koch, Peano, and Julia, only since the late 1960s, it has become a new field. This was due to the pioneering work of Benoit Mandelbrot of the IBM corporation and to the emergence of fast computers (Figure 5.17). The word *fractal* is used to describe figures with "infinite repetition of the same shape". Such an introduced fractal is the *Mandelbrot set* (Figure 5.15).[2]

Fractals are used today in data storage to compress large amounts of data into smaller, more manageable files. Fractal compression algorithms use self-similarity to identify patterns in data and encode them in a way that reduces the size of the file. This makes it easier to store, share, and transfer large amounts of data. Fractal compression is used in many applications, such as digital images, audio and video files, and 3D models.

It was observed by Barnsley [18] that many "fractal-like" objects (such as *Barnsley's fern*) can be obtained by plotting *iterations* of certain affine transformations.

---

[2] The Mandelbrot set and other fractals have their origins in the work of French mathematicians Pierre Fatou and Gaston Julia at the beginning of the twentieth century. The set was first defined and drawn in 1978 by R. W. Brooks and P. Matelski as part of a study of Kleinian groups. Benoit Mandelbrot in 1980 published an accurate visualization of this set at IBM's Thomas J. Watson Research Center in Yorktown Heights, New York.

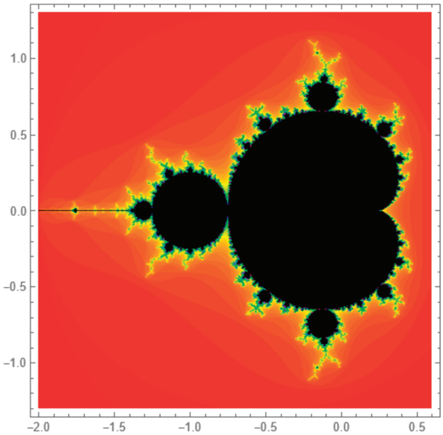

Figure 5.15: The Mandelbrot set.
The image was generated by the author by using Mathematica's MandelbrotSetPlot[] command.

Let us briefly describe the well-known fractal, the *Sierpinski triangle*, by using a sequence of affine transformations.

The Sierpinski triangle

Let f_2, f_2, f_3 be the three affine transformations from $\mathbf{R}^2$ to $\mathbf{R}^2$ given by

$$f_1(\mathbf{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \mathbf{x},$$

$$f_2(\mathbf{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix},$$

$$f_3(\mathbf{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}.$$

The Sierpinski triangle can be generated as follows. Starting with a triangle, say, the triangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, we pick a point inside it and plot it, say, the point $(\frac{1}{2}, \frac{1}{2})$. Then we randomly select one of f_1, f_2, f_3, say, f_i, and compute and plot $f_i\,(\frac{1}{2}, \frac{1}{2})$. Making this point as our new starting point, we repeat the process as often as desired. The resulting picture is a "fractal object" that looks like a triangle with triangular holes in it if enough points are plotted (Figure 5.16).

Let us outline the procedure that generated the Sierpinski triangle in the following algorithm. This algorithm produces a fractal image for *some* sets of plane affine matrix transformations.

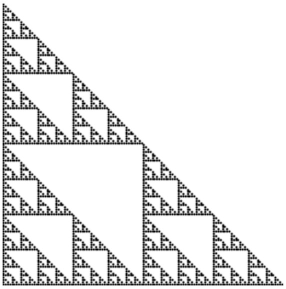

Figure 5.16: A Sierpinsky triangle.

Algorithm 5.5.1 (Fractal image generator). To produce a fractal image:

1. Start with an APPROPRIATE set of affine matrix transformations $S = \{f_1, f_2, \dots, f_n\}$ and an initial point (x_k, y_k).
2. Randomly choose an affine transformation of S, say, f_i.
3. Compute and plot the point $f_i(x_k, y_k)$. Set $(x_k, y_k) = f_i(x_k, y_k)$.
4. Go to Step 2. Repeat as much as desired.

Figure 5.17: Benoit Mandelbrot.
(Rama, CC BY-SA 2.0 FR <https://creativecommons.org/licenses/by-sa/2.0/fr/deed.en>, via Wikimedia Commons https://commons.wikimedia.org/wiki/File:Benoit_Mandelbrot_mg_1804-d.jpg.)
Benoit Mandelbrot (1924–2010) Polish-born French–American mathematician. He introduced the concept of fractals and coined the term. He was a research fellow at the Institute for Advanced Study in Princeton, and for IBM's Thomas J. Watson Research Center, and a Professor of Mathematical Sciences at Yale University.

5.6 Miniproject

5.6.1 Another fractal

Usually, fractal images cannot be plotted without the help of a computer. In this project, we study a fractal which, to some degree, can be visualized by hand plotting.

Consider the rectangles with the given vertices

$$\begin{aligned} A &: \quad (1,-1), (1,1), (-1,1), (-1,-1), \\ B &: \quad (1,0), (1,1), (-1,1), (-1,0), \\ C &: \quad (2,0), (2,1), (-2,1), (-2,0). \end{aligned}$$

Also, consider the affine transformations

$$R(\mathbf{x}) = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \mathbf{x}, \quad T(\mathbf{x}) = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}.$$

Let $A_1^R = R(A)$ (the image of rectangle A under R), $A_2^R = R(A_1^R)$, $A_3^R = R(A_2^R)$, $A_4^R = R(A_3^R)$. Likewise, let A_1^T, A_2^T, A_3^T, and A_4^T be the corresponding images under T. Likewise, we have the consecutive images of $B, B_1^R, B_2^R, B_3^R, B_4^R$ under R and $B_1^T, B_2^T, B_3^T, B_4^T$ under T. The images of C are defined the same way.

Problem A is designed to show you the effects of R and T on A, B, C and their iterated images.

Problem A.

1. Plot $A, A_1^R, A_2^R, A_3^R, A_4^R$ in one graph and $A, A_1^T, A_2^T, A_3^T, A_4^T$ in another.
2. Plot $B, B_1^R, B_2^R, B_3^R, B_4^R$ in one graph and $B, B_1^T, B_2^T, B_3^T, B_4^T$ in another.
3. Plot $C, C_1^R, C_2^R, C_3^R, C_4^R$ in one graph and $C, C_1^T, C_2^T, C_3^T, C_4^T$ in another.

Problem B is designed to show you the fractal image generated by applying R and T at the origin and iterating.

Problem B. Let $P(0,0)$. Find the two images P_1 and P_2 of P under R and T. Then find the images P_3 and P_4 of P_1 under R and T and the images P_5 and P_6 under T. Continue this process for as long as you please. Then plot *all* points found. It takes about 5 to 6 iterations to see a formation of a fractal object.

Problem C is designed to show you how the fractal image is affected if you start at a different point.

Problem C. Answer the questions of Problem B starting with the point $Q(0.5, 0.5)$.

5.7 Technology-aided problems and answers

In Exercises 1–8, let

$$\begin{aligned} T_1(x,y,z,w) &= (x + 2y + 3z + 4w, 2x + 3y + 4z + 5w, 3x + 4y + 5z + 6w), \\ T_2(x,y,z) &= (x + 2y + 3z + 4w, 2x + 2y + 3z + 4w, 3x + 3y + 3z + 4w), \\ T_3(x,y,z,w,t) &= (x + 2y + 3z + 4w, 2x + 2y + 3z + 4w, 3x + 3y + 3z + 4w, 4x + 4y + 4z + 4t). \end{aligned}$$

1. Which of $\mathbf{v}_1 = (-1, 0, 3, -2)$, $\mathbf{v}_2 = (18, -31, 8, 5)$, and $\mathbf{v}_3 = (1, -1, 8, 7)$ are in $\mathrm{Ker}(T_1)$?

2. Which of $\mathbf{w}_1 = (2, 7, 12)$, $\mathbf{w}_2 = (42, 59, 76)$, and $\mathbf{w}_3 = (42, 59, 77)$ are in $\mathrm{R}(T_1)$?

In Exercises 3–6, answer the question for each of T_1, T_2, and T_3.

3. Find the standard matrix.

4. Find a basis for the kernel. What is the nullity?

5. Find a basis for the range. What is the rank?

6. Verify the dimension theorem.

7. Which of the transformations T_1, T_2, and T_3 is one–to–one? onto? an isomorphism? None of these choices?

8. *True or False?*

$$\begin{array}{ll} \text{(a)} & \mathrm{R}(T_1) = \mathbf{R}^3. \\ \text{(b)} & \mathrm{R}(T_2) = \mathbf{R}^3. \\ \text{(c)} & \mathrm{R}(T_3) = \mathbf{R}^4. \end{array}$$

In Exercises 9–10, let a linear transformation T be such that

$$\begin{aligned} T(1,2,3,4) &= (1,0,-1,1), \\ T(1,3,5,7) &= (0,1,0,-1), \\ T(3,3,4,4) &= (1,1,1,-1), \\ T(4,4,4,5) &= (1,1,-1,1). \end{aligned}$$

9. Find the standard matrix of T.

10. Compute $T(2,2,-2,-2)$.

11. Let $T : P_1 \to P_2$ be given by

$$T(ax+b) = (3a-4b)x^2 + (a+3b)x - b.$$

Find a polynomial $p(x)$ such that $T(p(x)) = -x^2 + 4x - 1$. Verify your answer.

12. Find $L(2x^3 + 2x^2 - 2x - 2)$ if L is linear such that

$$\begin{aligned} L\left(x^3 + 2x^2 + 3x + 4\right) &= x^3 - x + 1, \\ L\left(x^3 + 3x^2 + 5x + 7\right) &= x^2 - 1, \\ L\left(3x^3 + 3x^2 + 4x + 4\right) &= x^3 + x^2 + x - 1, \\ L\left(4x^3 + 4x^2 + 4x + 5\right) &= x^3 + x^2 - x + 1. \end{aligned}$$

13. Find a basis for the null space of F, where

$$F(ax^3 + bx^2 + cx + d) = (a + 2b + 3c + 4d)x^2 + (2a + 3b + 4c + 5d)x + (3a + 4b + 5c + 6d).$$

14. Find the matrix M of $T : P_1 \to P_2$ with respect to $\mathcal{B} = \{x-1, x+1\}$ and $\mathcal{B}' = \{x^2 - 1, x+1, x-1\}$ if

$$T(ax+b) = (3a-4b)x^2 + (a+3b)x - b.$$

By using M determine whether or not T is one-to-one.

15. Let $\mathcal{B}$ and $\mathcal{U}$ be the following bases of M_{22} and P_3, respectively:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}, \quad \mathcal{U} = \left\{ x^3, x^2, x, 1 \right\}.$$

Let $T : M_{22} \to P_3$ be the linear transformation defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + (b-c)\,x + (c+d)\,x^2 + dx^3.$$

(a) Find the matrix A of T with respect to the bases $\mathcal{B}$ and $\mathcal{U}$.
(b) Use A from Part (a) to prove that T is an isomorphism.

5.7.1 Selected solutions with Mathematica

```
(*  Exercise 1. *)
T1[x_,y_,z_,w_]:={x+2y+3z+4w, 2x+3y+4z+5w, 3x+4y+5z+6w}
T2[x_,y_,z_,w_]:={x+2y+3z+4w, 2x+2y+3z+4w, 3x+3y+3z+4w}
T3[x_,y_,z_,w_]:={x+2y+3z+4w, 2x+2y+3z+4w, 3x+3y+3z+4w,4x+4y+4z+4w}
T1[-1,0,3,-2]              (* v1 in the kernel.        *)
T1[18,-31,8,5]             (* v2 in the kernel.        *)
T1[1,-1,8,7]               (* v3 not in the kernel.    *)
(* Standard matrix of T1: *)
M1=Transpose[{T1[1,0,0,0],T1[0,1,0,0],T1[0,0,1,0],T1[0,0,0,1]}]
(*  Exercise 2. *)
Mw1=RowReduce[Join[M1, {{2}, {7}, {12}}, 2]]
(* In Range(T1). Last column is nonpivot. *)
Mw2=RowReduce[Join[M1, {{42},{59},{76}}, 2]]
(* In Range(T1). Last column is nonpivot. *)
Mw3=RowReduce[Join[M1, {{42},{59},{77}}, 2]]
(* Not in Range(T1). Last column is pivot.*)
(*  Exercise 3. *)
M1  (* Standard matrix of T1 already found. The remaining:*)
M2=Transpose[{T2[1,0,0,0],T2[0,1,0,0],T2[0,0,1,0],T2[0,0,0,1]}]
M3=Transpose[{T3[1,0,0,0],T3[0,1,0,0],T3[0,0,1,0],T3[0,0,0,1]}]
(*  Exercise 4. *)
(* k1 has two vectors, so nullity 2.      *)
k1=NullSpace[M1]    (* kernel non-zero, not one-to-one. *)
(* Hence, not an isomorphism.             *)
(* k2 has one vector, so nullity 2.       *)
k2=NullSpace[M2]    (* kernel non-zero, T2 is not one-to-one. *)
(* Hence, not an isomorphism.             *)
k3=NullSpace[M3]    (* k3 has no vectors, so nullity 0.
T3 is one-to-one. This is from R^4 to R^4 so isomorphism. *)
(*  Exercises 5--8. *)
(* The first 2 columns form a basis for the range.   *)
r1=RowReduce[M1]   (* Rank is 2. 2 + 2 = number of columns. *)
(* The third row has no pivot, so not onto.          *)
(* The first 3 columns form a basis for the range.   *)
r2=RowReduce[M2]   (* Rank is 3. 3 + 1 = number of columns. *)
(* Each row has a pivot, so onto.              *)
(* All columns form a basis for the range.      *)
r3=RowReduce[M3]   (* Rank is 4. 4 + 0 = number of columns.*)
(* T3 is one-to-one and onto, hence, an isomorphism. *)
(* False. T1 is not onto.  *)
(* True. T2 is onto.       *)
(* True. T3 is onto.       *)
(* Exercises 9, 10. *)
M = {{1,1,3,4},{2,3,3,4},{3,5,4,4},{4,7,4,5}}    (*  The domain vectors.*)
n = {{1,0,1,1},{0,1,1,1},{-1,0,1,-1},{1,-1,-1,1}}(* The values.        *)
STM = n . Inverse[M]                             (* The standard matrix.*)
STM . {{2},{2},{-2},{-2}}                       (*  T(2,2,-2,-2) .     *)
```

```
(* Exercise 11. *)
Solve[{3 a-4 b == -1, a+3 b == 4, -b==-1},{a,b}]             (* a=b=1 *)
T[a_,b_] := (3 a-4 b)x^2+ (a+3 b)x-b
T[1,1]                              (* T(1,1) get the polynomial back *)
(* Exercise 12. *)
(* First we form a matrix with the coefficients of the given polynomials,*)
M = {{1,1,3,4},{2,3,3,4},{3,5,4,4},{4,7,4,5}}
(* then a matrix with the coefficients of their values.                *)
n = {{1,0,1,1},{0,1,1,1},{-1,0,1,-1},{1,-1,-1,1}}
M1 = RowReduce[Join[M, {{2},{2},{-2},{-2}},2]]
(* The last column of M1 has entries the coefficients of 2x^3+2x^2-2x-2*)
M2 = M1[[All, 5]]     (* in terms of the given polynomials.          *)
n . M2                (* Multiplication by n evaluates T(2,2,-2,-2). *)
(* Exercise 13. *)
(* Set F=0. We need to solve a+2b+3c+4d=0,2a+3b+4c+5d=0,3a+4b+5c+6d=0. *)
M = {{1,2,3,4}, {2,3,4,5}, {3,4,5,6}} (*  We compute a basis for the   *)
MN = NullSpace[M]           (* nullspace of the coefficient matrix.    *)
MN . {{x^3}, {x^2},{x},{1}} (* The answer in polynomial form.          *)
(* Exercise 14. *)
T[a_,b_] := {3 a-4 b, a+3 b, -b}
b2={{1,0,0},{0,1,1},{-1,1,-1}} (*The coefficients of B and the coefficients *)
aug = Join[b2,Transpose[{T[1,-1],T[1,1]}],2]          (* of T(x-1),T(x+1) *)
aug1=RowReduce[aug]           (* in terms of B' are computed by rref(aug) *)
M=aug1[[All, 4;;5]]         (* The last 2 columns form the matrix of T. *)
RowReduce[M]                (* Both columns are pivot columns, so 1-1   *)
NullSpace[M]             (* Another way, the nullspace has {} as basis. *)
(* Exercise 15. *)
T[a_, b_, c_, d_] := {a, b-c, c+d, d}  (* Define T as vector-valued.*)
(* Evaluate T at the basis matrices written as vectors. *)
MM=Transpose[{T[1,0,0,0],T[0,1,0,0],T[0,1,1,0],T[0,0,1,1]}]
b2={{0,0,0,1},{0,0,1,0},{0,1,0,0},{1,0,0,0}}
(* The coefficients of U as vectors*)
aug = Join[b2,MM,2]           (* the augmented matrix of the system *)
aug1=RowReduce[aug]                        (* Solve keep the last 4 *)
M=aug1[[All, 5;;8]]                             (* columns to get M *)
Det[M]            (* M has nonzero determinant, so T is isomorphism.*)
```

5.7.2 Selected solutions with MATLAB

```
%  Exercise 1.
% As usual, define the functions by editing and saving files named T1.m, T2.m,
% and T3.m in the current working. (The code follows.) When in MATLAB session
% evaluate each function as needed.
function [A] = T1(x,y,z,w)
A = [x+2*y+3*z+4*w; 2*x+3*y+4*z+5*w; 3*x+4*y+5*z+6*w]; end
function [A] = T2(x,y,z,w)
A = [x+2*y+3*z+4*w; 2*x+2*y+3*z+4*w; 3*x+3*y+3*z+4*w]; end
```

```
function [A] = T3(x,y,z,w)
A = [x+2*y+3*z+4*w;2*x+2*y+3*z+4*w;3*x+3*y+3*z+4*w;4*x+4*y+4*z+4*w]; end
T1(-1,0,3,-2)             % v1 in the kernel.
T1(18,-31,8,5)            % v2 in the kernel.
T1(1,-1,8,7)              % v3 not in the kernel.
M1=[T1(1,0,0,0) T1(0,1,0,0) T1(0,0,1,0) T1(0,0,0,1)]  % Standard matrix.
% Exercise 2.
Mw1=rref([M1 [2;7;12]])      % In Range(T1). Last column is pivot.
Mw2=rref([M1 [42;59;76]])    % In Range(T1). Last column is pivot.
Mw3=rref([M1 [42;59;77]])     % Not in Range(T1). Last column is non-pivot.
% Exercise 3.
M1              % Standard matrix of T1 already found. The remaining are:
M2=[T2(1,0,0,0) T2(0,1,0,0) T2(0,0,1,0) T2(0,0,0,1)]
M3=[T3(1,0,0,0) T3(0,1,0,0) T3(0,0,1,0) T3(0,0,0,1)]
% Exercise 4.
% k1 has two vectors, so nullity 2.
k1=null(M1)         % kernel non-zero, T1 is not one-to-one.
% Hence, not an isomorphism.
% k2 has one vector, so nullity 2.
k2=null(M2)         % kernel non-zero, T2 is not one-to-one.
% Hence, not an isomorphism.
k3=null(M3)         % k3 has no vectors, so nullity 0. T3 is one-to-one.
% Exercises 5--8.
% The first 2 columns form a basis for the range.
r1=rref(M1)         % Rank is 2. 2 + 2 = number of columns.
% The third row has no pivot, so not onto.
% The first 3 columns form a basis for the range.
r2=rref(M2)         % Rank is 3. 3 + 1 = number of columns.
% Each row has a pivot, so onto.
% All columns form a basis for the range.
r3=rref(M3)         % Rank is 4. 4 + 0 = number of columns.
% T3 is one-to-one and onto, hence, an isomorphism.
% False. T1 is not onto.
% True. T2 is onto.
% True. T3 is onto.
% Exercises 9, 10.
% First we form a matrix with the coefficients of the given polynomials,
M = [1 1 3 4; 2 3 3 4; 3 5 4 4; 4 7 4 5]
% then a matrix with the coefficients of their values.
N = [1 0 1 1; 0 1 1 1; -1 0 1 -1; 1 -1 -1 1]
% The last column of M1 has entries the coefficients of 2x^3+2x^2-2x-2
M1 = rref([M [2;2;-2;-2]])  % in terms of the given polynomials.
M1(:,5), N * ans            % Multiplication by N evaluates T(2,2,-2,-2).
% Exercise 11
% Solve 3 a-4b=-1, a+3b=4, -b=-1 for a and b to get
rref([3 -4 -1; 1 3 4; 0 -1 -1])    % a = b = 1
function [A] = T(a,b)           % In a function file we type the code:
A = [3*a-4*b; a+3*b; -b];
end
```

```
T(1,1)                          % T (1, 1) get the polynomial back.
%  Exercise 12.
% First we form a matrix with the coefficients of the given polynomials
M = [1 1 3 4; 2 3 3 4; 3 5 4 4; 4 7 4 5]
% then a matrix with the coefficients of their values.
n = [1 0 1 1; 0 1 1 1; -1 0 1 -1; 1 -1 -1 1]
M1=rref([M [2 2 -2 -2]'])
% The last column of M1 has entries the coefficients of 2x^3+2x^2-2x-2
M2 = M1(:,5)     % in terms of the given polynomials.
n * M2                % Multiplication by n evaluates T(2,2,-2,-2).
% Exercise 13.
% Set F=0. We need to solve a+2b+3c+4d=0,2a+3b+4c+5d=0,3a+4b+5c+6d=0.
M = [1 2 3 4; 2 3 4 5; 3 4 5 6]  %  We compute a basis for the
MN = null(M)           % nullspace of the coefficient matrix.
% Exercise 14.
function [A] = T(a,b)           % In a function file type the transformation.
A = [3*a-4*b; a+3*b; -b];
end
b2=[1,0,0; 0,1,1; -1,1,-1]         % The coefficients of B'.
aug = [b2,T(1,-1),T(1,1)]          % The coefficients of T(x-1),T(x+1)
rref(aug)                          % in terms of B' are computed by rref(aug).
tt=ans(:,4:5)                      % The last 2 columns form the matrix of T.
rref(tt)                           % Both columns are pivot so 1-1.
% Exercise 15.
function [A] = TT(a,b,c,d)     % In a function file type the transformation.
A=[a; b-c; c+d; d];
end                            % Define TT as vector-valued.
MM = [TT(1,0,0,0) TT(0,1,0,0) TT(0,1,1,0) TT(0,0,1,1)]
b2=[0 0 0 1; 0 0 1 0; 0 1 0 0; 1 0 0 0]
aug = [b2,MM]           % the augmented matrix of the system )
aug1=rref(aug)          %  Solve keep the last 4
M=[aug1(:,5:8)]         % columns to get M
det(M)            % M has nonzero determinant, so T is isomorphism.
```

5.7.3 Selected solutions with Maple

```
#  Exercise 1.
T1:=(x,y,z,w)->[x+2*y+3*z+4*w, 2*x+3*y+4*z+5*w, 3*x+4*y+5*z+6*w];
T2:=(x,y,z,w)->[x+2*y+3*z+4*w, 2*x+2*y+3*z+4*w, 3*x+3*y+3*z+4*w];
T3:=(x,y,z,w)->[x+2*y+3*z+4*w,2*x+2*y+3*z+4*w,3*x+3*y+3*z+4*w,4*x+4*y+4*z+4*w];
T1(-1,0,3,-2);        # v1 in the kernel.
T1(18,-31,8,5);       # v2 in the kernel.
T1(1,-1,8,7);         # v3 not in the kernel.
with(LinearALgebra);
M1:= <<T1(1,0,0,0)>|<T1(0,1,0,0)>|<T1(0,0,1,0)>|<T1(0,0,0,1)>>;
# Standard matrix.
# Exercise 2.
```

```
with(LinearAlgebra);
Mw1:= ReducedRowEchelonForm(<M1 | Vector([2,7,12])>);
# In Range(T1). Last column is pivot.
Mw2:= ReducedRowEchelonForm(<M1 | Vector([42,59,76])>);
# In Range(T1). Last column is pivot.
Mw3:= ReducedRowEchelonForm(<M1 | Vector([42,59,77])>);
# Not in Range(T1). Last column is non-pivot.
# Exercise 3.
M1:= <<T1(1,0,0,0)>|<T1(0,1,0,0)>|<T1(0,0,1,0)>|<T1(0,0,0,1)>>;
M2:= <<T2(1,0,0,0)>|<T2(0,1,0,0)>|<T2(0,0,1,0)>|<T2(0,0,0,1)>>;
M3:= <<T3(1,0,0,0)>|<T3(0,1,0,0)>|<T3(0,0,1,0)>|<T3(0,0,0,1)>>;
# Exercise 4.
k1:=NullSpace(M1);  # kernel non-zero, T1 is not one-to-one.
# k1 has two vectors, so nullity 2. Hence, not an isomorphism.
k2:=NullSpace(M2);  # kernel non-zero, T2 is not one-to-one.
# k2 has two vectors, so nullity 2. Hence, not an isomorphism.
k3:=NullSpace(M3);  # k3 has no vectors, nullity 0, one-to-one.
# Since T3:R^4->R^4 is one-to-one, it is an isomorphism.
# Exercises 5--8.
# The first 2 columns form a basis for the range.
r1:=ReducedRowEchelonForm(M1);   # Rank is 2. 2+ =number of columns.
# The third row has no pivot, so not onto.
# The first 3 columns form a basis for the range.
r2:=ReducedRowEchelonForm(M2);   # Rank is 3. 3+1=number of columns.
# Each row has a pivot, so onto.
# All columns form a basis for the range.
r3:=ReducedRowEchelonForm(M3);   # Rank is 4. 4+0 =number of columns.
# T3 is one-to-one and onto, hence, an isomorphism.
# Also,  r3:=range(T3);
# False. T1 is not onto.
# True. T2 is onto.
# True. T3 is onto.
# Exercises 9, 10.
M := Matrix([[1,1,3,4],[2,3,3,4],[3,5,4,4],[4,7,4,5]]); # The domain vectors.
n := Matrix([[1,0,1,1],[0,1,1,1],[-1,0,1,-1],[1,-1,-1,1]]);  # The values.
STM := n . M^(-1);                                  # The standard matrix.
STM . Vector([[2],[2],[-2],[-2]]);                  # T(2,2,-2,-2).
# Exercise 13.
# Set F=0. We need to solve a+2b+3c+4d=0,2a+3b+4c+5d=0,3a+4b+5c+6d=0.
M := Matrix(3,4,[1,2,3,4, 2,3,4,5, 3,4,5,6]);  # We compute a basis for
NM := NullSpace(M);           # the nullspace of the coefficient matrix.
# Exercise 14.
T := (a,b) -> [3*a-4*b, a+3*b, -b];   # We use [a,b,c] for ax^2+bx+c.
b2:=Matrix([[1,0,0],[0,1,1],[-1,1,-1]]); #The coefficients of B and
aug := <b2|Vector(T(1,-1))|Vector(T(1,1))>; # those of T(x-1),T(x+1).
aug1:= ReducedRowEchelonForm(aug); # in terms of B' are computed
# by rref(aug). The last 2 columns form the matrix of T.
M := SubMatrix(aug1, 1..3, 4..5);
ReducedRowEchelonForm(M);       # Both columns are pivot, so 1-1.
```

```
NullSpace(M);       # Another way, the nulsspace has {} as basis.
# Exercise 15.
T := (a,b,c,d) -> Vector([a, b-c, c+d, d]);#Define T as vector-valued.
MM :=<T(1,0,0,0)|T(0,1,0,0)|T(0,1,1,0)|T(0,0,1,1)>;
# Evaluate T at the basis matrices written as vectors.
b2:=Matrix([[0,0,0,1],[0,0,1,0],[0,1,0,0],[1,0,0,0]]);
aug:= <b2|MM>;
aug1:=ReducedRowEchelonForm(aug); # Solve and the
M := SubMatrix(aug1,1..4,5..8); # last 4 columns.
Determinant(M);
# M has nonzero determinant, so T is isomorphism.
```

6 Determinants

> The greatest mathematicians, as Archimedes, Newton, and Gauss, always united theory and applications in equal measure.
>
> Felix Klein, German mathematician (1849–1925).

Introduction

Determinants are among the most useful topics of linear algebra. There are numerous applications to engineering, physics, economics, mathematics, and other sciences. In geometry, they can be used to compute areas and volumes and to write equations of lines, circles, planes, spheres, and other geometric objects. They are also used to solve polynomial systems of equations.

Determinants first appear in 1683 in the work of the Japanese mathematician Takakazu Kowa Seki (Figure 6.1).

Figure 6.1: Takakazu Kowa Seki.
Upload by A Morozov – Gakken, Public Domain, https://commons.wikimedia.org/w/index.php?curid=710154.
Takakazu Kowa Seki (1642–1708) was a Japanese mathematician. He was adopted by a noble family named Seki Gorozayemon. He was the first to publish the definition of determinants up to size 4 in 1683. He discovered Bernoulli numbers before Bernoulli and solved a cubic equation using Horner's method a hundred years before Horner.

According to Dirichlet, determinants were also introduced, independently, by *Leibnitz* in Germany in a letter to L'Hôpital, dated 28 April 1693 (Figure 6.2). Although determinants appeared in the literature in the late 1600s (well before matrices![1]), the first work to systematically study them for their own sake was written in 1772 by Vandermonde. Main contributors in the theory of determinants were Laplace, Cauchy, Jacobi, Bezout, Sylvester, and Cayley.

We introduce determinants by using the cofactor expansion and examine their basic properties. We discuss the adjoint of a matrix and Cramer's rule. We also define determinants by using permutations.

We discuss two interesting mathematical applications, (a) algebraic equations of geometric objects and (b) solutions of nonlinear systems. Finally, as a special topic, we apply determinants to *image recognition*.

---

[1] For a brief history of the subject, see *Lessons Introductory to the Higher Modern Algebra* by George Salmon, D. D., 1885, Fifth Edition, Chelsea Publishing Company, pages 338–339.

https://doi.org/10.1515/9783111331850-006

Figure 6.2: Gottfried von Leibniz.
By Christoph Bernhard Francke – Herzog Anton Ulrich-Museum, Public Domain, https://commons.wikimedia.org/w/index.php?curid=53159699.
Gottfried Wilhelm von Leibniz (1646–1716) was a German mathematician, lawyer, and philosopher. He studied mathematics and physics under Huygens. He is the coinventor of calculus. In unpublished work, he developed a theory of determinants including Cramer's rule. He had a priority dispute with Newton over the invention of calculus.

6.1 Determinants: Basic concepts

In this section, we introduce the determinant of a square matrix along with examples and a discussion of a useful geometric property of the determinant.

6.1.1 Cofactor expansion

We define the determinant of a *square matrix* by using the *cofactor expansion*. The cofactor expansion is in fact a theorem whose proof is outlined in the exercises of Section 6.4. However, we use it here as quick introduction.

Let $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right]$. The determinant $\det(A)$ of A is the number

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Example 6.1.1. We have

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2.$$

We often write $|A|$ for $\det(A)$. This notation should not be confused with the absolute value of a number. So we may write

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2.$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

The determinant of A in terms of 2×2 determinants is the number

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Example 6.1.2. Find $\det(B)$ if

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -2 \\ 0 & 2 & -1 \end{bmatrix}.$$

Solution. We have

$$\begin{aligned}\det(B) &= 1\begin{vmatrix} 0 & -2 \\ 2 & -1 \end{vmatrix} - 2\begin{vmatrix} 1 & -2 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \\ &= 1 \cdot 4 - 2 \cdot (-1) + 0 \cdot 2 \\ &= 6.\end{aligned}$$

In the same manner, we can define determinants of 4 × 4 matrices:

$$\begin{aligned}\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} \\ &\quad + a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}.\end{aligned}$$

Example 6.1.3. Find $\det(C)$ if

$$C = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 0 \\ -2 & 1 & 0 & -2 \\ 1 & 0 & 2 & -1 \end{bmatrix}.$$

Solution. $\det(C)$ equals

$$\begin{aligned}&1\begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & -2 \\ 0 & 2 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 & 0 \\ -2 & 0 & -2 \\ 1 & 2 & -1 \end{vmatrix} + 0\begin{vmatrix} -1 & 1 & 0 \\ -2 & 1 & -2 \\ 1 & 0 & -1 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 & 2 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} \\ &= 1 \cdot 6 - 2 \cdot (-12) + 0 \cdot (-3) - 1 \cdot 0 = 30.\end{aligned}$$

We can continue similarly and define determinants of $n \times n$ matrices in terms of determinants of $(n-1) \times (n-1)$ matrices, called minors. The (i, j) *minor* M_{ij} of a matrix A is the determinant obtained by deleting the ith row and the jth column.

So far we have introduced the *cofactor expansion of a determinant about its first row*. Each entry of the first row is multiplied by the corresponding minor, and each such product is multiplied by ±1 depending on the position of the entry. The signed products are added together. In fact, instead of the first row, we can use *any* row or column as follows. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

First, we assign the sign $(-1)^{i+j}$ to the entry a_{ij} of A. This is a checkerboard pattern of ±s:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Then we pick a row or column and multiply each entry a_{ij} of it by the corresponding *signed* minor $(-1)^{i+j}M_{ij}$. Then we add all these products.

The signed minor $(-1)^{i+j}M_{ij}$ is called the (i,j) *cofactor* of A and is denoted by C_{ij},

$$C_{ij} = (-1)^{i+j}M_{ij}.$$

We have:

1. *Cofactor expansion about the ith row.* The determinant of A can be expanded about the ith row in terms of the cofactors as follows:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$

2. *Cofactor Expansion about the jth column.* The determinant of A can be expanded about the jth column in terms of the cofactors as follows:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

This method of computing determinants by using cofactors is called the *cofactor expansion*, or *Laplace expansion*, and it is attributed to Vandermonde and Laplace (Figure 6.6). The proof is discussed in the exercises of Section 6.4.

Example 6.1.4. Compute all cofactors of $\det(A)$ and find $\det(A)$ by using cofactor expansion about every row and column if

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 4 & 3 & -2 \\ -5 & 0 & 3 \end{bmatrix}.$$

Solution. We have

$$\begin{aligned}
M_{11} &= \det\begin{bmatrix} 3 & -2 \\ 0 & 3 \end{bmatrix} = 9, && C_{11} = (-1)^{1+1}M_{11} = 9,\\
M_{12} &= \det\begin{bmatrix} 4 & -2 \\ -5 & 3 \end{bmatrix} = 2, && C_{12} = (-1)^{1+2}M_{12} = -1\cdot 2 = -2,\\
M_{13} &= \det\begin{bmatrix} 4 & 3 \\ -5 & 0 \end{bmatrix} = 15, && C_{13} = (-1)^{1+3}M_{13} = 15,\\
M_{21} &= \det\begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix} = 6, && C_{21} = (-1)^{2+1}M_{21} = -1\cdot 6 = -6,\\
M_{22} &= \det\begin{bmatrix} -1 & 2 \\ -5 & 3 \end{bmatrix} = 7, && C_{22} = (-1)^{2+2}M_{22} = 7,\\
M_{23} &= \det\begin{bmatrix} -1 & 2 \\ -5 & 0 \end{bmatrix} = 10, && C_{23} = (-1)^{2+3}M_{23} = -1\cdot 10 = -10,\\
M_{31} &= \det\begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix} = -10, && C_{31} = (-1)^{3+1}M_{31} = -10,\\
M_{32} &= \det\begin{bmatrix} -1 & 2 \\ 4 & -2 \end{bmatrix} = -6, && C_{32} = (-1)^{3+2}M_{32} = (-1)(-6) = 6,\\
M_{33} &= \det\begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} = -11, && C_{33} = (-1)^{3+3}M_{33} = -11.
\end{aligned}$$

Thus

$$\begin{aligned}
\det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (-1)9 + 2(-2) + 2\cdot 15 = 17,\\
\det A &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = 4(-6) + 3\cdot 7 + (-2)(-10) = 17,\\
\det A &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (-5)(-10) + 0\cdot 6 + 3(-11) = 17,\\
\det A &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (-1)9 + 4(-6) + (-5)(-10) = 17,\\
\det A &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = 2(-2) + 3\cdot 7 + 0\cdot 6 = 17,\\
\det A &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} = 2\cdot 15 + (-2)(-10) + 3(-11) = 17.
\end{aligned}$$

1. If we multiply the entries of a row (or column) by the corresponding cofactors from *another* row (or column), the we get *zero*. For instance, in Example 6.1.4, we have

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = (-1)(-6) + 2\cdot 7 + 2(-10) = 0,$$

$$a_{11}C_{12} + a_{21}C_{22} + a_{31}C_{32} = (-1)(-2) + 4\cdot 7 + (-5)6 = 0.$$

2. The cofactor expansion implies that the determinant of any upper or lower triangular matrix is the product of its main diagonal entries. For example, repeated expansion about the first column yields

$$\begin{vmatrix} 4 & 5 & 6 & 2 \\ 0 & 7 & 8 & -4 \\ 0 & 0 & 9 & 5 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 4\begin{vmatrix} 7 & 8 & -4 \\ 0 & 9 & 5 \\ 0 & 0 & -1 \end{vmatrix} = 4\cdot 7\begin{vmatrix} 9 & 5 \\ 0 & -1 \end{vmatrix} = 4\cdot 7\cdot 9\cdot(-1) = -252.$$

3. We try to expand a determinant about the row or column with the most zeros. This avoids the computation of some of the minors.

6.1.2 Geometric property of the determinant

We discuss now a striking geometric property of the determinant.

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation with standard matrix A. Let $\mathcal{R}$ be a planar region of finite area, and let $T(\mathcal{R})$ be the image of $\mathcal{R}$ under T. Then

$$\text{Area}(T(\mathcal{R})) = \text{Area}(\mathcal{R})\,|\det(A)|.$$

For example, consider the effect of the shear

$$T(\mathbf{x}) = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$

on the unit square. The image is the rectangle with vertices $(0,0)$, $(3,0)$, $(1,2)$, and $(4,2)$. The area of the image is 6, which happens to be the determinant of the matrix (Figure 6.3).

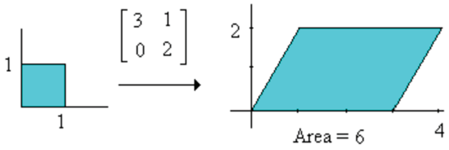

Figure 6.3: The area of the image is $|\det(A)| = 6$.

In general, if we apply

$$T(\mathbf{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x}$$

to the unit square, then the images of $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ are $(0,0)$, (a,c), (b,d), $(a+b, c+d)$, respectively. These define a parallelogram, provided that (a,c) is not proportional to (b,d). That is equivalent to saying that $ad - bc \neq 0$ or that the matrix is invertible. We leave it to the reader to prove that the area of the parallelogram is $|ad - bd|$ (Figure 6.4).

This geometric property of the determinant generalizes to space transformations and volumes of regions.

Let T be a space linear transformation with standard matrix A, and let $\mathcal{R}$ be a space region of finite volume. Then we have

$$\text{Vol}\big(T(\mathcal{R})\big) = \text{Vol}(\mathcal{R})\,|\det(A)|.$$

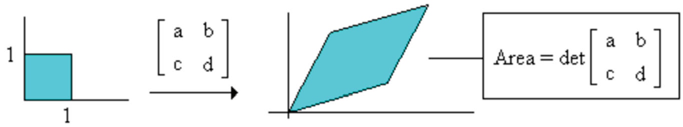

Figure 6.4: The area of the image is det(*A*).

6.1.3 The Sarrus scheme for 3 × 3 determinants

There is also a mnemonic device for memorizing the formula for a 3 × 3 determinant, called the *Sarrus scheme* (Figure 6.5). We add the first two columns to the right of the matrix and form the products of the entries covered by the arrows. The products of the arrows going from upper left to lower right are taken with a plus sign, and the others with a minus. Then all signed products are added:

$$\det(B) = a_{11}\, a_{22}a_{33} + a_{12}\, a_{23}a_{31} + a_{13}\, a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}\, a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

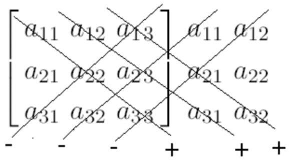

Figure 6.5: The Sarrus scheme for 3 × 3 determinants.

The Sarrus scheme only applies to 3 × 3 determinants. It *does not apply* to 2 × 2 or $n \times n$ determinants with $n \geq 4$.

Figure 6.6: **Pierre-Simon Laplace.**
By James Posselwhite – www.britannica.com, Public Domain, https://commons.wikimedia.org/w/index.php?curid=11128070.
Pierre-Simon Laplace (1749–1827) was a French mathematician, whose research included work on maxima–minima, integral calculus, and mechanics. He did fundamental work in astronomy and theory of probability. He wrote *Mécanique Céleste*, a master piece in celestial mechanics. Under Napoleon, he became the chancellor of the Senate and served as the Minister of the Interior. He became a count of the Empire, and later he was named a marquis.

Exercises 6.1

In Exercises 1–7, compute the determinants.

1. (a) $\begin{vmatrix} 1 & -1 \\ -4 & 4 \end{vmatrix}$, (b) $\begin{vmatrix} 0 & 3 \\ 6 & 2525 \end{vmatrix}$.

2. (a) $\begin{vmatrix} 0.5 & -0.7 \\ 1.2 & 3.4 \end{vmatrix}$, (b) $\begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$.

3. (a) $\begin{vmatrix} a & 3a \\ b & 3b \end{vmatrix}$, (b) $\begin{vmatrix} a & a & a \\ b & b & b \\ c & c & c \end{vmatrix}$.

4. (a) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$, (b) $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{vmatrix}$.

5. (a) $\begin{vmatrix} 2 & 0 & 0 \\ 191 & -6 & 0 \\ 312 & 755 & 10 \end{vmatrix}$, (b) $\begin{vmatrix} 1 & a & a^2 \\ a^3 & a^4 & a^5 \\ a^6 & a^7 & a^8 \end{vmatrix}$.

6. (a) $\begin{vmatrix} 4 & 2156 & 1569 \\ 0 & 5 & 7532 \\ 0 & 0 & -10 \end{vmatrix}$, (b) $\begin{vmatrix} 4 & 0 & 1 & 0 \\ 3 & 2 & 0 & -2 \\ 2 & 0 & 3 & 7 \\ 1 & 0 & 4 & 0 \end{vmatrix}$.

7. $\begin{vmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 \\ 1 & 3 & 0 & -5 & 1 \\ 1 & 4 & 0 & 1 & 0 \end{vmatrix}$.

8. For 2×2 matrices A and B, prove that

(a) $\det(A) = \det(A^T)$,

(b) $\det(AB) = \det(A)\det(B)$,

(c) $\det(A^{-1}) = 1/\det(A)$ if $\det(A) \neq 0$.

9. Write the cofactors of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

10. (a) Find the cofactors $C_{2,3}$ and $C_{3,1}$ of

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 5 & -4 \\ 7 & 0 & -6 \end{bmatrix}.$$

(b) Compute $\det(A)$ by using cofactor expansion about

(i) the second row;

(ii) the third column.

Determinants of elementary matrices

In Exercises 11–13, compute the determinants of the elementary matrices.

11. $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ r & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & r & 1 \end{bmatrix}$.

12. $E_3 = \begin{bmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix}$.

13. $E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $E_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

14. Based on your calculations from Exercises 11–13, form a statement about the determinants of the three kinds of elementary matrices: those obtained from I by (a) elimination, (b) scaling, and (c) interchange.

In Exercises 15–17, and for the E_i of Exercises 11–13, prove the identity

$$\det(E_i A) = \det(E_i)\det(A) = \det(A_i)$$

given that

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

and A_i are obtained from A by

$$\begin{array}{lc} A_1 & R_2 + rR_1 \to R_2, \\ A_2 & R_3 + rR_2 \to R_3, \\ A_3 & rR_1 \to R_1, \\ A_4 & rR_3 \to R_3, \\ A_5 & R_2 \leftrightarrow R_3, \\ A_6 & R_1 \leftrightarrow R_3. \end{array}$$

15. For $i = 1$ and $i = 2$.

16. For $i = 3$ and $i = 4$.

17. For $i = 5$ and $i = 6$.

Equations with determinants

In Exercises 18–21, solve for x.

18. (a) $\begin{vmatrix} 8-x & 0 \\ -1 & 3-x \end{vmatrix} = 0$, (b) $\begin{vmatrix} 1-x & 2 \\ -1 & 4-x \end{vmatrix} = 0$.

19. $\begin{vmatrix} 17 & 0 & 0 \\ 0 & x-3 & 1 \\ 0 & 4 & x-6 \end{vmatrix} = 0,$

20. $\begin{vmatrix} 1-x & 0 & 0 \\ 1 & 3-x & 0 \\ 0 & 1 & 5-x \end{vmatrix} = 0.$

21. $\begin{vmatrix} x & 4 \\ x & 6 \end{vmatrix} = \begin{vmatrix} x-1 & 1 \\ -2 & x-1 \end{vmatrix}.$

Volumes and determinants

Let R be the unit cube in $\mathbf{R}^3$ (shown in Figure 6.7), and let $T(\mathbf{x}) = A\mathbf{x}$ for the given A. In Exercises 22–23, compute the volume of the image of $T(R)$ and relate it to the determinant of A.

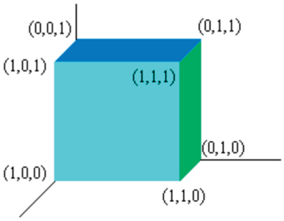
(0,0,1)
(0,1,1)
(1,0,1)
(1,1,1)
(0,1,0)
(1,0,0)
(1,1,0)

Figure 6.7: Unit cube.

22. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (Figure 6.8).

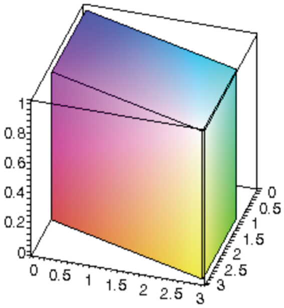
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
0
0.5
1
1.5
2
2.5
3

Figure 6.8: Image of unit cube.

23. $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (Figure 6.9).

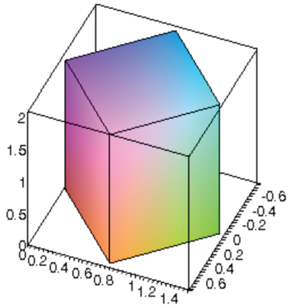

Figure 6.9: Image of unit cube.

6.2 Properties of determinants

Computing determinants by cofactor expansion is efficient if the matrix is small or if it has many zeros. A much better method is based on Gauss elimination. To see how we utilize elimination, we must first study the effects of the elementary row operations on determinants.

6.2.1 Elementary operations and determinants

The next theorem describes the basic properties of determinants. A step-by-step proof is discussed in Exercises 6.4.

Theorem 6.2.1. *Let A be an $n \times n$ matrix. Then*

1. *A and its transpose have the same determinant,* $\det(A) = \det(A^T)$. *For example,*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

2. *Let B be obtained from A by multiplying one of its rows (or columns) by a nonzero constant. Then* $\det(B) = k\det(A)$. *For example,*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ kb_1 & kb_2 & kb_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 & ka_3 \\ b_1 & b_2 & kb_3 \\ c_1 & c_2 & kc_3 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

3. *Let B be obtained from A by interchanging any two rows (or columns). Then* $\det(B) = -\det(A)$. *For example,*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 \\ c_3 & c_2 & c_1 \end{vmatrix}.$$

4. *Let B be obtained from A by adding a multiple of one row (or column) to another. Then* $\det(B) = \det(A)$. *For example,*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ ka_1 + b_1 & ka_2 + b_2 & ka_3 + b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

We conclude from Theorem 6.2.1 that

1. Elimination $R_i + cR_j \rightarrow R_i$, does not change the determinant;
2. Scaling $cR_i \rightarrow R_i$ scales the determinant by c;
3. Interchange $R_i \leftrightarrow R_j$ changes the sign of the determinant.

Example 6.2.2. By Theorem 6.2.1 we have

(a) $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$, $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = - \begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix}$,

(b) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix}$ by $-3R_1 + R_2 \rightarrow R_2$,

(c) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$ by $-C_1 + C_2 \rightarrow C_2$.

A common mistake. Property 4 of Theorem 6.2.1 is sometimes misused. A row (or column) is replaced by a multiple of another added to the first. *We do not scale the original row (or column). If we do, then the determinant is scaled.* For example, we have

$$\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 0 & \frac{5}{3} \end{vmatrix} = 5 \quad \text{by} \; -(1/3)R_1 + R_2 \rightarrow R_2.$$

However, the nonelementary operation $R_1 - 3R_2 \rightarrow R_2$ would yield the *wrong answer*

$$\begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} = -15.$$

Theorem 6.2.1 can be used to compute a determinant by row reduction and multiplication of the diagonal entries of the echelon form. The effects of interchanges and scalings should be taken into account.

Example 6.2.3. We have

$$
\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 & -1 & 8 \\ 0 & 0 & 4 & 2 & -1 \\ 0 & -5 & 5 & 3 & 7 \\ 0 & 0 & 0 & 1 & 6 \\ 1 & 2 & 3 & -2 & -9 \end{vmatrix}
&= \begin{vmatrix} 1 & 2 & 3 & -1 & 8 \\ 0 & 0 & 4 & 2 & -1 \\ 0 & -5 & 5 & 3 & 7 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -1 & -17 \end{vmatrix} \qquad \boxed{-R_1 + R_5 \to R_5} \\
&= - \begin{vmatrix} 1 & 2 & 3 & -1 & 8 \\ 0 & -5 & 5 & 3 & 7 \\ 0 & 0 & 4 & 2 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -1 & -17 \end{vmatrix} \qquad \boxed{R_2 \leftrightarrow R_3} \\
&= - \begin{vmatrix} 1 & 2 & 3 & -1 & 8 \\ 0 & -5 & 5 & 3 & 7 \\ 0 & 0 & 4 & 2 & -1 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & -11 \end{vmatrix} \qquad \boxed{R_4 + R_5 \to R_5} \\
&= 1 \cdot (-5) \cdot 4 \cdot 1 \cdot (-11) \\
&= -220.
\end{aligned}
$$

The method of Example 6.2.3 also yields a formula for the determinant. Let A be an $n \times n$ matrix that row reduces without scaling to the upper triangular matrix B. Note that reduction without any scaling is always possible. From the remaining two operations, only interchanges change the determinant by only a sign. Hence

$$\det(A) = (-1)^k \det(B),$$

where k is the number of interchanges in the reduction. If A is invertible, then $A \sim B \sim I$. So B has n pivots, say, $p_1, \dots, p_n$, all on the main diagonal. Therefore

$$\det(A) = (-1)^k \det(B) = (-1)^k p_1 p_2 \cdots p_n.$$

If on the other hand, A is noninvertible, then B has at least one row of zeros, so $\det(A) = \det(B) = 0$. Thus we have proved the following theorem.

Theorem 6.2.4. *Let A be an $n \times n$ matrix. Then*

$$\det(A) = \begin{cases} (-1)^k p_1 p_2 \cdots p_n & \text{if } A \text{ is invertible with pivots } p_i, \\ 0 & \text{if } A \text{ is noninvertible.} \end{cases}$$

Because pivots are always nonzero, Theorem 6.2.4 implies that a matrix A is invertible if and only if $\det(A) \neq 0$. This, combined with Theorem 3.3.10, yields the following important theorem.

Theorem 6.2.5. *Let A be an $n \times n$ matrix. The following are equivalent:*

1. *A is invertible;*
2. $\det(A) \neq 0$.

Example 6.2.6. Use determinants to check the system

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -6 & 2 \\ 5 & 4 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

for nontrivial solutions.

Solution. The coefficient matrix has determinant $63 \neq 0$, so it is invertible by Theorem 6.2.5. Hence $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Example 6.2.7. Use determinants to check the vectors

$$\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

for linear independence.

Solution. Let A be the matrix with columns these vectors. Then

$$\det(A) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 1 & 4 & -1 \end{vmatrix} = 0.$$

Hence A is noninvertible by Theorem 6.2.5. So it has linearly dependent columns.

Next, we draw useful conclusions from Theorem 6.2.1 and cofactor expansions.

Theorem 6.2.8. *Let A be an $n \times n$ matrix.*

1. *If A has a row (or column) of zeros, then* $\det(A) = 0$. *For example,*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & 0 \end{vmatrix} = 0.$$

2. *If A has two rows (or columns) that are equal, then* $\det(A) = 0$. *For example,*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & a_1 \\ b_1 & b_2 & b_1 \\ c_1 & c_2 & c_1 \end{vmatrix} = 0.$$

3. *If A has two rows (or columns) that are multiples of each other, then* $\det(A) = 0$. *For example,*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ ka_1 & ka_2 & ka_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0\,, \qquad \begin{vmatrix} a_1 & a_2 & ka_1 \\ b_1 & b_2 & kb_1 \\ c_1 & c_2 & kc_1 \end{vmatrix} = 0.$$

4. *If a row (or column) of A is the sum of multiples of two other rows (or columns), then* $\det(A) = 0$. *For example,*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ ka_1 + lc_1 & ka_2 + lc_2 & ka_3 + lc_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0\,, \qquad \begin{vmatrix} a_1 & a_2 & ka_1 + la_2 \\ b_1 & b_2 & kb_1 + lb_2 \\ c_1 & c_2 & kc_1 + lc_2 \end{vmatrix} = 0.$$

Proof of 1 and 2.

1. We expand the determinant about the zero row (or column) to get 0.
2. Let the ith and jth rows be equal. We replace the jth row with the difference of the ith and jth row, which is zero. The resulting determinant is equal to the original one, and it is zero by Part 1. We may repeat this argument with columns instead of rows. □

Example 6.2.9. By Part 4 of Theorem 6.2.8 we have

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 1 & -1 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 - 2 \cdot 1 & 3 - 2 \cdot 2 & 3 - 2 \cdot 3 \end{vmatrix} = 0.$$

6.2.2 Matrix operations and determinants

Let us see how determinants are affected by the matrix operations: $A + B$, kA, and AB.

In general,

$$\det(A + B) \neq \det(A) + \det(B).$$

On the positive side, we have the following fact.

Theorem 6.2.10. *If every entry in any row (or column) of a determinant is the sum of two others, then the determinant is the sum of two others. For example,*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 + d_1 & c_2 + d_2 & c_3 + d_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}.$$

Proof. Exercise. (*Hint:* If the ith row is $[c_{i1} + d_{i1}, \dots, c_{in} + d_{in}]$, then cofactor expand about it.) ☐

Next, we compute the determinant of a scaled matrix.

Theorem 6.2.11. *Let A be an $n \times n$ matrix, and let k be any scalar. Then*

$$\det(kA) = k^n \det(A).$$

Proof. By Part 3 of Theorem 6.2.1 we have

$$\begin{aligned}\det(kA) &= \det\,[k\mathbf{a}_1\ k\mathbf{a}_2\ \cdots\ k\mathbf{a}_n] \\ &= k\det\,[\mathbf{a}_1\ k\mathbf{a}_2\ \cdots\ k\mathbf{a}_n] \\ &= k^2\det\,[\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ k\mathbf{a}_n] \\ &\cdots \\ &= k^n\det(A).\end{aligned}$$ ☐

Next, we turn to the computation of the determinant of a product, $\det(AB)$. We have the following important property.

Theorem 6.2.12 (Cauchy's theorem). *The determinant of the product of two $n\times n$ matrices is the product of the determinants of the factors:*

$$\det(AB) = \det(A)\det(B).$$

A proof of Cauchy's theorem is outlined in the exercises of this section.[2]

! Repeated application of Cauchy's theorem yields

$$\det(A_1 \cdots A_k) = \det(A_1)\cdots\det(A_k).$$

Example 6.2.13. Verify that $\det(AB) = \det(A)\det(B)$, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 0 \\ -5 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution. We have $\det(A) = -3$ and $\det(B) = 10$. So $\det(A)\det(B) = -30$. Also,

$$\det(AB) = \det\begin{bmatrix} -5 & 0 & 0 \\ -5 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix} = -30.$$

---

[2] Cauchy's theorem was also discovered by Gauss for the particular cases of 2 × 2 and 3 × 3 matrices.

Theorem 6.2.12 has the following implication.

Theorem 6.2.14. *If A is invertible, then*

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Exercises 6.2

In Exercises 1–7, evaluate the determinants by inspection.

1. (a) $\begin{vmatrix} 1 & 0 & 0 \\ 100 & 10 & 0 \\ 1000 & 10000 & 100 \end{vmatrix}$, (b) $\begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 4 \\ 4 & 4 & 8 \end{vmatrix}$.

2. $\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$.

3. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$.

4. $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix}$.

5. $\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}$.

6. $\begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 2 & 2 & 0 & 0 & 0 \\ 3 & 6 & 3 & 0 & 0 \\ 4 & 6 & 6 & 4 & 0 \\ 5 & 5 & 5 & 5 & -1 \end{vmatrix}$.

7. $\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 \\ 0 & 6 & 3 & 0 & 3 \\ 0 & 6 & 6 & 4 & 5 \\ -1 & 5 & 5 & 5 & 5 \end{vmatrix}$.

In Exercises 8–15, use

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$$

and explain the identities without computation.

8. $\begin{vmatrix} a & b & c \\ d & e & f \\ 2g & 2h & 2i \end{vmatrix} = 6.$

9. $\begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} = -3.$

10. $\begin{vmatrix} a-4c & b & c \\ d-4f & e & f \\ g-4i & h & i \end{vmatrix} = 3.$

11. $\begin{vmatrix} 2b-4c & b & c \\ 2e-4f & e & f \\ 2h-4i & h & i \end{vmatrix} = 0.$

12. $\begin{vmatrix} c & b & -a \\ f & e & -d \\ i & h & -g \end{vmatrix} = 3.$

13. $\begin{vmatrix} a & b & c \\ d-a & e-b & f-c \\ 3g & 3h & 3i \end{vmatrix} = 9.$

14. $\begin{vmatrix} -1 & a & e & i \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{vmatrix} = -3.$

15. $\begin{vmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{vmatrix} = 24.$

16. Explain without computing why the substitutions $x = 0, 2$ make the determinant $\begin{vmatrix} x & x & 2x \\ 0 & 1 & 0 \\ 2 & 2 & 4 \end{vmatrix}$ zero.

17. Explain without computing why the following determinants are equal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & -b & c \\ -d & e & -f \\ g & -h & i \end{vmatrix}.$$

In Exercises 18–22, evaluate the determinants by row reduction.

18. $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}.$

19. $\begin{vmatrix} 1 & -2 & 5 \\ -2 & 6 & -4 \\ 3 & -5 & 0 \end{vmatrix}.$

20. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{vmatrix}$.

21. $\begin{vmatrix} 1 & -1 & 2 & 0 & 0 \\ 0 & 1 & 2 & -2 & 7 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & 0 & 3 \\ 0 & 3 & 0 & 0 & 1 \end{vmatrix}$.

22. $\begin{vmatrix} 2 & 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & -2 & 3 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 4 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{vmatrix}$.

23. Prove the following properties of determinants:

(a) $\begin{vmatrix} a_1 & a_2 & ka_3 \\ b_1 & b_2 & kb_3 \\ c_1 & c_2 & kc_3 \end{vmatrix} = k \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

(b) $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$.

24. Prove the following properties of determinants:

(a) $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

(b) $\begin{vmatrix} a_1 + kb_1 & a_2 + kb_2 & a_3 + kb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

25. Complete the proof of Theorem 6.2.8.

26. Prove Theorem 6.2.10.

In Exercises 27–29, use Theorem 6.2.5 to find which matrices are invertible.

27. $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$.

28. $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

29. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

In Exercises 30–31, find all values of k such that the matrices are noninvertible.

30. $\begin{vmatrix} k & k-1 & 1 \\ 0 & k+1 & 4 \\ k & 0 & k \end{vmatrix}$.

31. $\begin{vmatrix} k & k^2 & 0 \\ 0 & k^3 & 4 \\ -k & 0 & k \end{vmatrix}$.

32. Prove the identity

$$\det(AB) = \det(BA).$$

(This is true even if $AB \neq BA$.)

33. If B is invertible, then prove that

$$\det\left(B^{-1}AB\right) = \det(A).$$

34. Let A be a 3×3 matrix with $\det(A) = -2$. Compute

$$\begin{aligned} &\text{(a)}\quad \det\left(A^3\right), \quad \text{(b)}\quad \det\left(A^{-1}\right), \quad \text{(c)}\quad \det\left(A^{-3}\right), \\ &\text{(d)}\quad \det\left(A^T\right), \quad \text{(e)}\quad \det\left(AA^T\right), \quad \text{(f)}\quad \det(-3A). \end{aligned}$$

35. Prove that the square of any determinant $\det(A)^2$ can be expressed as the determinant of a symmetric matrix.[3]

36. For a 2×2 matrix A, prove that $\det(A + I) = \det(A) + 1$ if and only if $\operatorname{tr}(A) = 0$.

37. Let A be a skew-symmetric matrix of size $n \times n$ with odd n. Prove that $\det(A) = 0$.

In Exercises 38–42, prove the identities.

38. $\begin{vmatrix} 1 & 1 \\ a^2 & b^2 \end{vmatrix} = (a+b)(b-a)$.

39. $\begin{vmatrix} a & b & b \\ b & a & b \\ b & a & a \end{vmatrix} = (a+b)(a-b)^2$.

40. $\begin{vmatrix} a & b & b & b \\ b & a & b & b \\ b & a & a & b \\ b & a & a & a \end{vmatrix} = (a+b)(a-b)^3$.

41. $\begin{vmatrix} 1 & -1 & 0 & 0 \\ a & b & -1 & 0 \\ a^2 & ab & b & -1 \\ a^3 & a^2b & ab & b \end{vmatrix} = (a+b)^3$.

---

[3] The claim of this exercise is due to Lagrange.

42. $\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = \left(a^2 + b^2 + c^2 + d^2\right)^2.$

43. Prove that the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is divisible by $a + b + c$.

44. Given that each of 91 234, 84 332, 57 797, 95 497, 37 497 is divisible by 29, prove that the following determinant is also divisible by 29 (calculation of the determinant is not necessary):

$$\begin{vmatrix} 9 & 1 & 2 & 3 & 4 \\ 8 & 4 & 3 & 3 & 2 \\ 5 & 7 & 7 & 9 & 7 \\ 9 & 5 & 4 & 9 & 7 \\ 3 & 7 & 4 & 9 & 7 \end{vmatrix}.$$

A proof of Cauchy's theorem

The following exercises outline a proof of the basic formula $\det(AB) = \det(A)\det(B)$ by using Theorem 6.2.1.

45. Let E be an elementary matrix. Use Theorem 6.2.1 to prove

	If *E* is obtained from *I* by
$\det(E) = 1$	$R_i + cR_j \to R_i$
$\det(E) = c$	$cR_i \to R_i$
$\det(E) = -1$	$R_i \leftrightarrow R_j$

46. Let A and E be $n \times n$ matrices with elementary E. Use Exercise 45 to prove that

$$\det(EA) = \det(E)\det(A).$$

47. Let A and B be $n \times n$ matrices with noninvertible A. Prove that

$$\det(AB) = \det(A)\det(B).$$

(*Hint:* Use Theorems 3.3.11 and 6.2.5.)

48. Let A and B be $n \times n$ matrices with invertible A. Prove that

$$\det(AB) = \det(A)\det(B).$$

(*Hint:* Use Exercise 46.)

49. For the block matrix, prove that $\det \begin{bmatrix} A & \vdots & B \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & C \end{bmatrix} = \det(A)\det(C)$. The blocks are compatible square matrices.

6.3 The adjoint; Cramer's rule

In this section, we introduce the *adjoint* of a square matrix and use it to prove a formula for the inverse of an invertible matrix. We also use the adjoint to deduce a formula for the solution of a consistent square system with invertible coefficient matrix. This formula is known as *Cramer's rule*.

6.3.1 Adjoint and inverse

Definition 6.3.1. Let A be an $n \times n$ matrix. The matrix whose (i,j) entry is the cofactor C_{ij} of A is the *matrix of cofactors* of A. Its transpose is the *adjoint of* A and is denoted by $\operatorname{Adj}(A)$,

$$\operatorname{Adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Example 6.3.2. Find the adjoint of

$$A = \begin{bmatrix} -1 & 2 & 2 \\ 4 & 3 & -2 \\ -5 & 0 & 3 \end{bmatrix}.$$

Solution. In Example 6.1.4, Section 6.1, we found the cofactors of A to be

$$\begin{aligned} &C_{11} = 9, \quad C_{12} = -2, \quad C_{13} = 15, \\ &C_{21} = -6, \quad C_{22} = 7, \quad C_{23} = 10, \\ &C_{31} = -10, \quad C_{32} = 6, \quad C_{33} = -11. \end{aligned}$$

Hence

$$\operatorname{Adj}(A) = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} 9 & -6 & -10 \\ -2 & 7 & 6 \\ 15 & -10 & -11 \end{bmatrix}$$

In Chapter 3, we discussed an algorithm but gave *no formula* for the computation of A^{-1}. This was done by the row reduction of $[A : I]$. Here we can use determinants to give an *explicit formula* for A^{-1}. This is done in Theorem 6.3.4, but we first need some preparation, the following theorem.

Theorem 6.3.3. *Let A be an n × n matrix. Then*

$$A\,\mathrm{Adj}(A) = \det(A)I_n = \mathrm{Adj}(A)\,A.$$

Proof. Consider the product

$$\mathrm{Adj}(A)\,A = \begin{bmatrix} C_{11} & \cdots & C_{j1} & \cdots & C_{n1} \\ \vdots & & \vdots & & \vdots \\ C_{1i} & \cdots & C_{ji} & \cdots & C_{ni} \\ \vdots & & \vdots & & \vdots \\ C_{1n} & \cdots & C_{jn} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}.$$

The (i,j) entry is

$$C_{1i}a_{1j} + C_{2i}a_{2j} + \cdots + C_{ni}a_{nj}.$$

This sum can be viewed as the determinant cofactor expansion about the jth column of the matrix A' obtained from A by replacing the jth column with the ith one. If $i = j$, then the sum is $\det(A)$, because in this case, $A = A'$. If $i \neq j$, then the sum is 0, because A has a repeated column by Theorem 6.2.8. Therefore

$$\mathrm{Adj}(A)\,A = \begin{bmatrix} \det(A) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \det(A) \end{bmatrix} = \det(A)I_n.$$

A similar argument shows that $A\,\mathrm{Adj}(A) = \det(A)I_n$. □

Theorem 6.3.4. *Let A be an invertible matrix. Then*

$$A^{-1} = \frac{1}{\det(A)}\mathrm{Adj}(A).$$

Proof. Multiplying the relation $A\,\mathrm{Adj}(A) = \det(A)I_n$ of Theorem 6.3.3 on the left by A^{-1}, we get

$$\mathrm{Adj}(A) = A^{-1}\det(A).$$

Since A is invertible, $\det(A) \neq 0$ by Theorem 6.2.5. Hence

$$\frac{1}{\det(A)}\mathrm{Adj}(A) = A^{-1}.$$ □

Example 6.3.5. Let A be as in Example 6.3.2. Compute A^{-1} by applying Theorem 6.3.4.

Solution. We have $\det(A) = 17$. Hence

$$A^{-1} = \frac{1}{\det(A)}\operatorname{Adj}(A) = \frac{1}{17}\begin{bmatrix} 9 & -6 & -10 \\ -2 & 7 & 6 \\ 15 & -10 & -11 \end{bmatrix} = \begin{bmatrix} \frac{9}{17} & -\frac{6}{17} & -\frac{10}{17} \\ -\frac{2}{17} & \frac{7}{17} & \frac{6}{17} \\ \frac{15}{17} & -\frac{10}{17} & -\frac{11}{17} \end{bmatrix}.$$

Numerical consideration

Computing $\operatorname{Adj}(A)$ involves the calculation of n^2 determinants of size $(n-1)\times(n-1)$. For $n = 10$, we need one hundred 9×9 determinants. Because of this type of computational intensity, Theorem 6.3.4 is rarely used to find A^{-1}. Row reduction of $[A : I]$ *is the method of choice.*

6.3.2 Cramer's rule

Cramer's rule gives a formula that solves a square consistent linear system in terms of determinants. Gaussian elimination offers only an algorithm to solve linear systems. This formula is named after Gabriel Cramer, who published it in 1750 (Figure 6.10). Colin Maclaurin had published particular cases of the formula in 1748.

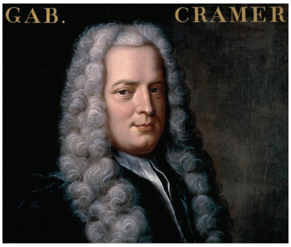

Figure 6.10: Gabriel Cramer. Bibliothéque de Geneve; Fotografie: C. Poite, Public Domain, https://commons.wikimedia.org/w/index.php?curid=7747185. Gabriel Cramer (1704–1752) was born in Switzerland and died in France. He taught mathematics at the Académie de Clavin in Geneva. His most famous work is the book *Introduction à l'analyse des lignes courbes algébraiques*. There he solves a square linear system of five equations by using Cramer's rule.

Let $A\mathbf{x} = \mathbf{b}$ be a square system with

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.$$

Let A_i denote the matrix obtained from A by replacing the ith column with $\mathbf{b}$,

$$A_i = \begin{bmatrix} a_{11} & \cdots & a_{1,i-1} & b_1 & a_{1,i+1} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,i-1} & b_n & a_{n,i+1} & \cdots & a_{nn} \end{bmatrix}.$$

Cramer's rule gives *an explicit formula* for the solution of a consistent square system.

Theorem 6.3.6 (Cramer's rule). *If* $\det(A) \neq 0$, *then the system* $A\mathbf{x} = \mathbf{b}$ *has a unique solution* $\mathbf{x} = (x_1, \ldots, x_n)$ *given by*

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}.$$

Proof. Because $\det(A) \neq 0$, the matrix A is invertible by Theorem 6.2.5. Hence $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$. The inverse A^{-1} can be computed by Theorem 6.3.4:

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\operatorname{Adj}(A)\,\mathbf{b} = \frac{1}{\det(A)}\begin{bmatrix} C_{11}b_1 + C_{21}b_2 + \cdots + C_{n1}b_n \\ \vdots \\ C_{1n}b_1 + C_{2n}b_2 + \cdots + C_{nn}b_n \end{bmatrix}.$$

So the ith component x_i of $\mathbf{x}$ equals the ith component of the right-hand side:

$$x_i = \frac{1}{\det(A)}(C_{1i}b_1 + C_{2i}b_2 + \cdots + C_{ni}b_n).$$

Since A and A_i differ only by the ith column, the cofactors of that column are the same. Hence $C_{1i}b_1 + C_{2i}b_2 + \cdots + C_{ni}b_n$ is $\det(A_i)$ by cofactor expansion about its ith column. Therefore

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad i = 1, \ldots, n. \qquad \square$$

Example 6.3.7. Use Cramer's rule to solve the system

$$\begin{aligned} x_1 + x_2 - x_3 &= 2, \\ x_1 - x_2 + x_3 &= 3, \\ -x_1 + x_2 + x_3 &= 4. \end{aligned}$$

Solution. We compute the determinant of the coefficient matrix A and the determinants of

$$A_1 = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -1 & 1 \\ 4 & 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & 1 \\ -1 & 4 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ -1 & 1 & 4 \end{bmatrix}$$

to get $\det(A) = -4$, $\det(A_1) = -10$, $\det(A_2) = -12$, $\det(A_3) = -14$. Hence

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{5}{2}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = 3, \quad x_3 = \frac{\det(A_3)}{\det(A)} = \frac{7}{2}.$$

Exercises 6.3

In Exercises 1–4, use Theorem 6.3.4 to find the inverses of the given matrices.

1. $\begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$.

2. $\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

4. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}$.

5. Let A be a square matrix with *integer* entries. Prove that
(a) det(A) is an integer;
(b) Adj(A) has integer entries;
(c) If det(A) divides exactly all the entries of Adj(A), then A^{-1} has integer entries;
(d) If det(A) $= \pm 1$, then A^{-1} has integer entries.

In Exercises 6–7, use Cramer's rule to solve the systems.

6.
$$x + y = 1,$$
$$x - y = 1.$$

7.
$$x + y + z = 1,$$
$$x - y + z = 1,$$
$$x + y - z = 1.$$

8. Use Cramer's rule to solve the following system for x and y:

$$\begin{aligned}(\cos\theta)\,x - (\sin\theta)\,y &= 2\cos\theta - 3\sin\theta,\\ (\sin\theta)\,x + (\cos\theta)\,y &= 2\sin\theta + 3\cos\theta.\end{aligned}$$

9. Use Cramer's rule to solve the following system with $ad - bc = -1$:

$$\begin{aligned}ax_1 + bx_2 &= k_1,\\ cx_1 + dx_2 &= k_2.\end{aligned}$$

10. In the following system solve for y only:

$$\begin{aligned}x - y - z &= 0,\\ -x - y + z &= 2,\\ x + y - 2z &= 1.\end{aligned}$$

11. Given that the determinant of the coefficient matrix is nonzero, solve only for x_5 (calculation of determinants can be avoided!):

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 4 & 5 \\ 3 & 3 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

12. Let A be an $n \times n$ matrix. Prove that

$$\det\bigl(\mathrm{Adj}(A)\bigr) = \bigl(\det(A)\bigr)^{n-1}.$$

13. Let A be a 4×4 matrix with $\det(A) = 3$. Find $\det(\mathrm{Adj}(A))$. (Use Exercise 12.)

14. Prove that an $n \times n$ matrix A is invertible if and only if $\mathrm{Adj}(A)$ is invertible. (Use Exercise 12.)

15. For any invertible $n \times n$ matrices A and B, prove that
(a) $\mathrm{Adj}(AB) = \mathrm{Adj}(A)\mathrm{Adj}(B)$;
(b) $\mathrm{Adj}(BAB^{-1}) = B\,\mathrm{Adj}(A)\,B^{-1}$.

16. Let A and B be $n \times n$ matrices with invertible A. Prove that

$$AB = BA \implies \mathrm{Adj}(A)\,B = B\,\mathrm{Adj}(A).$$

6.4 Determinants with permutations

In this section, we study a definition of the determinant by using permutations. Although not computationally efficient, this approach can be used to prove all properties of determinants, including cofactor expansion and Theorem 6.2.1.

6.4.1 Permutations

The cofactor expansion computes a determinant in terms of smaller size *determinants*. We now explore a definition of the determinant, called the *complete expansion*, that is not inductive.[4]

Definition 6.4.1. A *permutation* τ of the set of integers $\{1, 2, \ldots, n\}$ is a rearrangement of these integers. More precisely, a permutation is a *mapping* $\tau : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that any two different numbers from $\{1, 2, \ldots, n\}$ have different images. We write

$$\tau = (j_1, j_2, \ldots, j_n)$$

---

[4] This method of studying determinants is due to Bezout and to Laplace.

to mean that the numbers $1, \dots, n$ map respectively to the numbers $j_1, \dots, j_n$. The permutation $(1, 2, \dots, n)$ is called the *identity* permutation.

For example, by $\tau = (3, 2, 4, 1)$ we denote the permutation τ of $\{1, 2, 3, 4\}$ such that

$$\tau(1) = 3, \quad \tau(2) = 2, \quad \tau(3) = 4, \quad \tau(4) = 1.$$

Example 6.4.2. All permutations of $\{1, 2, 3\}$ are

$$(1,2,3),\ (2,1,3),\ (3,1,2),\ (1,3,2),\ (2,3,1),\ (3,2,1).$$

The number of permutations of $\{1, 2, \dots, n\}$ is $n!$ (pronounced "*n factorial*"). This is the product

$$n! = 1 \cdot 2 \cdot \cdots \cdot n.$$

This can be seen as follows: To fill the first position, there are n choices, because anyone of the numbers can be used. For the second position, there are $n-1$ choices, because one number has been already used in the first position. So to fill the first two positions, there are $n \cdot (n-1)$ choices. This number grows rapidly with n. For example, $11! = 39916800$.

Definition 6.4.3. Let $\tau = (j_1, \dots, j_n)$ be any permutation of $\{1, \dots, n\}$. We say that τ has an *inversion* (j_i, j_k) if a larger integer j_i precedes a smaller one j_k. The permutation τ is called *even* if it has an even total number of inversions. Otherwise, τ is called *odd*. The *sign* of τ, denoted by $\operatorname{sign}(\tau)$, is 1 if τ is even and is -1 if τ is odd:

$$\operatorname{sign}(\tau) = \begin{cases} 1 & \text{if } \tau \text{ is even,} \\ -1 & \text{if } \tau \text{ is odd.} \end{cases}$$

Note that the identity permutation $(1, 2, \dots, n)$ is considered as even with sign 1.

For example, $(1, 3, 2, 4)$ has one inversion $(3, 2)$. So it is odd with sign -1.

The permutation $(4, 2, 1, 3)$ has four inversions $(4, 2)$, $(4, 1)$, $(4, 3)$, $(2, 1)$. So it is even with sign 1.

Example 6.4.4. For the permutations of $\{1, 2, 3\}$, we have

Permutation	**Inversions**	**Even/Odd**	**Sign**
$(1,2,3)$	None	even	1
$(1,3,2)$	$(3,2)$	odd	-1
$(2,1,3)$	$(2,1)$	odd	-1
$(2,3,1)$	$(2,1), (3,1)$	even	1
$(3,1,2)$	$(3,1), (3,2)$	even	1
$(3,2,1)$	$(3,2), (3,1), (2,1)$	odd	-1

If r is the number of inversions of a permutation τ, then $\operatorname{sign}(\tau) = (-1)^r$.

Now we are ready to discuss the complete expansion of the determinant. We do is the following.

1. We form all products each consisting of n entries of A coming from *different* rows and columns. These are called *elementary products* and can be found with the help of permutations.
2. We assign a sign to each elementary product, and we add all signed products.

Let us be more specific. Suppose we want to expand $\det(A)$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

To get all elementary products, we form

$$a_{1\_}\, a_{2\_}\, a_{3\_},$$

and the blanks are filled with the permutations of $\{1, 2, 3\}$. For example, $a_{13}\, a_{21}\, a_{32}$ corresponds to permutation $(3, 1, 2)$. By following this process we ensure that no two entries come from the same row or column. The sign of each elementary term is the sign of the corresponding permutation. The sign of $a_{13}\, a_{21}\, a_{32}$ is 1, because $(3, 1, 2)$ is even. The sign of $a_{12}a_{21}a_{33}$ is -1, since $(2, 1, 3)$ is odd. Using all permutations shown in Example 6.4.2, we get

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

In general, we may *define* an $n \times n$ determinant using permutations as follows.

Definition 6.4.5. If A is an $n \times n$ matrix with entries a_{ij}, then the *complete expansion* of the determinant of A is

$$\det(A) = \sum_{\tau} \operatorname{sign}(\tau)\, a_{1j_1}\, a_{2j_2} \,\dots\, a_{nj_n}, \tag{6.1}$$

where the $n!$-term sum is over all permutations $\tau = (j_1, \dots, j_n)$ of $\{1, \dots, n\}$.

Example 6.4.6. Use a complete expansion to find $\begin{vmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \\ 6 & 0 & 7 \end{vmatrix}$.

Solution. Each elementary product should have no factors from the same column or row. So the only nonzero terms are

$$3 \cdot 5 \cdot 7 \quad \text{and} \quad 4 \cdot 5 \cdot 6,$$

corresponding to $a_{11}\,a_{22}\,a_{33}$ and $a_{13}\,a_{22}\,a_{31}$ and hence to the permutations $(1, 2, 3)$ and $(3, 2, 1)$. The first permutation is even, and the second is odd. So the signs are 1 and −1, respectively. Therefore the determinant is

$$3 \cdot 5 \cdot 7 - 4 \cdot 5 \cdot 6 = 105 - 120 = -15.$$

Example 6.4.7. Use a complete expansion to find the determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 6 & 0 & 7 & 0 \\ 8 & 0 & 0 & 0 & 9 \end{vmatrix}.$$

Solution. A nonzero elementary product can have a factor of 1 or 2 from the first row. If it starts with 1, then the last factor is 9 and not 8, because 1 and 8 are in the same column. Likewise, if a product starts with 2, then the last factor is 8. So we only have products of the form

$$1 \cdot \_ \cdot \_ \cdot \_ \cdot 9 \quad \text{and} \quad 2 \cdot \_ \cdot \_ \cdot \_ \cdot 8.$$

The rest of the factors come from the submatrix

$$\begin{bmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \\ 6 & 0 & 7 \end{bmatrix}$$

used in Example 6.4.6. The possible products here are $3 \cdot 5 \cdot 7$ and $4 \cdot 5 \cdot 6$. Hence we get a total of four nonzero products, namely

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot 9, \quad 1 \cdot 4 \cdot 5 \cdot 6 \cdot 9, \quad 2 \cdot 3 \cdot 5 \cdot 7 \cdot 8, \quad 2 \cdot 4 \cdot 5 \cdot 6 \cdot 8.$$

We have

Elementary Product	Permutation	Even/Odd	Sign	Value
$1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = 945$	$(1, 2, 3, 4, 5)$	even	+	945
$1 \cdot 4 \cdot 5 \cdot 6 \cdot 9 = 1080$	$(1, 4, 3, 2, 5)$	odd	−	−1080
$2 \cdot 3 \cdot 5 \cdot 7 \cdot 8 = 1680$	$(5, 2, 3, 4, 1)$	odd	−	−1680
$2 \cdot 4 \cdot 5 \cdot 6 \cdot 8 = 1920$	$(5, 4, 3, 2, 1)$	even	−	1920

So the determinant is

$$945 - 1080 - 1680 + 1920 = 105.$$

6.4.2 Computational consideration

Computing determinants by using equation (6.1) is not practical. For example, for a 11×11 determinant, we would need 11! terms, each consisting of 11 factors. This requires a total of $11! \cdot 10$ multiplications plus $11! - 1$ additions. This is a total of 439,084,799 operations (Figure 6.11).

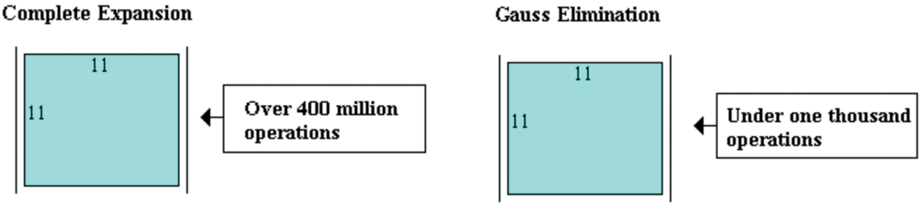

Figure 6.11: Gauss elimination is better than complete expansion!

Gauss elimination on the other hand requires

$$\frac{2n\,(n-1)\,(2n-1)}{6} + \frac{n\,(n-1)}{2} + (n-1)$$

operations for an $n \times n$ determinant. For $n = 11$, this number is only 835.

Exercises 6.4
In Exercises 1–2, determine the sign and classify the permutations as odd and even.

1. $(2,1,3,4)$, $(1,4,2,3)$,
$(1,5,2,4,3)$, $(1,4,3,5,2)$.

2. $(3,1,4,2)$, $(4,2,1,3)$,
$(3,4,2,1,5)$, $(4,2,5,1,3)$.

In Exercises 3–5 compute the determinants of the matrices by using the complete expansion.

3. (a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 0 \end{bmatrix}$

4. (a) $\begin{bmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 0 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & 0 & 0 \end{bmatrix}$.

Permutation matrices

A *permutation matrix* is a square matrix consisting of 1s and 0s such that there is exactly one 1 in each row and each column. Permutation matrices were also studied in Exercises 3.3. The following matrices are permutation matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A permutation matrix gives rise to one and only one permutation as follows: For each row of the matrix, write the column number of the entry with 1. All these numbers form the entries of the corresponding permutation. For example, the permutations corresponding to A, B, C are $(1, 3, 2)$, $(1, 3, 2, 4)$, and $(2, 1, 4, 3, 5)$, respectively.

6. Write all 3×3 permutation matrices. For each such matrix, write its corresponding permutation.

7. Write the permutation matrices corresponding to the permutations

$$p = (4, 1, 2, 3), \quad q = (3, 2, 4, 1).$$

8. Prove that the sign of a permutation equals the determinant of the corresponding permutation matrix.

9. Use Exercise 8 to find the signs of the permutations

$$p = (1, 4, 2, 3), \quad q = (3, 2, 1, 4).$$

In the next two sections, we outline the proofs of Theorem 6.2.1 and the cofactor expansion by using permutations. For this material, A and B are two $n \times n$ matrices with respective entries a_{ij} and b_{ij}.

Proof of Theorem 6.2.1

10. Prove that if B is obtained from A by multiplying one of its rows by a nonzero constant k, then $\det(B) = k\det(A)$.

11. Prove that if we interchange any two *consecutive* entries in a permutation, then the number of inversions increases or decreases by 1.

12. Use Exercise 11 to prove that if we interchange *any* two entries in a permutation, then the number of inversions changes by an *odd* integer.

13. Use Exercise 12 to prove that if we interchange *any* two entries in a permutation, then the new and old permutations have opposite signs.

14. Use Exercise 13 to prove that if B is obtained from A by interchanging any two rows, then $\det(B) = -\det(A)$.

15. Use Exercise 14 to prove that if two rows of A are equal, then $\det(A) = 0$.

16. Use Exercise 15 to prove that if B is obtained from A by adding a multiple of one row to another, then $\det(B) = \det(A)$.

17. Prove that $\det(A) = \det(A^T)$. (*Hint:* $\det(A^T) = \sum \pm a_{j_1 1} \dots a_{j_n n}$. Rearrange $a_{j_1 1} \dots a_{j_n n}$ in the form $a_{1 l_1} \dots a_{n l_n}$ and compare the signs of $(j_1, \dots, j_n)$ and $(l_1, \dots, l_n)$.)

18. Prove Theorem 6.2.1.

Proof of cofactor expansion
Recall that the terms of the sum

$$\det(A) = \sum \pm a_{1j_1} \dots a_{ij_i} \dots a_{nj_n}$$

contain exactly one entry from each row and each column. Hence the factor a_{i1} of the ith row occurs in exactly $(n-1)!$ terms, whereas a_{i2} occurs in $(n-1)!$ terms, distinct from the first ones, and finally a_{in} occurs in $(n-1)!$ terms, distinct from the preceding ones. Since the sum of all these terms is $\det(A)$, we may write

$$\det(A) = a_{i1}D_{i1} + a_{i2}D_{i2} + \cdots + a_{in}D_{in},$$

where, D_{ij} is the sum in $\det(A)$ that is left after we factor a_{ij} out of all terms that contain it. In the next two exercises, we prove that $D_{ij} = C_{ij}$, the (i,j)th cofactor of A, thus proving the cofactor expansion of $\det(A)$ about the ith row:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}. \tag{6.2}$$

Notation. We denote by $A(i,j)$ the matrix obtained from A by deleting the ith row and the jth column.

19. Prove that $D_{11} = C_{11}$. (*Hint:*

$$D_{11} = \sum \pm a_{2j_2} \dots a_{nj_n},$$

where the sum is over all permutations of the form $(j_2, \dots, j_n)$, since $j_1 = 1$. But this is the determinant of $A(1,1)$.)

20. Prove that $D_{ij} = C_{ij}$. (*Hint:* Let A' be the matrix obtained from A by $i-1$ successive interchanges of adjacents rows and $j-1$ successive interchanges of adjacent columns that may bring a_{ij} into the top left position maintaining the relative orders of the other elements. Then $\det(A) = (-1)^{i+j}\det(A')$. Note that $a_{ij} = a'_{11}$ and that $\det(A(i,j)) = \det(A'(1,1))$. Now use Exercise 19.)

21. Prove equation (6.2), which is the cofactor expansion of $\det(A)$ about the ith row.

22. Prove the following formula, which is the cofactor expansion of $\det(A)$ about the jth column:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

(*Hint:* Use Exercises 17 and 21.)

23. If A is a complex square matrix, then prove that $\det(A^H) = \overline{\det(A)}$.

6.5 Applications: Geometry, polynomial systems

We discuss two important mathematical applications of determinants. The first is about algebraic equations of geometric objects. The second is a method of solving nonlinear systems.

6.5.1 Equations of geometric objects

In analytic geometry, determinants play a main role in computations of areas and volumes and also in finding equations of geometric objects, such as straight lines, circles, parabolas, planes, spheres, etc. We see that algebraic equations of geometric objects can be expressed elegantly in terms of determinants.

Example 6.5.1 (Line through two points). Let l be a line in $\mathbf{R}^2$ passing through two given points with coordinates (x_1,y_1) and (x_2,y_2) (Figure 6.12).

(a) Find an equation for l in terms of the points.

(b) Find an equation for the line passing through the points $(1,2)$ and $(-2,0)$.

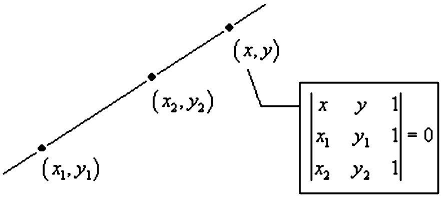
(x,y)
(x_2,y_2)
(x_1,y_1)
$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Figure 6.12: Determinants are used to determine equations of lines.

Solution.

(a) Let $ax + by + c = 0$ be the equation of the line. The points lie on the line, so their coordinates should satisfy this equation. Hence

$$\begin{aligned} ax + by + c &= 0, \\ ax_1 + by_1 + c &= 0, \\ ax_2 + by_2 + c &= 0. \end{aligned}$$

This is a homogeneous system with unknowns a, b, c. It has a nontrivial solution if and only if

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{6.3}$$

(b) Substitution of the points into equation (6.3) yields

$$\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ -2 & 0 & 1 \end{vmatrix} = 0 \Rightarrow 2x - 3y + 4 = 0.$$

So an equation for the line is $2x - 3y + 4 = 0$.

Example 6.5.2 (Circle through three points). Let C be a circle in $\mathbf{R}^2$ passing through three noncolinear points with coordinates (x_1,y_1), (x_2,y_2), and (x_3,y_3).

(a) Find an equation for C in terms of the points. It is sufficient to write a formula in the form of an unexpanded determinant.

(b) Find an equation for C if the points are $(1,4)$, $(3,2)$, and $(-1,2)$.

(c) Find the center and radius of the circle of Part (b).

Solution.

(a) Let $(x-a)^2+(y-b)^2=r^2$ be the equation of the circle of radius r centered at (a,b). This equation expanded can be written in the form $A(x^2+y^2)+Bx+Cy+D=0$. (Note that $A=1$.) If we plug in the points (x_1,y_1), (x_2,y_2), (x_3,y_3), then we get three more equations, and the homogeneous system with unknowns A, B, C, D has nontrivial solutions (since $A=1$) if and only if the coefficient determinant is zero (Figure 6.13),

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{6.4}$$

We expand the determinant to get the equation of the circle.

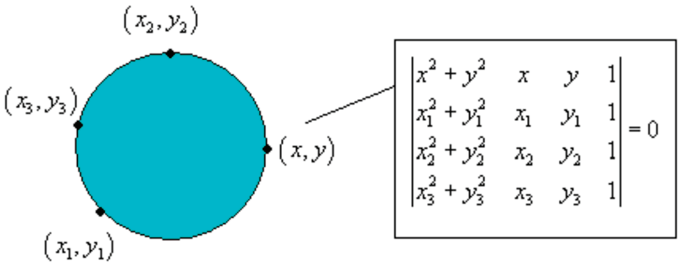
(x_2,y_2)
(x_3,y_3)
(x,y)
(x_1,y_1)
$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$

Figure 6.13: Determinants are used to determine equations of circles.

(b) Substitution of the points into equation (6.4) yields

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 17 & 1 & 4 & 1 \\ 13 & 3 & 2 & 1 \\ 5 & -1 & 2 & 1 \end{vmatrix} = 0 \Rightarrow -8x^2-8y^2+32y+16x-32=0.$$

(c) Division by -8 and completion of the two squares gives

$$(x-1)^2+(y-2)^2=4.$$

Therefore we have a circle of radius 4 with center at $(1,2)$.

Example 6.5.3 (Plane through three points). Let $\mathcal{P}$ be a plane in $\mathbf{R}^3$ passing through three noncolinear points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3).

(a) Find an equation for $\mathcal{P}$ in terms of the points.

(b) Find an equation for $\mathcal{P}$ if the points are $(1, 1, 7)$, $(3, 2, 6)$, and $(-2, -2, 4)$.

Solution.

(a) Let $ax + by + cz + d = 0$ be the equation of the plane. After substitution of the three points and seeking nontrivial solutions of the homogeneous system in unknowns a, b, c, d, we get the following equation (Figure 6.14):

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \tag{6.5}$$

(b) Substitution of the points into equation (6.5) and expansion yield the equation $2x - 3y + z - 6 = 0$.

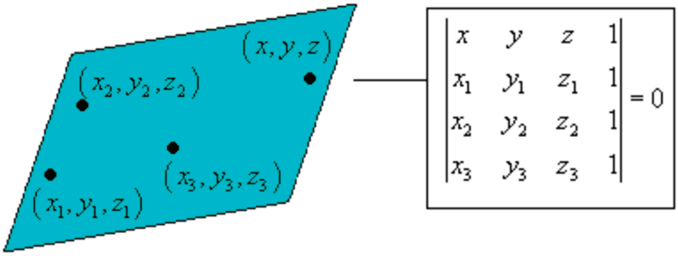
(x, y, z)
(x_2, y_2, z_2)
(x_3, y_3, z_3)
(x_1, y_1, z_1)
$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

Figure 6.14: Determinants are used to determine equations of planes.

6.5.2 Elimination theory, resultants, and polynomial systems

Elimination theory is about eliminating a number of unknowns out of a system of polynomial equations. Usually, the goal is to find the common solutions of these equations.[5] A polynomial system is nonlinear if it has polynomials of degree higher than one. Nonlinear polynomial systems appear very often in practice. Unfortunately, they are usually hard to solve. Some solution methods of such systems use determinants. Two early contributors to solutions of nonlinear systems were *Leonard Euler* (Figure 6.17) and *James Joseph Sylvester* (Figure 6.15).

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[5] Elimination theory, studied early on by Euler and Bezout, flourished between 1850 and 1920 with Sylvester, Cayley, Dixon, and Macaulay. Then it went out of fashion, until recently. Now there is much renewed interest, partly due to the existence of symbolic mathematical software. It is used to solve polynomial systems. See S. S. Abyankar's 1976 paper titled "Historical ramblings in algebraic geometry and related algebra", American Mathematical Monthly **83**(6), pp. 409–448.

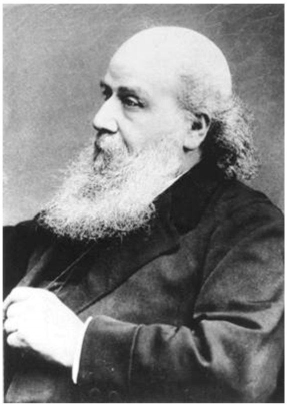

Figure 6.15: James Joseph Sylvester.
By from: http://en.wikipedia.org/wiki/Image:Untitled04.jpg, Public Domain, https://commons.wikimedia.org/w/index.php?curid=268041.
James Joseph Sylvester (1814–1897) was a British mathematician known for his contributions to algebra and combinatorics. He studied law and mathematics. He introduced the term "matrix" and made significant advancements in the theory of invariants. Sylvester had a successful academic career in both England and the United States and played a vital role in the development of modern mathematics.

Suppose for simplicity that we have a system of two general quadratics in one variable x,

$$\begin{aligned} a_1x^2 + b_1x + c_1 &= 0, \\ a_2x^2 + b_2x + c_2 &= 0. \end{aligned}$$

Let $p_1 = a_1x^2 + b_1x + c_1$ and $p_2 = a_2x^2 + b_2x + c_2$. We find a necessary and sufficient condition for the existence of a common solution. If p_1 and p_2 have a common root, then they must have a common linear factor, say, Q. Let $q_1 = p_1/Q$ and $q_2 = p_2/Q$ be the two linear quotients, and let $q_1 = A_1x + B_1$ and $q_2 = -A_2x - B_2$ (the minus signs in q_2 yield a more symmetric formula). Then $Q = p_1/q_1 = p_2/q_2$ or $p_1q_2 = p_2q_1$. Explicitly, we have

$$(a_1x^2 + b_1x + c_1)(-A_2x - B_2) = (a_2x^2 + b_2x + c_2)(A_1x + B_1).$$

Expansion and collection of terms in powers of x give

$$\begin{aligned} &(a_1A_2 + a_2A_1)\,x^3 + (b_2A_1 + b_1A_2 + a_1B_2 + a_2B_1)\,x^2 \\ &\quad + (c_1A_2 + c_2A_1 + b_2B_1 + b_1B_2)\,x + c_1B_2 + c_2B_1 = 0. \end{aligned}$$

Because this polynomial equation is valid for all x, the coefficients of x^3, x^2, x^1, x^0 should be zero. Therefore

$$\begin{aligned} &a_1\,A_2 + a_2\,A_1 = 0, \\ &b_2\,A_1 + b_1\,A_2 + a_1\,B_2 + a_2\,B_1 = 0, \\ &c_1\,A_2 + c_2\,A_1 + b_2\,B_1 + b_1\,B_2 = 0, \\ &c_1\,B_2 + c_2\,B_1 = 0. \end{aligned}$$

This is a homogeneous system in A_2, B_2, A_1, B_1. The system has nontrivial solutions if and only if the coefficient determinant is zero. So

$$\begin{vmatrix} a_1 & 0 & a_2 & 0 \\ b_1 & a_1 & b_2 & a_2 \\ c_1 & b_1 & c_2 & b_2 \\ 0 & c_1 & 0 & c_2 \end{vmatrix} = 0.$$

This determinant is called the *Sylvester resultant*[6] of p_1 and p_2. It has size four. Its columns consist of the coefficients of the two polynomials padded by zeros. In general, the Sylvester resultant of two polynomials of degrees m and n is formed similarly and has size $m + n$. For example, consider the system

$$\begin{aligned} a_1x^2 + b_1x + c_1 &= 0, \\ a_2x^3 + b_2x^2 + c_2x + d_2 &= 0. \end{aligned}$$

Then the Sylvester resultant of this system is of size $2 + 3 = 5$ and is given by

$$\begin{vmatrix} a_1 & 0 & 0 & a_2 & 0 \\ b_1 & a_1 & 0 & b_2 & a_2 \\ c_1 & b_1 & a_1 & c_2 & b_2 \\ 0 & c_1 & b_1 & d_2 & c_2 \\ 0 & 0 & c_1 & 0 & d_2 \end{vmatrix}.$$

This determinant is zero if and only if the system has a common solution.

Theorem 6.5.4 (Vanishing of the Sylvester resultant). *Let f and g be two polynomials in x. The system $f = 0, g = 0$ has a solution if and only if the Sylvester resultant is zero, provided that not both coefficients of the highest powers of x are zero.*

Example 6.5.5. Without solving the equations, show that the following system has a solution:

$$\begin{aligned} x^2 - 5x + 6 &= 0, \\ x^2 + 2x - 8 &= 0. \end{aligned}$$

Solution. The Sylvester resultant of the system is

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ -5 & 1 & 2 & 1 \\ 6 & -5 & -8 & 2 \\ 0 & 6 & 0 & -8 \end{vmatrix} = 0.$$

Hence the system has a common root by Theorem 6.5.4.

---

[6] Originally due to Euler. The current formulation is due to Sylvester.

Example 6.5.5 only served as an illustration of the Sylvester resultant method. In this case, we can just solve each quadratic. In general, this fails, so we use the Sylvester resultant. It is quite powerful. It applies even when the coefficients are polynomials in another variable. This allows us to solve some multivariate polynomial systems.

Example 6.5.6. Solve the system

$$\begin{aligned} &x^2 + y^2 - 1 = 0, \\ &x^2 - 2x + y^2 - 2y + 1 = 0. \end{aligned}$$

Solution. Let us view this system as a system in y with coefficients polynomials in x:

$$\begin{aligned} &y^2 + (x^2 - 1) = 0, \\ &y^2 - 2y + (x^2 - 2x + 1) = 0. \end{aligned}$$

By Theorem 6.5.4, if there is a solution, then the Sylvester resultant is zero:

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ x^2 - 1 & 0 & x^2 - 2x + 1 & -2 \\ 0 & x^2 - 1 & 0 & x^2 - 2x + 1 \end{vmatrix} = 0.$$

Expansion of the determinant and simplification yields

$$8x^2 - 8x = 0.$$

Therefore $x = 0, 1$. If $x = 0$, then the first equation implies $y = -1, 1$. However, substitution into the second equation implies that y can only be 1. Likewise, substitution $x = 1$ into the system implies that $y = 0$. We conclude that there are two common solutions $(1, 0)$ and $(0, 1)$.

Note that the system in Example 6.5.6 can be rewritten as $x^2 + y^2 = 1$ and $(x - 1)^2 + (y - 1)^2 = 1$. These are the equations of the circles of radius 1 centered at $(0, 0)$ and $(1, 1)$, respectively. Their intersection is the solution, and it is geometrically obvious in this case (Figure 6.16).

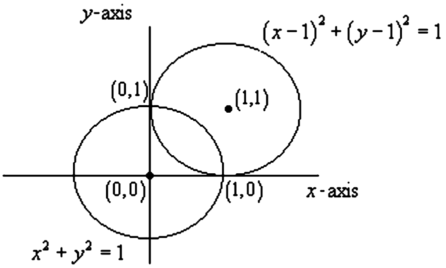

Figure 6.16: The solutions are the intersections of two circles.

Figure 6.17: Leonhard Euler.
By Jakob Emanuel Handmann, Public Domain,
https://commons.wikimedia.org/w/index.php?curid=1001511.
Leonhard Euler (1707–1783) was born in Basel, Switzerland, and died in St. Petersburg, Russia. He was one of the greatest and most prolific mathematicians of all time. His published works exceed 80 volumes. During the last seventeen years of his life, he produced over half of his work, although totally blind. He made fundamental contributions to all branches of mathematics, to acoustics, cartography, education, magnetism, rational and celestial mechanics, astronomy, navigation, theory of music, etc.

Exercises 6.5

In Exercises 1–2, find the equation of the line passing through P and Q.

1. $P(-1,2)$ and $Q(1,1)$.

2. $P(-3,-5)$ and $Q(4,7)$.

In Exercises 3–4, determine whether the points P, Q, and R are on the same line.

3. $P(-2,0)$, $Q(0,1)$, and $R(2,1)$.

4. $P(-1,2)$, $Q(0,0)$, and $R(2,-1)$.

In Exercises 5–6, find the equation, center, and radius of the circle passing through P, Q, and R.

5. $P(0,0)$, $Q(-1,-1)$, and $R(0,-2)$.

6. $P(7,7)$, $Q(1,1)$, and $R(-3,2)$.

In Exercises 7–8, use determinants to find the equation of the parabola of the form $y = ax^2 + bx + c$ passing through P, Q, and R.

7. $P(0,4)$, $Q(1,3)$, and $R(-1,9)$.

8. $P(2,2)$, $Q(3,1)$, and $R(1,7)$.

In Exercises 9–10, find the equation of the plane passing through P, Q, and R.

9. $P(1,1,1)$, $Q(0,-1,1)$, $R(4,3,-1)$.

10. $P(5,4,3)$, $Q(-1,2,2)$, $R(4,4,4)$.

11. Prove that an equation of the sphere through four noncoplanar points $P_i(x_i, y_i, z_i)$, $i = 1, \dots, 4$, is given by

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

12. Use Exercise 11 to find the equation of the sphere passing through the points $(1,2,7)$, $(5,2,3)$, $(1,6,3)$, $(1,2,-1)$.

13. (*Cayley–Menger determinant*) Let $P_i(x_i, y_i)$, $i = 1,\dots,4$, be any four points of $\mathbf{R}^2$. Let d_{ij} be the square distance from P_i to P_j. So $d_{ij} = (x_i - x_j)^2 + (y_i - y_j)^2$. A famous relation involving the square distances is that the following Cayley–Menger determinant is zero:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12} & d_{13} & d_{14} \\ 1 & d_{21} & 0 & d_{23} & d_{24} \\ 1 & d_{31} & d_{32} & 0 & d_{34} \\ 1 & d_{41} & d_{42} & d_{43} & 0 \end{vmatrix} = 0.$$

This is type of relation is normally hard to prove. However, there is an unexpectedly easy proof. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1^2 + y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2 + y_2^2 & 2x_2 & 2y_2 & 1 \\ x_3^2 + y_3^2 & 2x_3 & 2y_3 & 1 \\ x_4^2 + y_4^2 & 2x_4 & 2y_4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & -x_1 & -y_1 & x_1^2 + y_1^2 \\ 1 & -x_2 & -y_2 & x_2^2 + y_2^2 \\ 1 & -x_3 & -y_3 & x_3^2 + y_3^2 \\ 1 & -x_4 & -y_4 & x_4^2 + y_4^2 \end{bmatrix}.$$

If C is the matrix of the Cayley–Menger determinant, first, prove that

$$C = AB^T,$$

then argue that C cannot be invertible, because A and B have rank at most 4.

14. Use the Sylvester resultant to solve the system

$$\begin{aligned} x^2 + y^2 - 1 &= 0, \\ x^2 + 2x + y^2 - 2y + 1 &= 0. \end{aligned}$$

15. Use the Sylvester resultant to solve the system

$$\begin{aligned} x^2 + y^2 - 1 &= 0, \\ x^2 - 2x + y^2 - 1 &= 0. \end{aligned}$$

16. Use the Sylvester resultant to find only the real roots of the system

$$\begin{aligned} x^2 - 2x + y^2 - 1 &= 0, \\ x^3 + y^2 - 1 &= 0. \end{aligned}$$

6.6 Special topic: Image recognition

In this essay, we examine a recent advance in image recognition. The material is within the bounds of current knowledge, but the details may be challenging. Still, it is worth reading about a real-life application of linear algebra.

First, we need some background in projective geometry. Projective geometry grew out of the needs of perspective projections in painting during the Renaissance era.

6.6.1 Introduction to projective geometry

Euclidean geometry describes objects as they are. Rigid motions do not change lengths, angles, or parallelism. In contrast, projective geometry describes objects as they appear to the human eye or to a camera. In particular, lengths and angles get distorted when we look at objects or take a picture. The image as it appears is called *perspective*.

A characteristic of perspective images is that parallel lines appear to intersect at a seemingly distant point, called a *point at infinity* (Figure 6.18).

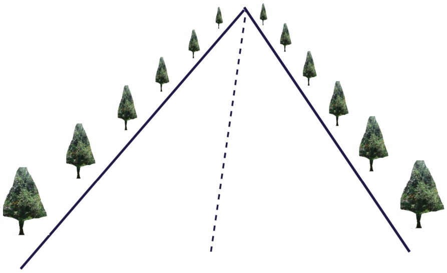

Figure 6.18: Perspective images of parallel lines intersect at a point at infinity.

The projective plane

Let us describe now a model for projective geometry in the plane. We consider a plane $\mathcal{P}$, and we want to examine points and lines on $\mathcal{P}$.

To study perspective images, we add one more dimension as follows: We choose a three-dimensional rectangular coordinate system $x_1x_2x_3$ and place $\mathcal{P}$ at distance 1 unit above the x_1x_2-plane. So $\mathcal{P}$ has equation $x_3 = 1$. We identify each point P of $\mathcal{P}$ with the line (or *ray*) through the origin and P. If (x_1, x_2, x_3) is a nonzero vector on this line, then all points $(\lambda x_1, \lambda x_2, \lambda x_3)$, $\lambda \neq 0$, of the line are identified with P. So if P originally had coordinates (x_1, x_2) in $\mathcal{P}$, then it now has coordinates $(x_1, x_2, 1)$ in this model or, more generally, coordinates $(\lambda x_1, \lambda x_2, \lambda)$ with $\lambda \neq 0$ (Figure 6.19).

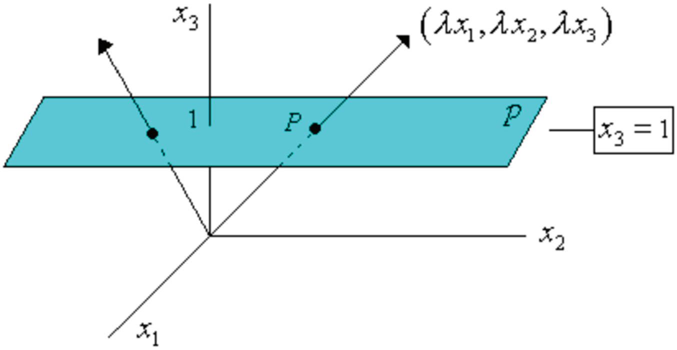

Figure 6.19: A model for the projective plane.

Given the correspondence between rays through the origin and points of $\mathcal{P}$, it is clear how to describe straight lines of $\mathcal{P}$. Imagine a line l in $\mathcal{P}$ and a moving point in it. Since each point corresponds to a ray, as the point on l scans l, the corresponding ray scans a *plane* through the origin that contains l. So we can completely describe l by the plane through the origin that contains this line. Any such plane is uniquely determined by a normal vector. So a line l can be described by a nonzero vector (a, b, c) (Figure 6.20).

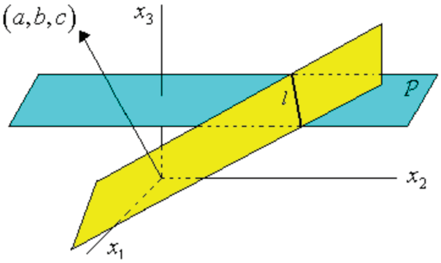

Figure 6.20: Lines in $\mathcal{P}$ are viewed as planes through **0** or as normals to such planes.

In this setting, we have:

1. A point $P(x_1, x_2, x_3)$ is on the line (a, b, c) if

$$ax_1 + bx_2 + cx_3 = 0, \tag{6.6}$$

   because (a, b, c) is a normal to the plane determined by the line.

2. Two points $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$ define a line $\mathbf{a} = (a, b, c)$ by

$$\mathbf{a} = \mathbf{p} \times \mathbf{q}, \tag{6.7}$$

   because the cross-product is normal to the plane containing $\mathbf{p}$ and $\mathbf{q}$.

3. Two lines $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ intersect at the point $\mathbf{p}$ with

$$\mathbf{p} = \mathbf{a} \times \mathbf{b}. \tag{6.8}$$

Examination of equations (6.6), (6.7), and (6.8) shows that there is complete symmetry between point coordinates and line coordinates. In fact, algebraically, we cannot distinguish between lines and points!

The points $(x_1, x_2, 0)$ on the x_1x_2-plane correspond to rays that do not intersect the plane $\mathcal{P}$. These points are called *ideal points* or *points at infinity*. All points at infinity define the line corresponding to the plane $x_3 = 0$. This line is completely determined by the normal $(0, 0, 1)$ to the plane and is called *the line at infinity* (Figure 6.21).

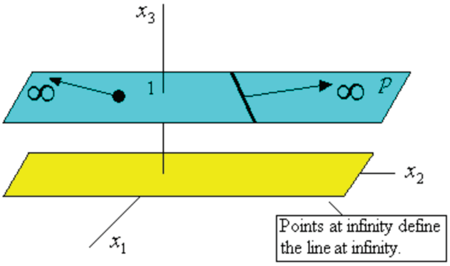

Figure 6.21: Points and line at infinity.

Now taking all the points of the plane $\mathcal{P}$ plus all points at infinity, we get the two-dimensional (real) *projective plane* $\mathbf{P}^2$. It consists of all nonzero 3-vectors with the identification that two vectors $\mathbf{a}$ and $\mathbf{b}$ are equal if they define the same line through the origin or, in other words, if $\mathbf{a} = c\mathbf{b}$ for some nonzero scalar c. In the language of linear algebra, $\mathbf{P}^2$ is the set of one-dimensional subspaces of $\mathbf{R}^3$. We use the notation $(x_1 : x_2 : x_3)$ for the points of $\mathbf{P}^2$, and for simplicity, we even use the vector notation keeping in mind that not all x_1, x_2, x_3 are zero and that $\mathbf{x}$ is determined up to nonzero scalar multiple. The coordinates $(x_1 : x_2 : x_3)$ are called *homogeneous coordinates*.

Our original plane $\mathcal{P}$ can be viewed as a copy of $\mathbf{R}^2$ embedded in $\mathbf{P}^2$. It consists of all the points $(x_1 : x_2 : x_3)$ with $x_3 \neq 0$. Equivalently, it consists of the points of the form $(x_1/x_3 : x_2/x_3 : 1) = (y_1 : y_2 : 1)$.

The projective space

Projective geometry can be extended to three dimensions and in fact to any number of dimensions. In three dimensions, we define the projective space $\mathbf{P}^3$ as the set of points with homogeneous coordinates $(x_1 : x_2 : x_3 : x_4)$, where the x_is define a nonzero vector, and two points $\mathbf{x} = (x_1 : x_2 : x_3 : x_4)$ and $\mathbf{y} = (y_1 : y_2 : y_3 : y_4)$ are equal if the 4-vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is a nonzero scalar multiple of the 4-vector $\mathbf{y} = (y_1, y_2, y_3, y_4)$. So, in the language of linear algebra, $\mathbf{P}^3$ is the set of one-dimensional subspaces of $\mathbf{R}^4$.

A copy of $\mathbf{R}^3$ is embedded in $\mathbf{P}^3$ as the set of points $(x_1 : x_2 : x_3 : x_4)$ with $x_4 \neq 0$. Equivalently, it consists of the points of the form $(x_1 : x_2 : x_3 : 1)$. The points of the form $(x_1 : x_2 : x_3 : 0)$ are called *ideal points* or the *points at infinity*.

A straight line in $\mathbf{P}^3$ is defined as the span of two distinct points. A plane in $\mathbf{P}^3$ is the set of points $(x_1 : x_2 : x_3 : x_4)$ defined by a hyperplane

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0. \tag{6.9}$$

A plane is completely determined by one of the normals (a_1, a_2, a_3, a_4) or, more precisely, of $(a_1 : a_2 : a_3 : a_4)$ of the defining hyperplane.

6.6.2 Projective transformations

In Euclidean geometry, we are interested in transformations that preserve lengths of vectors and angles between vectors. In projective geometry, we are interested in transformations that preserve *collineation*. This means that points on the same line are mapped to points on the same line. Such transformations are called *projective transformations* (Figure 6.22). They are also called *projectivities*, or *collineations*, or *homographies*.

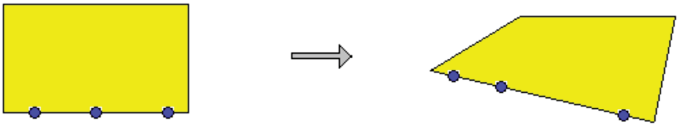

Figure 6.22: Projective transformations send collinear points to collinear points.

The following fact tells us exactly which transformations are projective.

> A mapping $\mathbf{y} = T(\mathbf{x})$ is a projective transformation if and only if it is a linear transformation of homogeneous coordinates for an *invertible* matrix A, i. e.,
>
> $$\mathbf{y} = T(\mathbf{x}) = A\mathbf{x}.$$

This fact applies equally well to mappings from $\mathbf{P}^2$ to $\mathbf{P}^2$ and also to mappings from $\mathbf{P}^3$ to $\mathbf{P}^3$. In the case of $\mathbf{P}^2$ the matrix A has size 3×3 and is determined up to nonzero scalar multiple. So it can be defined by $3^2 - 1 = 8$ entries. In the case of $\mathbf{P}^3$, A is 4×4 and requires the specification of 15 entries.

Note that a projective transformation $\mathbf{y} = T(\mathbf{x})$ is *invertible*. Its inverse T^{-1} has the property that in Cartesian coordinates, $T \circ T^{-1}(\mathbf{x}) = \lambda\mathbf{x}$ and $T^{-1} \circ T(\mathbf{x}) = \mu\mathbf{x}$ for $\lambda, \mu \neq 0$. In fact, if $T(\mathbf{x}) = A\mathbf{x}$ and $T^{-1}(\mathbf{x}) = B\mathbf{x}$, then AB is a nonzero scalar product of the identity, $AB = \lambda I$. If we work in homogeneous coordinates, we may simply let $B = A^{-1}$.

6.6.3 Projective invariants

Projective transformations are ideal for studying plane or space images as they appear to the eye or to a pinhole camera. They describe the distortions of objects as the perspective view changes. We consider two projective objects to be equivalent if there is a projective transformation that takes one to the other.

In image recognition, we are interested in recognizing an object independently of its position and a possible perspective distortion. We study features (usually numbers) associated with objects that are independent of a projective transformation distortion. These features are called *projective invariants* and are of great interest in projective geometry and its applications.

Suppose that with a pinhole camera we take a two-dimensional picture of a three-dimensional object as shown in Figure 6.23. If we consider the object to be in $\mathbf{P}^3$ and the image in $\mathbf{P}^2$, then the relation between corresponding points $\mathbf{p}$ on the object and points $\mathbf{q}$ in the image is given by

$$\mathbf{q} = C\mathbf{p},$$

where C is a 3×4 matrix of rank 3, called the *(generalized) camera matrix*. The camera matrix depends on physical characteristics of the camera such as the focal length. We say that a camera is *calibrated* if C is known. In practice the calibration of a camera can be a difficult problem.

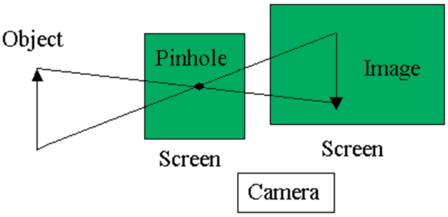

Figure 6.23: A pinhole camera image.

A basic question in image recognition is the following: *If we take a picture of an object, then how can we identify this object?* Usually, we are given a large database of main features of images of known objects. Here by “main features” we mean a set of distinct points that identify key elements of the object.

In the following, we show that there are polynomial relations between projective invariants of the object points and the image points, so that when satisfied, then very likely we have a match between the photographed object and a database image. If they are not satisfied, then for sure we do not have a match. These relations are called the *object-image equations*. It takes at least six points to get meaningful object-image equations (Figure 6.24).

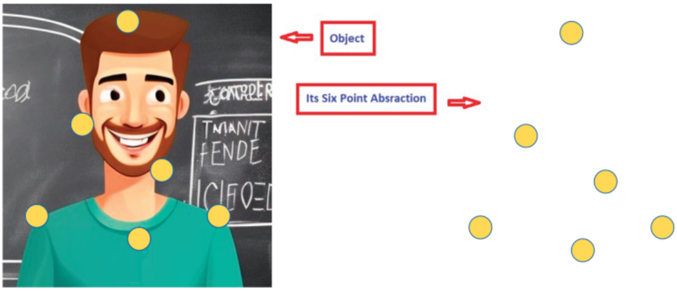

Figure 6.24: Six point image abstraction. (Image generated by **openart.ai**.)

We select six points $P_1, \dots, P_6$ on the three-dimensional object. Let $Q_1, \dots, Q_6$ be their images in the photograph. We assume that the points P_i are in *general position*, meaning that every four of them have linearly independent coordinate vectors or, equivalently, that every four of them are not coplanar in $\mathbf{P}^3$. Now since a projective transformation distorts the appearance of an object without really changing it, we may transform the P_is to points with “easy” coordinates. The points with easy coordinates we choose are the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_4$ in $\mathbf{R}^4$ and the vector $\mathbf{e} = (1, 1, 1, 1)$.

There is a unique space projective transformation T that maps the first five Ps to $\mathbf{e}$s as follows:

$$T(P_1) = \mathbf{e}_1, \quad T(P_2) = \mathbf{e}_2, \quad T(P_3) = \mathbf{e}_3, \quad T(P_4) = \mathbf{e}_4, \quad T(P_5) = \mathbf{e}.$$

In fact, we can find a formula for such a transformation by using Cramer’s rule. We let the reader verify the following formula in homogeneous coordinates:

$$T(\mathbf{x}) = \left(\frac{|\mathbf{x}\,\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4|}{|\mathbf{x}_5\mathbf{x}_2\mathbf{x}_3\mathbf{x}_4|}, \frac{|\mathbf{x}_1\,\mathbf{x}\,\mathbf{x}_3\mathbf{x}_4|}{|\mathbf{x}_1\mathbf{x}_5\mathbf{x}_3\mathbf{x}_4|}, \frac{|\mathbf{x}_1\mathbf{x}_2\,\mathbf{x}\,\mathbf{x}_4|}{|\mathbf{x}_1\mathbf{x}_2\mathbf{x}_5\mathbf{x}_4|}, \frac{|\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\,\mathbf{x}|}{|\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_5|} \right).$$

Now let the image of P_6 under T be $T(P_6) = T(\mathbf{x}_6) = (p_1, p_2, p_3, 1)$ in homogeneous coordinates. We claim that the numbers p_1, p_2, p_3 are *projective invariants* for the six points $P_1, \dots, P_6$ in $\mathbf{P}^3$. In other words, the numbers p_1, p_2, p_3 remain the same if the six points P_i undergo a projective transformation. Indeed, let P_i' be the image of P_i under some projectivity M, and let p_1', p_2', p_3' be the numbers obtained from the image $T(P_6')$. Then the transformation TMT^{-1} leaves $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, $\mathbf{e}$ fixed up to scalar multiple, and therefore it is of the form λI_4 for some nonzero scalar λ. We conclude that $p_i = p_i'$, $i = 1, \dots, 6$.

For the images Q_i of P_i, we have a similar situation in $\mathbf{P}^2$. If $\mathbf{u}_i$ are the coordinate vectors of Q_i and if $\mathbf{u}$ is any vector in $\mathbf{P}^2$, then the projective transformation

$$L(\mathbf{u}) = \left(\frac{|\mathbf{u}\,\mathbf{u}_2\mathbf{u}_3|}{|\mathbf{u}_4\mathbf{u}_2\mathbf{u}_3|}, \frac{|\mathbf{u}_1\,\mathbf{u}\,\mathbf{u}_3|}{|\mathbf{u}_1\mathbf{u}_4\mathbf{u}_3|}, \frac{|\mathbf{u}_1\mathbf{u}_2\,\mathbf{u}|}{|\mathbf{u}_1\mathbf{u}_2\mathbf{u}_4|} \right)$$

maps Q_i to $\mathbf{e}_i$ for $i = 1,2,3$ and Q_4 to $\mathbf{e} = (1,1,1)$. Let the images of Q_5 and Q_6 under L be $L(Q_5) = L(\mathbf{u}_5) = (q_1, q_2, 1)$ and $L(Q_6) = L(\mathbf{u}_6) = (q_3, q_4, 1)$. Then just as in the three-dimensional case, the numbers q_1, q_2, q_3, q_4 are projective invariants of six points in $\mathbf{P}^2$.

6.6.4 The object-image equations

We now look for a relation between the invariants p_i and q_i. Since we have replaced the points P_i with $\mathbf{e}_i$ ($i = 1, \ldots, 4$) and $\mathbf{e}$ and the images Q_i with $\mathbf{e}_i$ ($i = 1, \ldots, 4$) and $\mathbf{e}$, we may assume that the camera matrix $C = [c_{ij}]$ maps points as follows.

$$C\mathbf{e}_1 = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C\mathbf{e}_2 = \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C\mathbf{e}_3 = \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} = \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C\mathbf{e}_4 = \begin{bmatrix} c_{14} \\ c_{24} \\ c_{34} \end{bmatrix} = \lambda_4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$C\mathbf{e} = \begin{bmatrix} c_{11} + c_{12} + c_{13} + c_{14} \\ c_{21} + c_{22} + c_{23} + c_{24} \\ c_{31} + c_{32} + c_{33} + c_{34} \end{bmatrix} = \lambda_5 \begin{bmatrix} q_1 \\ q_2 \\ 1 \end{bmatrix},$$

$$C \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{bmatrix} = \begin{bmatrix} p_1c_{11} + p_2c_{12} + p_3c_{13} + c_{14} \\ p_1c_{21} + p_2c_{22} + p_3c_{23} + c_{24} \\ p_1c_{31} + p_2c_{32} + p_3c_{33} + c_{34} \end{bmatrix} = \lambda_6 \begin{bmatrix} q_3 \\ q_4 \\ 1 \end{bmatrix}.$$

The first four equations imply that c_{12}, c_{21}, c_{13}, c_{31}, c_{23}, c_{32} are 0, $c_{14} = c_{24} = c_{34} = \lambda_4$, $c_{11} = \lambda_1$, $c_{22} = \lambda_2$, and $c_{33} = \lambda_3$. The last two equations yield the following homogeneous system in λ_i:

$$\begin{gathered} \lambda_1 + \lambda_4 - \lambda_5 q_1 = 0, \quad \lambda_2 + \lambda_4 - \lambda_5 q_1 = 0, \quad \lambda_3 + \lambda_4 - \lambda_5 = 0, \\ \lambda_1 p_1 + \lambda_4 - \lambda_6 q_3 = 0, \quad \lambda_2 p_2 + \lambda_4 - \lambda_6 q_4 = 0, \quad \lambda_3 p_3 + \lambda_4 - \lambda_6 = 0. \end{gathered}$$

The system has nontrivial solutions if and only if the coefficient determinant is zero,

$$\begin{vmatrix} 1 & 0 & 0 & 1 & -q_1 & 0 \\ 0 & 1 & 0 & 1 & -q_2 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ p_1 & 0 & 0 & 1 & 0 & -q_3 \\ 0 & p_2 & 0 & 1 & 0 & -q_4 \\ 0 & 0 & p_3 & 1 & 0 & -1 \end{vmatrix} = 0.$$

Expansion yields the useful *object-image equations*

$$\begin{gathered}-q_2p_2 + q_4p_3 + p_1q_2p_2 - p_1q_4p_3 + q_1p_1 - q_3p_3 - p_2q_1p_1 + \\ p_2q_3p_3 - q_1p_1q_4 + q_3q_2p_2 + p_3q_1p_1q_4 - p_3q_3q_2p_2 = 0,\end{gathered}$$

which should be satisfied, if there is a match between the object and the candidate for a true image.

More information on this interesting application can be found in [19] and [20].

6.7 Miniprojects

6.7.1 Vandermonde determinants

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{bmatrix}.$$

Notice that the entries in each column are the powers $2^0 = 1, 2^1 = 2, 2^2 = 4$ for the first column, $3^0 = 1, 3^1 = 3, 3^2 = 9$ for the second column, and $5^0 = 1, 5^1 = 5, 5^2 = 25$ for the third column. A matrix with this property is called a Vandermonde matrix.

Definition 6.7.1. An $n \times n$ matrix A_n is a *Vandermonde matrix* if there are numbers $x_1, x_2, \dots, x_n$ such that

$$A_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}.$$

There is a simple formula for the determinant of a Vandermonde matrix.

Theorem 6.7.2 (Vandermonde's determinant). *The determinant V_n of the Vandermonde matrix A_n is given by*

$$V_n = \det(A_n) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

In other words,

$$\begin{aligned}V_n = (x_n - x_{n-1})(x_n - x_{n-2})\cdots(x_n - x_2)(x_n - x_1)\times\\(x_{n-1} - x_{n-2})(x_{n-1} - x_{n-3})\cdots(x_{n-1} - x_2)(x_{n-1} - x_1)\times\\\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\(x_4 - x_3)(x_4 - x_2)(x_4 - x_1)\times\\(x_3 - x_2)(x_3 - x_1)\times\\(x_2 - x_1).\end{aligned}$$

Problem A.

(a) Verify Theorem 6.7.2 for the following Vandermonde matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 9 & 25 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{bmatrix}.$$

(b) Use Theorem 6.7.2 to compute the determinant of the following Vandermonde matrices:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 10 & 11 & 12 \\ 100 & 121 & 144 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{bmatrix}.$$

(c) Use Theorem 6.7.2 to compute the determinant of the following matrices:

$$A = \begin{bmatrix} 1 & 5 & 25 \\ 1 & 9 & 81 \\ 1 & 12 & 144 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 5 & 25 & 125 \\ 1 & 7 & 49 & 343 \end{bmatrix}.$$

Problem B. Find a necessary and sufficient condition that a Vandermonde matrix has determinant zero.

Problem C. Prove Theorem 6.7.2.

(*Hints:*

(a) Perform the operations

$$\begin{aligned}R_n - x_n R_{n-1} &\to R_n,\\R_{n-1} - x_n R_{n-2} &\to R_{n-1},\\R_{n-2} - x_n R_{n-3} &\to R_{n-2},\\&\vdots\\R_2 - x_n R_1 &\to R_2\end{aligned}$$

to get a matrix whose ith column is

$$\begin{bmatrix} 1 \\ x_i - x_n \\ x_i(x_i - x_n) \\ x_i^2(x_i - x_n) \\ \vdots \\ x_i^{n-2}(x_i - x_n) \end{bmatrix} \quad \text{if} \quad i < n \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{if} \quad i = n.$$

(b) Expand the resulting determinant about the nth column to obtain the $(1, n)$th cofactor $C_{1,n} = (-1)^n M_{1,n}$.

(c) Use Theorem 6.2.1, Section 6.2, to factor the product

$$(x_1 - x_n)(x_2 - x_n) \cdots (x_{n-1} - x_n)$$

out of the minor $M_{1,n}$. The leftover determinant is just V_{n-1}. Continue this process.)

6.7.2 Bezout resultant

This project is a quick introduction to the increasingly important resultant of Bezout[7] (1774) of a system of two polynomials in one variable. We discuss *Cayley's statement* of the Bezout method.

Let $f(x)$ and $g(x)$ be two polynomials. We want to find a necessary condition that the system $f(x) = 0, g(x) = 0$ has a common solution. Let a be a second variable independent of x. By $f(a)$ and $g(a)$ we denote the polynomials $f(x)$ and $g(x)$ with x replaced by a. Consider the 2×2 determinant

$$\Delta(x, a) = \begin{vmatrix} f(x) & g(x) \\ f(a) & g(a) \end{vmatrix} = f(x)g(a) - g(x)f(a).$$

Note that Δ is zero for any common solution x of the system $f(x) = 0, g(x) = 0$. Moreover, $\Delta = 0$ if $x = a$. Therefore $x - a$ divides $\Delta(x, a)$ exactly. Hence the quotient

$$\delta(x, a) = \frac{\Delta(x, a)}{x - a} = \frac{f(x)g(a) - g(x)f(a)}{x - a}$$

is zero for any solution of the original system and is a polynomial in a and x. For any common zero of the system, say $x = x_0$, $\delta(x_0, a)$ is zero for all a, and therefore the coefficients of the powers of a in $\delta(x, a)$ are identically zero. Setting the coefficients of the powers of a in $\delta(x,a)$ equal to zero results in a homogeneous system in x. This system has nontrivial solutions if the determinant of the coefficient matrix is zero. This last determinant is called the *Bezout resultant* of the system. If the original system has a common zero, then the Bezout resultant is zero.

---

[7] Especially, the multivariate version due to Dixon (1908).

For example, let

$$f(x) = x^2 - 5x + 6,$$
$$g(x) = x^2 + 2x - 8.$$

Then

$$\begin{aligned}\delta(x,a) &= \frac{1}{x-a}\begin{vmatrix} x^2 - 5x + 6 & x^2 + 2x - 8 \\ a^2 - 5a + 6 & a^2 + 2a - 8 \end{vmatrix} \\ &= \frac{1}{x-a}((x^2 - 5x + 6)(a^2 + 2a - 8) - (x^2 + 2x - 8)(a^2 - 5a + 6)),\end{aligned}$$

which simplifies to

$$\delta(x,a) = 7ax - 14x - 14a + 28.$$

For any common zero of the system, $\delta(x,a) = 0$ for all a. Hence the coefficients of all powers of a must be zero. Since the coefficient of a^0 is $28 - 14x$ and the coefficient of a^1 is $-14 + 7x$, we have

$$28 - 14x = 0,$$
$$-14 + 7x = 0.$$

The determinant of the coefficient matrix of this system is the Bezout resultant. The system has a common solution since the Bezout resultant is zero,

$$\begin{vmatrix} 28 & -14 \\ -14 & 7 \end{vmatrix} = 0.$$

If the two polynomials f and g are of the same degree, then the Bezout and Sylvester resultants are identical. The Bezout resultant is often preferred since the size of the determinant Δ (= $\max(\deg(f), \deg(g))$) is much smaller than that of the Sylvester determinant (= $\deg(f) + \deg(g)$).

Problem A. The Bezout method can be used to eliminate one variable out of a system of two polynomial equations in two variables. Consider

$$x^2 + y^2 - 1 = 0,$$
$$x^2 - 2x + y^2 - 2y + 1 = 0$$

as a system in y with coefficients polynomials in x,

$$\begin{aligned} &y^2 + (x^2 - 1) = 0, \\ &y^2 - 2y + (x^2 - 2x + 1) = 0. \end{aligned}$$

(a) Use the Bezout method to eliminate y.
(b) Set the Bezout resultant equal to zero and solve for x.
(c) Back substitute into the original system and find all the common zeros.

Problem B. Repeat the process of Problem A for the following system and conclude that there are no common solutions:

$$\begin{aligned} &x^2 + y^2 - 1 = 0, \\ &x^2 - 6x + y^2 - 2y + 6 = 0. \end{aligned}$$

6.8 Technology-aided problems and answers

Let

$$A = \begin{bmatrix} 6 & 7 & 1 \\ 6 & -7 & 2 \\ 6 & 7 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \\ 13 & 15 & 17 \end{bmatrix}.$$

1. Compute the determinants of A, B, and C. Which of these matrices are invertible?

2. Find $\det(B)$ by cofactor expansion about the second column.

3. Compute the following and explain in each case why the answer should be zero:
(a) $\det(A^5) - (\det(A))^5$;
(b) $\det(A) - \det(A^T)$;
(c) $\det(AB) - \det(A)\det(B)$.

4. Compute the following and explain in each case why the answer should be zero:
(a) $\det(5B) - 125\det(B)$;
(b) $\det(B^{-1}) - 1/\det(B)$;
(c) $\det(B^{-2}) - 1/\det(B)^2$.

5. Compute and compare $\det(C)$ and $\det(C^T)$. Repeat with $\det(C^T C)$ and $\det(C)^2$. Explain the comparisons.

Let M_n be the following matrix sequence:

$$M_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}, \quad \ldots.$$

6. Define M_n as a function in n. Use this function to compute $\det(M_2)$, $\det(M_3)$, … . Do you see a pattern for $\det(M_n)$?

7. Compute $\det(S^T S)$ for several 3×2 matrices S. Find a connection between $S^T S$ being invertible and the linear dependence or independence of the columns of S.

8. Solve the system (a) by matrix inversion and (b) by Cramer's rule. (c) Compute the adjoint of the coefficient matrix.

$$\begin{aligned} -85x - 55y - 37z &= -306, \\ -35x + 97y + 50z &= 309, \\ 79x + 56y + 49z &= 338. \end{aligned}$$

9. Let $A\mathbf{x} = \mathbf{b}$ be a square system. Write the code for two functions CramerDisplay and CramerSolve, each having three arguments, A, $\mathbf{b}$, and i. CramerDisplay is to return the matrix A_i obtained from A by replacing the ith column with $\mathbf{b}$. CramerSolve is to solve for x_i by Cramer's rule. Test the code by displaying A_1, A_2, and A_3 and solving the system of Exercise 8.

10. If the appropriate command is available, then find *all* permutations of $\{1, 2, 3, 4\}$. Check that the number of permutations found is the correct one.

11. If the appropriate command is available, then find the signs of the permutations $\{1, 4, 2, 3\}$ and $\{4, 2, 3, 1\}$.

12. Find a random permutation of $\{1, 2, 3, 4\}$.

13. Use determinants to define a function $f(x_1, y_1, x_2, y_2, x_3, y_3)$ that computes the equation in x and y of a circle passing through three points (x_1, y_1), (x_2, y_2), (x_3, y_3).

14. Use f above to find the equation of the circle C_1 through $(-1, 1)$, $(1, 1)$, $(2, 4)$.

15. Find the point(s) with x-coordinate -2 on the circle C_1 above. Also, prove that C_1 has no points with x-coordinate -3.

16. Use determinants to define a function $g(x_1, y_1, x_2, y_2, x_3, y_3)$ that tests whether or not the four points with coordinates (x, y), (x_1, y_1), (x_2, y_2), (x_3, y_3) lie on the same circle.

17. Use g above to check whether the points $A(-2, 2)$, $B(-2, 4)$, and $C(-1, 2)$ lie on the circle C_1 above.

18. Let

$$\begin{aligned} p_1 &= x^2 + x - 2, \\ p_2 &= x^2 + 2x - 3, \\ p_3 &= x^2 + 3x + 2. \end{aligned}$$

Compute the resultants of the polynomial pairs (p_1, p_2), (p_1, p_3), and (p_2, p_3). Which pairs have a common solution?

6.8.1 Selected solutions with Mathematica

```
A = {{6,7,1},{6,-7,2},{6,7,3}}                              (* DATA *)
B = {{1/3,1/4,1/5},{1/4,1/4,1/5},{1/5,1/5,1/5}}
C1 = {{1,3,5},{7,9,11},{13,15,17}}
```

```
(* C is used in differential eqn. contants.*)
(* Exercises 1-4. *)
Det[A]
Det[B]
Det[C1]        (* C is the only non-invertible, since Det[C]=0.*)
(* Watch for the reverse numbering in the Minors command.*)
-(1/4) Minors[B,2][[3,2]]+(1/4) Minors[B,2][[2,2]]
-(1/5) Minors[B,2][[1,2]]
Det[MatrixPower[A, 5]]-Det[A]^5
Det[A]-Det[Transpose[A]]
Det[A.B]-Det[A] Det[B]
Det[5B]-125Det[B]
Det[Inverse[B]]-1/Det[B]
Det[MatrixPower[B,-2]]-1/Det[B]^2
(* Exercise 5. *)
Det[C1]
Det[Transpose[C1]]
Det[Transpose[C1].C1]
Expand[% - Det[C1]^2]           (* Expand the difference to get 0. *)
(* Exercise 6. *)
m[n_]:= Table[If[i==j, i, 1], {i,1,n},{j,1,n}]      (* M_n.  *)
{Det[m[2]],Det[m[3]],Det[m[4]],Det[m[5]]}           (* Etc.. *)
(*  Pattern:   det (M_n) = (n-1)!   *)
(* Exercise 7 - Comment. *)
(* (S^T)S is invertible only if the columns of S are lin. independent.*)
(* Exercise 8. *)
A={{-85,-55,-37},{-35,97,50},{79,56,49}}
b={{-306},{309},{338}}
sol=Inverse[A] . b                  (* (a) *)
A1 = Join[b, A[[All,2;;3]],2]  (* Replaces column 1 of A with b *)
A2 = Join[A[[All,{1}]],b,A[[All,{3}]],2] (* Replaces column 2 with b *)
A3 = Join[A[[All, 1 ;; 2]], b, 2]   (* Replaces column 3 of A with b *)
x=Det[A1]/Det[A]                    (* (b) *)
y=Det[A2]/Det[A]                    (* (b) *)
z=Det[A3]/Det[A]                    (* (b) *)
adj=Det[A] Inverse[A]               (* (c) *)
(* Exercise 9. *)
CramerDisplay[A_,b_,i_] := Module[{AA=A, j},
For[j=1,j<=Length[A],j++,      (* Replacing the ith *)
AA[[j,i]]=b[[j,1]]]; AA]      (* column with b.    *)
CramerSolve[A_,b_,i_] := Det[CramerDisplay[A,b,i]]/Det[A]
CramerDisplay[A,b,1]
(* Also CramerDisplay[A,b,2] and CramerDisplay[A,b,3]*)
CramerSolve[A,b,1]
(* Also CramerSolve[A,b,2] and CramerSolve[A,b,3]*)
(* Exercises 10, 11, 12 *)
Permutations[{1,2,3,4}]   (* The permutations of {1,2,3,4}.*)
Length[%]                 (* The number of computed permutations. *)
4!                            (* The expected answer. *)
```

```
Signature[{1,4,2,3}]         (* The Sign function. *)
Signature[{4,2,3,1}]
Permutations[{1,2,3,4}][[RandomInteger[{1, 24}]]]
(* Random permutation *)
(* Exercise 13. *)
Clear[x,y,z]              (* Clear the values from Exer. 8.*)
circleeqn[x1_,y1_,x2_,y2_,x3_,y3_]:=Det[{{x^2+y^2,x,y,1},
{x1^2+y1^2,x1,y1,1},
{x2^2+y2^2,x2,y2,1},
{x3^2+y3^2,x3,y3,1}}]
(* Exercise 14. *)
ce=circleeqn[-1,1,1,1,2,4]          (* The equation of the circle. *)
(* Exercise 15 *)
ce /. {x -> -2}          (* We substitute x=-2 set equal to zero  *)
Solve[%==0,y]   (*  and solve for y to get the points (-2,2), (-2,4) *)
ce /. {x -> -3}       (*  If we repeat with x=-3 and solve for y  *)
Solve[%==0,y]                        (* we get complex roots. *)
(* Exercise 18. - Partial *)
p1=-2 + x + x^2
p2=-3 + 2 x + x^2
Resultant[p1,p2,x]  (* We also need to declare the variable x. Etc. *)
```

6.8.2 Selected solutions with MATLAB

```
A = [6 7 1; 6 -7 2; 6 7 3]                     % DATA
B = [1/3 1/4 1/5; 1/4 1/4 1/5; 1/5 1/5 1/5]
C = [1 3 5; 7 9 11; 13 15 17]
% Exercises 1-4.
det(A), det(B), det(C)   % C is the only non-invertible, since det(C)=0.
-(1/4)*det(B([2 3],[1 3]))+(1/4)*det(B([1 3],[1 3]))
-(1/5)*det(B([1 2],[1 3]))
det(A^5)-det(A)^5
det(A)-det(A.')
det(A*B)-det(A)*det(B)
det(5*B)-125*det(B)
det(inv(B))-1/det(B)
det(B^(-2))-1/det(B)^2
% Exercise 5.
det(C)        % Although det(C)=det(C') due to floating point error
det(C')       % we get slightly different answers.
det(C'*C)(* Although det(C'*C)=det(C)^2 due to floating point error
det(C)^2      % we get slightly different answers.
% Exercise 6.
% Create a script file named m.m having the following lines
function a = m(n)
for i=1:n,
for j=1:n,
if i==j
a(i,j)=i;
else a(i,j)=1;
end
end
end
% Then type
det(m(2)), det(m(3)), det(m(4)), det(m(5))          % Etc..
% Pattern: det (M_n) = (n-1)!
% Exercise 7 - Comment.
% (S^T)S is invertible only if the columns of S are lin. independent.
% Exercise 8.
A=[-85 -55 -37; -35 97 50; 79 56 49]
b=[-306;309;338]
sol=A\b                    %  (a)
A1=[b A(:,2:3)]            %  column b and columns 2 and 3 of A
A2=[A(:,1) b A(:,3)]       %  column 1 of A column b column 3 of A
A3=[A(:,1:2) b]            %  columns 2 and 3 of A and column b
x=det(A1)/det(A)                    %  (b)
y=det(A2)/det(A)                    %  (b)
z=det(A3)/det(A)                    %  (b)
adj=det(A)*inv(A)                   %  (c)
% Exercise 9.
% In a file called CramerD.m type and save the code:
function [B] = CramerD (A,b,i)
B = [A(:,1:i-1) b A(:,i+1:length(A))];
end
% In a file called CramerS.m type and save the code:
function [B] = CramerS (A,b,i)
B = det([A(:,1:i-1) b A(:,i+1:length(A))])/det(A);
end
% Then in MATLAB session type:
CramerD(A,b,1),CramerD(A,b,2),CramerD(A,b,3)
CramerS(A,b,1),CramerS(A,b,2),CramerS(A,b,3)
% Exercise 10.
perms([1 2 3 4])             (* All permutations of [1,2,3,4]. *)
length(perms([1 2 3 4]))     (* There are 24 of them. *)
% Exercise 11
x = [1 4 3 2];               % This can be done indirectly
y = eye(numel(x));           % by forming the corresponding
signature = det( y(:,x) ); % permutation matrix
% and computing its determinant.
% The same with the second permutation.
% Exercise 12.
randperm(4)                 % A random permutation of {1,2,3,4}.
% Exercises 13,14.
% Create a script file called ”circ.m” with the following contents
function [a] = circ(x,y,x1,y1,x2,y2,x3,y3)    % x,y are used as
a = det([x^2+y^2 x y 1;         % arguments of the function
x1^2+y1^2 x1 y1 1;         % because MATLAB will not
x2^2+y2^2 x2 y2 1;         % accept them as symbolic
x3^2+y3^2 x3 y3 1]);       % variables
% then return to your MATLAB session and type
circ(-2,2,-1,1,1,1,2,4)          % evaluation of the function
circ(-2,4,-1,1,1,1,2,4)          % at all four points
circ(-1,2,-1,1,1,1,2,4)    % not zero so (-1,2) is not on the circle.
% Exercise 18.  Find the det. of the manually entered the Sylvester matrix.
det([-2 1 1 0; 0 -2 1 1; -3 2 1 0; 0 -3 2 1])
% 4.163336342344336e-16 actually zero. Etc.
```

6.8.3 Selected solutions with Maple

```
with(LinearAlgebra);
A := Matrix([[6,7,1],[6,-7,2],[6,7,3]]);    # DATA
B := Matrix([[1/3,1/4,1/5],[1/4,1/4,1/5],[1/5,1/5,1/5]]);
C := Matrix([[1,3,5],[7,9,11],[13,15,17]]);
# Exercises 1-4.
Determinant(A); Determinant(B); Determinant(C);
# C is the only non-invertible, since Determinant(C)=0.
-(1/4)*Minor(B,1,2)+(1/4)*Minor(B,2,2)-(1/5)*Minor(B,3,2);
Determinant(A^5)-Determinant(A)^5;
Determinant(A)-Determinant(Transpose(A));
Determinant(A.B)-Determinant(A)*Determinant(B);
Determinant(5*B)-125*Determinant(B);
Determinant(MatrixInverse(B))-1/Determinant(B);
Determinant(B^(-2))-1/Determinant(B)^2;
# Exercise 5.
Determinant(C); Determinant(Transpose(C));
Determinant(Transpose(C).C); Determinant(C)^2;
# Exercise 6.
m:=proc(n) Matrix(n,n, (i,j)->if i=j then i else 1 fi) end: # M_n.
Determinant(m(2)); Determinant(m(3)); Determinant(m(4));  # Etc..
#  Pattern:   Determinant (M_n) = (n-1)!
# Exercise 7 - Comment.
# (S^T)S is invertible only if the columns of S are lin.independent.
# Exercise 8
A:=Matrix([[-85,-55,-37],[-35,97,50],[79,56,49]]);
b:=Vector([-306,309,338]);
sol:=MatrixInverse(A).b;
A1:=<b|SubMatrix(A,[1..3],[2..3])>;  # Replace column 1 of A with b.
A2:=<SubMatrix(A,[1..3],[1..1])|b|SubMatrix(A,[1..3],[3..3])>; # Etc.
A3:=<SubMatrix(A,[1..3],[1..2])|b>;
x:=Determinant(A1)/Determinant(A);                  # (b)
y:=Determinant(A2)/Determinant(A);                  # (b)
z:=Determinant(A3)/Determinant(A);                  # (b)
adj:=Adjoint(A);                   # Adjoint
```

```
# Exercise 9.
CramerDisplay := proc (A,b,i) local AA, j; AA:= copy(A);
for j from 1 to RowDimension(A) do
AA[j,i]:=b[j] od:                     # Replacing the ith
AA                             # column with b.
end:
CramerSolve := proc (A,b,i)
Determinant(CramerDisplay(A,b,i))/Determinant(A) end:
CramerDisplay(A,b,1); CramerDisplay(A,b,2);
CramerDisplay(A,b,3);
CramerSolve(A,b,1); CramerSolve(A,b,2); CramerSolve(A,b,3);
# Exercises 10,12.
with(combinat);              # Loading the combinat package.
permute(4);                  # The permutations of {1,2,3,4}.
nops(%);                     # The number of computed permutations.
4!;                          # The expected answer.
randperm(4);                 # A random permutation of {1,2,3,4}.
# Exercise 13.
restart;
with(LinearAlgebra);
circleeqn := proc(x1,y1,x2,y2,x3,y3)
Determinant(Matrix(4,4,[x^2+y^2,x,y,1,
x1^2+y1^2,x1,y1,1,
x2^2+y2^2,x2,y2,1,
x3^2+y3^2,x3,y3,1]))
end:
# Exercise 14.
ce:=circleeqn(-1,1,1,1,2,4);     # The equation of the circle.
# Exercise 15.
subs(x=-2,ce);   # We substitute x=-2 set equal to zero
solve(% = 0, y); # and solve for y to get the points (-2,2),(-2,4)
subs(x=-3,ce);   # If we repeat with x=-3 and solve for y
solve(% = 0, y);  # we get complex roots.
# Exercise 18 - Partial
p1:=-2 + x + x^2;
p2:=-3 + 2 x + x^2;
resultant(p1,p2,x);     # We also need to declare the variable x.
```

7 Eigenvalues

> The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful (Metaphysica 3-1078b).
>
> Aristotle, Greek philosopher (384–322, BCE).

Figure 7.1: Augustin-Louis Cauchy.
Public domain – Library of Congress. From an illustration in: Das neunzehnte Jahrhundert in Bildnissen / Karl Werckmeister, ed. Berlin: Kunstverlag der photographische gesellschaft, 1901, vol. V, no. 581. Augustin-Louis Cauchy (1789–1857) was a prominent and prolific French mathematician. He is regarded as one of the most influential mathematicians of the nineteenth century, known for his significant contributions to analysis, complex analysis, number theory, and elasticity theory. He was a pioneer in the rigorous development of calculus and helped establish the foundations of modern mathematical analysis. Cauchy is the first known mathematician who made systematic use of eigenvalues.

Introduction

Eigenvalues and eigenvectors are among the most useful topics of linear algebra. They are used in several areas of mathematics, mechanics, electrical engineering, hydrodynamics, aerodynamics, etc. In fact, it is rather hard to find an applied area where eigenvalues are not used. Some specific uses include the following.

Automobile vibration analysis, which is important in driving safety and comfort.

Building vibration analysis, which is useful to the study of the effect of earthquakes to buildings.

Surface approximation, which is the conversion of scattered three-dimensional data points into a surface. These are used in *computer-aided geometric design*.

Automatic feedback control for dynamical systems used in HVAC thermostats, robotic factories, satellite positioning, etc.

Historically, examples of use of eigenvalues are found in Euler's study of quadratic forms, in Lagrange's studies of celestial mechanics, and in D'Alembert's study of the motion of a string with masses attached to it. However, it was Cauchy in 1826 who first used eigenvalues systematically (Figure 7.1). He did so to convert quadratic forms to sums of squares. Later, Sturm used the concept of eigenvalue in the context of solutions of systems of differential equations.

https://doi.org/10.1515/9783111331850-007

7.1 Eigenvalues and eigenvectors

If A is an $n \times n$ matrix and $\mathbf{v}$ is an n-vector, then the vector $A\mathbf{v}$ is usually seemingly unrelated to the original vector $\mathbf{v}$. A case of great interest occurs when $A\mathbf{v}$ is a scalar multiple of $\mathbf{v}$.

Definition 7.1.1. Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{v}$ is called an *eigenvector* of A if for some scalar λ,

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{7.1}$$

The scalar λ (which can be zero) is called an *eigenvalue* of A *corresponding to* (or *associated with*) the eigenvector $\mathbf{v}$.

Geometrically, if $\mathbf{v}$ is an eigenvector of A, then $\mathbf{v}$ and $A\mathbf{v}$ are on the same line through the origin.

Example 7.1.2. Let

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

(a) Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A.
(b) What are the eigenvalues corresponding to $\mathbf{v}_1$ and $\mathbf{v}_2$?

Solution. We have

$$A\mathbf{v}_1 = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 3\mathbf{v}_1,$$

$$A\mathbf{v}_2 = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\mathbf{v}_2.$$

Therefore $\mathbf{v}_1$ is an eigenvector with corresponding eigenvalue $\lambda = 3$, and $\mathbf{v}_2$ is an eigenvector with corresponding eigenvalue $\lambda = -2$.

Example 7.1.3. Use geometric arguments to find all the eigenvalues and eigenvectors of $A = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

Solution. $A\mathbf{x}$ is the reflection of $\mathbf{x}$ about the line $y = x$ (why?). The only vectors that remain on the same line after rotation are the vectors along the lines $y = x$ and $y = -x$. These without the origin are the only eigenvectors. For $\mathbf{v}$ along the line $y = x$, we have $A\mathbf{v} = 1\mathbf{v}$, so $\mathbf{v}$ is an eigenvector with corresponding eigenvalue 1. For $\mathbf{v}$ along the line $y = -x$, we have $A\mathbf{v} = -1\mathbf{v}$, so $\mathbf{v}$ is an eigenvector with corresponding eigenvalue -1 (Figure 7.2).

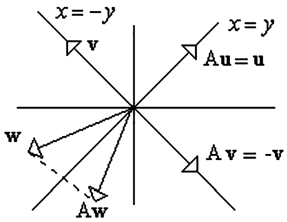

Figure 7.2: Reflection about the line $y = -x$.

Theorem 7.1.4. *Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. Let E_λ be the set of all eigenvectors of A corresponding to λ and the zero n-vector. Then E_λ is a subspace of $\mathbf{R}^n$.*

Proof. It suffices to prove that if $\mathbf{v}_1, \mathbf{v}_2 \in E_\lambda$, then $\mathbf{v}_1 + \mathbf{v}_2 \in E_\lambda$ and $c\mathbf{v}_1 \in E_\lambda$ for $c \in \mathbf{R}$. We have

$$\begin{aligned} A(\mathbf{v}_1 + \mathbf{v}_2) &= A\mathbf{v}_1 + A\mathbf{v}_2 \\ &= \lambda\mathbf{v}_1 + \lambda\mathbf{v}_2 \\ &= \lambda(\mathbf{v}_1 + \mathbf{v}_2). \end{aligned}$$

Hence, if $\mathbf{v}_1 + \mathbf{v}_2 \neq \mathbf{0}$, then $\mathbf{v}_1 + \mathbf{v}_2$ is an eigenvector of A with eigenvalue λ. So $\mathbf{v}_1 + \mathbf{v}_2 \in E_\lambda$. If $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$, then $\mathbf{v}_1 + \mathbf{v}_2$ is in E_λ by the definition of E_λ. Furthermore,

$$\begin{aligned} A(c\mathbf{v}_1) &= cA\mathbf{v}_1 \\ &= c\lambda\mathbf{v}_1 \\ &= \lambda(c\mathbf{v}_1). \end{aligned}$$

So again $c\mathbf{v}_1 \in E_\lambda$. Hence E_λ is a subspace of $\mathbf{R}^n$. □

Definition 7.1.5. The subspace E_λ of $\mathbf{R}^n$ of Theorem 7.1.4 consisting of the zero vector and the eigenvectors of A with eigenvalue λ is called an *eigenspace* of A. It is the eigenspace with eigenvalue λ. The dimension of E_λ is called the *geometric multiplicity* of λ.

Example 7.1.6. Referring to Example 7.1.2, determine geometrically
(a) the eigenspaces E_3 and E_{-2} of A,
(b) the effect of the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ on E_3 and E_{-2}.

Solution.
(a) By Example 7.1.2, E_3 is a subspace that contains the eigenvector $\mathbf{v}_1 = [2\ 1]^T$. Therefore E_3 includes the line l_1 through the origin and $(2, 1)$. It contains no other vector, or else $E_3 = \mathbf{R}^2$, which is not possible (why?). So E_3 is l_1. Likewise, E_{-2} is the line l_2 through the origin and $(1, -2)$ (Figure 7.3).

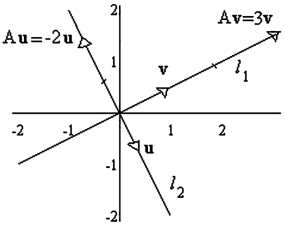

Figure 7.3: Stretching by a factor of 3 along eigenline l_1. Reflection and stretching by a factor of 2 along eigenline l_2.

(b) If $\mathbf{x} \in E_3$, then $T(\mathbf{x}) = A\mathbf{x} = 3\mathbf{x}$. So T *stretches* the vectors of line E_3 by a factor of $\lambda = 3$. If $\mathbf{x} \in E_{-2}$, then $T(\mathbf{x}) = -2\mathbf{x}$, so the vectors of line E_{-2} are reflected about the origin and then stretched by a factor of 2.

7.1.1 Computation of eigenvalues and eigenvectors

The geometric approach to eigenvectors is illuminating but limited to certain matrices of sizes 2×2 and 3×3. Let us introduce a general algebraic approach.

Theorem 7.1.7. *Let A be a square matrix.*

1. *A vector $\mathbf{v}$ is an eigenvector of A corresponding to an eigenvalue λ if and only if $\mathbf{v}$ is a nontrivial solution of the system*

$$(A - \lambda I)\mathbf{v} = \mathbf{0}. \tag{7.2}$$

2. *A scalar λ is an eigenvalue of A if and only if*

$$\det(A - \lambda I) = 0. \tag{7.3}$$

Proof. 1. We have

$$\begin{aligned} A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A\mathbf{v} &= \lambda I\mathbf{v} \\ \Rightarrow A\mathbf{v} - \lambda I\mathbf{v} &= \mathbf{0} \\ \Rightarrow (A - \lambda I)\mathbf{v} &= \mathbf{0}. \end{aligned}$$

Hence $\mathbf{v}$ is an eigenvector if and only if it is a nontrivial solution of the homogeneous system $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

2. The homogeneous linear system (7.2) has a nontrivial solution if and only if the determinant of the coefficient matrix is zero. Thus λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$. ☐

Definition 7.1.8. Equation (7.3) is called the *characteristic equation* of A. The determinant $\det(A - \lambda I)$ is a polynomial of degree n in λ and is called the *characteristic polynomial* of A. The matrix $A - \lambda I$ is called the *characteristic matrix* of A. If an eigenvalue λ is a root of the characteristic equation of multiplicity k, then we say that λ has *algebraic multiplicity* k.

Part 1 of Theorem 7.1.7 in fact states that the null space of the characteristic matrix $A - \lambda I$ is the eigenspace E_λ corresponding to the eigenvalue λ,

$$E_\lambda = \operatorname{Null}(A - \lambda I). \tag{7.4}$$

In Examples 7.1.9–7.1.11, we compute the eigenvalues and eigenvectors and find bases for each eigenspace of the given matrix A. In addition, we find the algebraic and geometric multiplicities of each eigenvalue.

Example 7.1.9. $A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix}$.

Solution. The characteristic equation is

$$\begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 0-\lambda & 2 \\ -1 & 3 & -1-\lambda \end{vmatrix} = -\lambda^3 + 9\lambda = -\lambda(\lambda-3)(\lambda+3) = 0.$$

Hence the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = 3, \quad \lambda_3 = -3.$$

Next, we find the eigenvectors, bases, and multiplicities.

(a) For $\lambda_1 = 0$, we have

$$[A - 0I : \mathbf{0}] = \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The general solution is $(2r, r, r)$ for $r \in \mathbf{R}$. Hence

$$E_0 = \left\{ \begin{bmatrix} 2r \\ r \\ r \end{bmatrix}, r \in \mathbf{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\},$$

and the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ defines a basis $\{\mathbf{v}_1\}$ of E_0.

Since $\lambda_1 = 0$ is a single root of the characteristic equation, the algebraic multiplicity is 1. Since the dimension of E_0 is 1, the geometric multiplicity is 1.

(b) For $\lambda_2 = 3$, we have

$$[A - 3I : \mathbf{0}] = \begin{bmatrix} -2 & -1 & -1 & 0 \\ -1 & -3 & 2 & 0 \\ -1 & 3 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The general solution is $(-r, r, r)$ for $r \in \mathbf{R}$. Hence

$$E_3 = \left\{ \begin{bmatrix} -r \\ r \\ r \end{bmatrix}, r \in \mathbf{R} \right\} = \mathrm{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

So $\mathbf{v}_2 = \left[\begin{smallmatrix} -1 \\ 1 \\ 1 \end{smallmatrix}\right]$ defines a basis $\{\mathbf{v}_2\}$ of E_3.
Both the algebraic and geometric multiplicities of $\lambda_2 = 3$ are 1.

(c) For $\lambda_3 = -3$, we have

$$[A - (-3)I : \mathbf{0}] = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 3 & 2 & 0 \\ -1 & 3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{11} & 0 \\ 0 & 1 & \frac{7}{11} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The general solution is $(\frac{r}{11}, -\frac{7r}{11}, r)$ for $r \in \mathbf{R}$. Hence

$$E_{-6} = \left\{ \begin{bmatrix} \frac{r}{11} \\ -\frac{7r}{11} \\ r \end{bmatrix}, r \in \mathbf{R} \right\} = \mathrm{Span} \left\{ \begin{bmatrix} \frac{1}{11} \\ -\frac{7}{11} \\ 1 \end{bmatrix} \right\}.$$

Any nonzero vector of E_{-3} is a basis. For example, $\mathbf{v}_3 = \left[\begin{smallmatrix} 1 \\ -7 \\ 11 \end{smallmatrix}\right]$ defines the basis $\{\mathbf{v}_3\}$ of E_{-3}.
Both the algebraic and geometric multiplicities of $\lambda_3 = -3$ are 1.
Note that all three eigenspaces are straight lines through the origin (Figure 7.4).

Example 7.1.10. $A = \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$.

Solution. The characteristic equation is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = -\lambda\,(1-\lambda)^2 = 0.$$

Hence the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 1.$$

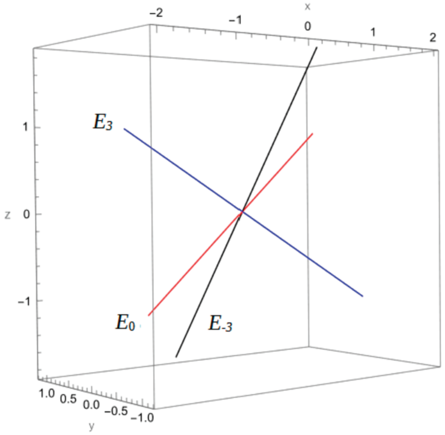

Figure 7.4: Eigenspaces E_0, E_3, and E_{-3}.

(a) For $\lambda_1 = 0$, we have

$$[A - 0I : \mathbf{0}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The general solution is $(r, 0, 0)$ for $r \in \mathbf{R}$. Hence

$$E_0 = \left\{ \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix}, r \in \mathbf{R} \right\} = \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\},$$

and the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ defines the basis $\{\mathbf{v}_1\}$ of E_0.
Both the algebraic and geometric multiplicities of $\lambda_1 = 0$ are 1.

(b) For $\lambda_2 = \lambda_3 = 1$, we have

$$[A - 1I : \mathbf{0}] = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The general solution is (r, s, r) for $r \in \mathbf{R}$. Since $(r, s, r) = r(1, 0, 1) + s(0, 1, 0)$, we have

$$E_1 = \left\{ \begin{bmatrix} r \\ s \\ r \end{bmatrix}, r \in \mathbf{R} \right\} = \mathrm{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

The spanning eigenvectors $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent. So, $\{\mathbf{v}_2, \mathbf{v}_3\}$ is a basis for E_1.

Both the algebraic and geometric multiplicities of $\lambda_2 = \lambda_2 = 1$ are 2.
Note that E_0 is the x-axis and E_1 is the plane through the points $(0, 0, 0)$, $(1, 0, 1)$, and $(0, 1, 0)$ (Figure 7.5).

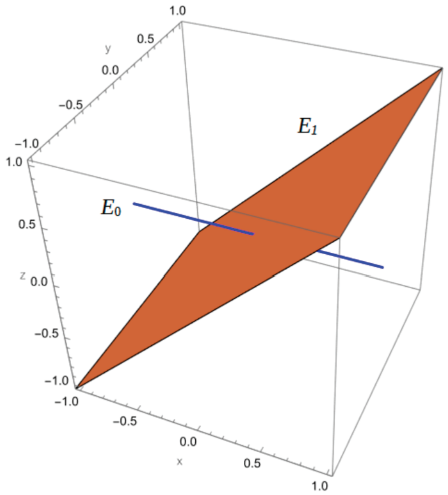

Figure 7.5: Eigenspaces E_0 and E_1.

Example 7.1.11. $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$.

Solution. The eigenvalues are

$$\lambda_1 = \lambda_2 = 0\,, \quad \lambda_3 = -2,$$

and the eigenspaces are

$$E_0 = \text{Span}\left\{\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}\right\}, \quad E_{-2} = \text{Span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\},$$

from which we can find bases and the multiplicities.

Note that although the algebraic multiplicity of $\lambda = 0$ is 2, the geometric multiplicity is only 1.

If $A = [a_{ij}]$ is a triangular matrix, then so is $A - \lambda I$. Hence, in this case,

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)\cdots(a_{nn} - \lambda).$$

We conclude that the eigenvalues of a triangular matrix are the diagonal entries.

Example 7.1.12. The eigenvalues of the upper triangular $A = \left[\begin{smallmatrix} 1 & 999 \\ 0 & -4 \end{smallmatrix}\right]$ are the diagonal entries $\lambda = 1, -4$.

Example 7.1.13 (Complex eigenvalues). $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ represents the 90° counterclockwise rotation about the origin. So, geometrically, A should have no eigenvectors. However, the algebraic approach yields *complex eigenvalues*

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

and complex eigenspaces

$$E_i = \mathrm{Span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}, \quad E_{-i} = \mathrm{Span}\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}.$$

The scalars in the spans are now complex numbers.

Theorem 7.1.14. *Let A be an $n \times n$ matrix. The following are equivalent.*

1. *A is invertible.*
2. *0 is not an eigenvalue of A.*

Proof. A is invertible if and only if

$$\begin{aligned} \det(A) \neq 0 &\Leftrightarrow \det(A - 0I) \neq 0 \\ &\Leftrightarrow 0 \text{ is not an eigenvalue of } A. \end{aligned}$$ □

7.1.2 Eigenvalues of linear operators

Recall that a linear operator is a linear transformation of a vector space into itself.

We may define eigenvalues and eigenvectors for linear operators. If V is a vector space and $T : V \to V$ is a linear operator, then a nonzero vector $\mathbf{v}$ is an *eigenvector* of T if

$$T(\mathbf{v}) = \lambda \mathbf{v}$$

for some scalar λ. Just as before, we call λ an *eigenvalue* of T.

Example 7.1.15. Let V be any vector space, and let c be a fixed scalar. Find the eigenvalues and eigenvectors of the homothety

$$T : V \to V\,, \quad T(\mathbf{v}) = c\mathbf{v}.$$

Solution. Because $T(\mathbf{v}) = c\mathbf{v}$, every nonzero vector is an eigenvector. The corresponding eigenvalue is $\lambda = c$.

Example 7.1.16 (Requires calculus). Let $V = C^1(\mathbf{R})$ be the vector space of all real-valued differentiable functions on $\mathbf{R}$. Let $\frac{d}{dx} : V \to V$ be the differentiation operator,

$$\frac{d}{dx} : V \to V, \quad \frac{d}{dx}(f) = \frac{df}{dx}.$$

Let r be a fixed scalar. Prove that e^{rx} in V is an eigenvector of $\frac{d}{dx}$. Find the corresponding eigenvalue.

Solution. We leave to the reader the verification that $\frac{d}{dx}$ is a linear operator. Because

$$\frac{d}{dx}(e^{rx}) = re^{rx},$$

it follows that e^{rx} is an eigenvector of $\frac{d}{dx}$ and r is the corresponding eigenvalue (Figure 7.6).

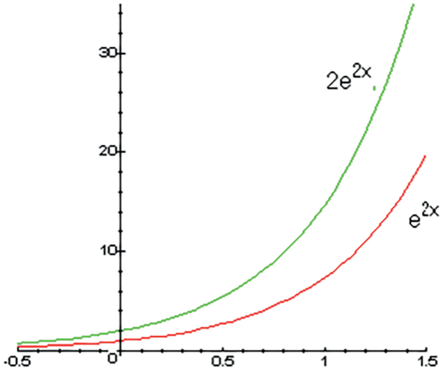

Figure 7.6: e^{2x} is an eigenvector of $\frac{d}{dx}$ with eigenvalue 2.

7.1.3 Numerical note

For large matrices, the characteristic equation is of high degree, which makes it hard to find, or even estimate, the eigenvalues. In addition, the reduction of the characteristic matrix may introduce cumulative round-off errors. For these reasons, other methods of computation are used in practice. Some of these are the iterative methods, such as the power method studied in Sections 7.3 and 7.6. These methods first approximate an eigenvector and then compute the corresponding eigenvalue.

Exercises 7.1

In Exercises 1–4, let $A = \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix}$, and let

$$\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

1. Prove that **u** and **v** are eigenvectors of A. Find the corresponding eigenvalue(s).

2. Is $\mathbf{u} + \mathbf{v}$ an eigenvector of A? Explain.

3. Find the reasoning flaw in the following false statements: The vectors $b\mathbf{u}$ and $c\mathbf{v}$ are eigenvectors of A, hence, $b\mathbf{u} + c\mathbf{v}$ is also an eigenvector of A. Therefore all 2-vectors are eigenvectors of A.

4. Are **u** and **v** eigenvectors of an echelon form of A? Explain.

In Exercises 5–6, find the eigenvalues and eigenvectors of each matrix by using *geometric* arguments.

5. (a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

6. (a) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

For the matrices of Exercises 7–12, find
(i) the characteristic polynomial,
(ii) the eigenvalues,
(iii) bases of eigenvectors for all eigenspaces, and
(iv) the algebraic and geometric multiplicities of each eigenvalue.

7. (a) $\begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$, (b) $\begin{bmatrix} 3 & 6 \\ 9 & 0 \end{bmatrix}$.

8. (a) $\begin{bmatrix} -2 & 17 \\ 17 & -2 \end{bmatrix}$, (b) $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$.

9. (a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

10. (a) $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

11. (a) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & -1 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

12. (a) $\begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$, (b) $\begin{bmatrix} 6 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{bmatrix}$.

13. Find the eigenvalues by factoring the characteristic polynomial of the matrix $\begin{bmatrix} 3 & 0 & 0 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 4 & 1 & 2 \\ 0 & -2 & 1 & 1 \end{bmatrix}$.

14. Without any computation, find the eigenvalues of the following matrices:

$$\text{(a)} \quad \begin{bmatrix} -4 & 0 \\ 888 & 5 \end{bmatrix}, \quad \text{(b)} \quad \begin{bmatrix} a+b & d \\ 0 & a-c \end{bmatrix}.$$

15. Without calculating, find one eigenvector of A and its eigenvalue, if

$$A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}.$$

16. Without computing, explain why $\lambda = 6$ is an eigenvalue of

$$A = \begin{bmatrix} 7 & 1 & 1 & 1 \\ 1 & 7 & 1 & 1 \\ 1 & 1 & 7 & 1 \\ 1 & 1 & 1 & 7 \end{bmatrix}.$$

Also, find a basis for E_6.

17. Without computing, explain why $\lambda = a - b$ is an eigenvalue of

$$A = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}.$$

Then find a basis for E_{a-b} if $b \neq 0$.

18. Prove that any square matrix A and its transpose A^T have the same characteristic polynomial and the same eigenvalues.

19. Prove that the matrix A has 0 as an eigenvalue if and only if $\text{Null}(A) \neq \{\mathbf{0}\}$. In this case, prove that $\text{Null}(A) = E_0$.

20. Let $\mathbf{v}$ be an eigenvector of a matrix A with eigenvalue 2. Find one solution of the system $A\mathbf{x} = \mathbf{v}$.

21. Let $\mathbf{v}$ be an eigenvector of a matrix A with eigenvalue -7. Find one solution of the system $A\mathbf{x} = 3\mathbf{v}$.

In Exercises 23–24, let $\mathbf{v}$ be an eigenvector of A with eigenvalue λ.

22. (*Power*) Prove that $\mathbf{v}$ is also an eigenvector of A^k with eigenvalue λ^k.

23. (*Inverse*) If A is invertible, then prove that $\mathbf{v}$ is also an eigenvector of A^{-1} with eigenvalue λ^{-1}.

24. (*Shift of origin*) Let c be any scalar. Prove that $\mathbf{v}$ is an eigenvector of $A - cI$ with eigenvalue $\lambda - c$.

25. (*Similar matrices (Cauchy)*) Let A and B be two $n \times n$ similar matrices. Recall the definition from Section 5.3 that A is *similar* to B if there is an invertible matrix P such that $P^{-1}AP = B$. Prove the following.
(a) A and B have the same characteristic polynomial.
(b) A and B have the same eigenvalues.

(c) If $\mathbf{v}$ is an eigenvector of B with eigenvalue λ, then $P\mathbf{v}$ is an eigenvector of A with eigenvalue λ.
(d) If $\mathbf{u}$ is an eigenvector of A with eigenvalue λ, then $P^{-1}\mathbf{u}$ is an eigenvector of B with eigenvalue λ.

26. Let A and B have size $n \times n$, and let A be invertible. Prove that AB and BA have the same characteristic polynomial.

27. Find matrices A and B such that none of the eigenvalues of the sum $A + B$ is a sum of eigenvalues of A and B.

28. Find matrices A and B such that none of the the eigenvalues of AB is a product of eigenvalues of A and B.

29. Prove that the standard basis n-vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ are eigenvectors of any diagonal $n \times n$ matrix A. Find the corresponding eigenvalues.

30. Suppose that A has every nonzero n-vector as an eigenvector to the same eigenvalue. Prove that A is a scalar matrix.

31. Prove that the real matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has real eigenvalues if and only if $(a - d)^2 + 4bc \geq 0$.

32. (*Nilpotent*) A square matrix A is called *nilpotent* if $A^k = \mathbf{0}$ for some positive integer k. Let A be a nilpotent matrix. Prove that 0 is its only eigenvalue.

33. Prove that if A is nilpotent (see Exercise 32), then the geometric multiplicity of 0 equals the nullity of A.

34. Let $\lambda_1, \dots, \lambda_n$ be all the eigenvalues (repeated if multiple) of an $n \times n$ matrix A. Prove that

$$\begin{aligned} \operatorname{tr}(A) &= \lambda_1 + \lambda_2 \cdots + \lambda_n, \\ \det(A) &= \lambda_1 \lambda_2 \cdots \lambda_n. \end{aligned}$$

35. The matrix $\begin{bmatrix} 111 & 222 \\ 222 & -222 \end{bmatrix}$ has 222 as an eigenvalue. Without computing, find its second eigenvalue. Then use the two eigenvalues to find the determinant.

36. Prove that the geometric multiplicity of an eigenvalue λ is less than or equal to its algebraic multiplicity. (*Hint:* Extend a basis of eigenvectors of λ to a basis of $\mathbf{R}^n$. Let A' be the matrix of $T(\mathbf{x}) = A\mathbf{x}$ relative to this basis. Then $A = P^{-1}A'P$ for some invertible matrix P. Now use Exercise 25.)

37. Prove that any square matrix is the sum of two invertible matrices. (*Hint:* Choose c that is not an eigenvalue of $\pm A$ and consider $A \pm cI$.)

Companion matrix of polynomial
Let $p(x)$ be the polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

The following $n \times n$ matrix, denoted by $C(p)$, is called the *companion matrix* of p:

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}.$$

38. Find the companion matrix $C(p)$ of $p(x) = x^2 + 2x - 15$ and then find the characteristic polynomial of $C(p)$.

39. Prove that the companion matrix $C(p)$ of $p(x) = x^2 + ax + b$ has the characteristic polynomial $\lambda^2 + a\lambda + b$.

40. Prove that the companion matrix $C(p)$ of $p(x) = x^3 + ax^2 + bx + c$ has the characteristic polynomial $-(\lambda^3 + a\lambda^2 + b\lambda + c)$. Also, prove that for any eigenvalue λ of $C(p)$, the vector $(1, \lambda, \lambda^2)$ is an eigenvector of $C(p)$.

41. Find a nontriangular matrix with eigenvalues 4 and −5. (*Hint:* Use Exercise 39.)

42. Find a nontriangular matrix with eigenvalues 4, −5, and −2. (*Hint:* Use Exercise 40.)

43. Find a matrix with eigenvectors $(1, 2, 4)$, $(1, 3, 9)$, and $(1, 4, 16)$. (*Hint:* Consider the companion matrix of $(x - 2)(x - 3)(x - 4)$ and use Exercise 40.)

44. Prove by induction that the companion matrix $C(p)$ of $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has the characteristic polynomial $(-1)^n p(\lambda)$.

Eigenvalues of linear operators

45. Find the eigenvalues and eigenvectors of the projection p of $\mathbf{R}^3$ onto the xy-plane.

46. Find the eigenvalues and eigenvectors of the linear transformation that takes the unit square to the rectangle shown in Figure 7.7.

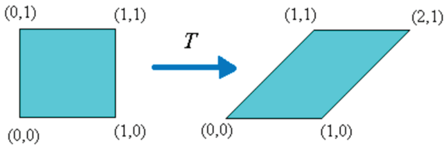

Figure 7.7: Eigenvalues of transformation.

47. Find the eigenvalues and eigenvectors of $T : P_2 \longrightarrow P_2$, $T(a + bx) = b + ax$.

48. (*Requires calculus*) Find the eigenvalues and eigenvectors of differentiation $\frac{d}{dx} : P_2 \longrightarrow P_2$.

7.2 Diagonalization

Matrix arithmetic with diagonal matrices is easier than with any other matrices. This is most notable in matrix multiplication. For example, a diagonal matrix D does not mix the rows of A in a product DA (or columns in AD):

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ 3d & 3e & 3f \end{bmatrix}.$$

Moreover, it is easy to compute the powers D^k:

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}.$$

In this section, we study matrices that can be transformed to diagonal matrices and use the advantageous arithmetic. Eigenvalues provide criteria that identify these matrices.

7.2.1 Diagonalization of matrices

Recall from Section 5.3 that an $n \times n$ matrix A is *similar* to an $n \times n$ matrix B if there is an invertible matrix P such that $P^{-1}AP = B$.

Definition 7.2.1. If an $n \times n$ matrix A is similar to a *diagonal* matrix D, then it is called *diagonalizable*. We also say that A *can be diagonalized*. This means that there exists an invertible $n \times n$ matrix P such that $P^{-1}AP$ is a diagonal matrix D,

$$P^{-1}AP = D.$$

The process of finding matrices P and D is called *diagonalization*. We say that P and D *diagonalize* A.

The answer of how to diagonalize a matrix is provided in the next theorem.

Theorem 7.2.2 (Criterion for diagonalization). *Let A be an $n \times n$ matrix.*

1. *A is diagonalizable if and only if it has n linearly independent eigenvectors.*
2. *If A is diagonalizable with $P^{-1}AP = D$, then the columns of P are eigenvectors of A, and the diagonal entries of D are the corresponding eigenvalues (Figure 7.8).*
3. *If $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are linearly independent eigenvectors of A with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, then A can be diagonalized by*

$$P = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n] \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}.$$

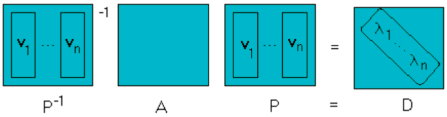

Figure 7.8: Diagonalization of A.

Proof. Let P be any matrix with columns any n-vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, and let D be any diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$. Then

$$AP = A[\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n] = [A\mathbf{v}_1\ A\mathbf{v}_2\ \cdots\ A\mathbf{v}_n] \tag{7.5}$$

and

$$[\lambda_1\mathbf{v}_1\ \lambda_2\mathbf{v}_2\ \cdots\ \lambda_n\mathbf{v}_n] = [\mathbf{v}_1\ \mathbf{v}_2\ \cdots\ \mathbf{v}_n]\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = PD. \tag{7.6}$$

If A is diagonalizable by $P^{-1}AP = D$, then $AP = PD$. So $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$, $i = 1, \dots, n$, by (7.5) and (7.6). Therefore the λ_is are eigenvalues of A and the $\mathbf{v}_i$s are the corresponding eigenvectors. This proves Part 2. It also proves the direct implication in Part 1, because P is invertible, and therefore its columns are linearly independent.

Now suppose that A has n linearly independent eigenvectors, say, $\mathbf{v}_1, \dots, \mathbf{v}_n$ (the columns of P). If $\lambda_1, \dots, \lambda_n$ are the corresponding eigenvalues, then $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$, $i = 1, \dots n$. If D is diagonal with diagonal entries $\lambda_1, \dots, \lambda_n$, then $AP = PD$ by (7.5) and (7.6). Because P is square with linearly independent columns, it is invertible. Hence $P^{-1}AP = D$, and A is diagonalizable. This proves Part 3 and the converse implication in Part 1. □

Theorem 7.2.3. *Let A be an $n \times n$ matrix. The following are equivalent.*

1. *A is diagonalizable.*
2. *$\mathbf{R}^n$ has a basis of eigenvectors of A.*

Proof. This is implied by Theorem 7.2.2, since n linearly independent n-vectors form a basis of $\mathbf{R}^n$. □

Example 7.2.4. Check that $A = \begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$ is diagonalizable and find P and D that diagonalize it.

Solution. We compute to get $\lambda_1 = -3$, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 3$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. We have two linearly independent eigenvectors, so A is diagonalizable by Theorem 7.2.2. We may use

$$P = \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Although not necessary, we can verify that $P^{-1}AP = D$.

If A is diagonalizable, then P and D are not unique. They depend on the choice of basic eigenvectors and the order of the eigenvalues and eigenvectors.

Example 7.2.5. Is

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

diagonalizable? If it is, then find P and D that diagonalize it.

Solution. In Example 7.1.10, we found that $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$, and

$$E_0 = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}, \quad E_1 = \text{Span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}.$$

We see that A has three linearly independent eigenvectors, so it is diagonalizable by Theorem 7.2.2. We may take

$$P = \begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & 1 & 0\end{bmatrix}, \quad D = \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}.$$

Theorem 7.2.6. *Let $\lambda_1, \dots, \lambda_l$ be any distinct eigenvalues of an $n \times n$ matrix A.*

1. *Then any corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_l$ are linearly independent.*
2. *If $\mathcal{B}_1, \dots, \mathcal{B}_l$ are bases for the corresponding eigenspaces, then*

   $$\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_l$$

   is linearly independent.
3. *Let l be the number of* all *distinct eigenvalues of A. Then A is diagonalizable if and only if $\mathcal{B}$ in Part 2 has exactly n elements.*

Proof. 1. If the $\mathbf{v}$s are not linearly independent, let $\mathbf{v}_k$ be the first that can be written as a linear combination of the preceding ones. So we have

$$\mathbf{v}_k = a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1} \tag{7.7}$$

with scalars a_i not all zero and $\mathbf{v}_1, \dots, \mathbf{v}_{k-1}$ linearly independent. We multiply on the left by A to get

$$\begin{aligned} A\mathbf{v}_k &= A(a_1\mathbf{v}_1 + \cdots + a_{k-1}\mathbf{v}_{k-1}) \\ &= a_1A\mathbf{v}_1 + \cdots + a_{k-1}A\mathbf{v}_{k-1}. \end{aligned}$$

Hence

$$\lambda_k\mathbf{v}_k = a_1\lambda_1\mathbf{v}_1 + \cdots + a_{k-1}\lambda_{k-1}\mathbf{v}_{k-1} \tag{7.8}$$

Now multiplying (7.7) by $-\lambda_k$ and adding it to (7.8), we get

$$a_1(\lambda_1 - \lambda_k)\mathbf{v}_1 + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k)\mathbf{v}_{k-1} = \mathbf{0}.$$

Therefore $a_1(\lambda_1 - \lambda_k) = \cdots = a_{k-1}(\lambda_{k-1} - \lambda_k) = 0$ by the linear independence of $\mathbf{v}_1, \dots, \mathbf{v}_{k-1}$. One of the as, say, a_i, is nonzero, so $\lambda_i - \lambda_k = 0$ or $\lambda_i = \lambda_k$, which contradicts the assumption that the λs are distinct. We conclude that all $\mathbf{v}_1, \dots, \mathbf{v}_l$ must be linearly independent.

2. For simplicity, we consider two distinct eigenvalues λ_1, λ_2 and two bases $\mathcal{B}_1 = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ and $\mathcal{B}_2 = \{\mathbf{w}_1, \dots, \mathbf{w}_q\}$ for E_{λ_1} and E_{λ_2}, the general case being similar. We prove that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent. Let

$$c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p + d_1\mathbf{w}_1 + \cdots + d_q\mathbf{w}_q = 0.$$

Then $\mathbf{u} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$ is either an eigenvector of λ_1 or zero. Likewise, $\mathbf{w} = d_1\mathbf{w}_1 + \cdots + d_q\mathbf{w}_q$ is either an eigenvector of λ_2 or zero. In addition, $\mathbf{u} + \mathbf{v} = \mathbf{0}$. If both $\mathbf{u}$ and $\mathbf{w}$ were eigenvectors, then they would be linearly independent by Part 1. This contradicts that their sum is zero. We conclude that

$$\mathbf{u} = \mathbf{w} = \mathbf{0}.$$

Hence $c_1 = \cdots = c_p = 0$ and $d_1 = \cdots = d_q = 0$ by the linear independence of $\mathcal{B}_1$ and $\mathcal{B}_2$. Therefore $\mathcal{B}$ is linearly independent.

3. If $\mathcal{B}$ has n vectors, then they are linearly independent by Part 2. Hence A is diagonalizable. Conversely, if A is diagonalizable, then it has n linearly independent eigenvectors. If exactly n_i of these eigenvectors correspond to the eigenvalue λ_i, then $\mathcal{B}_i$ has at least n_i elements, because the eigenvectors are linearly independent. We conclude that $\mathcal{B}$ has at least n and hence exactly n elements. ☐

Example 7.2.7. Is $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$ diagonalizable?

Solution. In Example 7.1.11, we found that $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -2$, and

$$E_0 = \text{Span}\left\{\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}\right\}, \quad E_{-2} = \text{Span}\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right\}.$$

This time, A has at most 2 (< 3) linearly independent eigenvectors, so it is *not* diagonalizable by Part 2 of Theorem 7.2.6.

We can now draw some interesting corollaries from Theorem 7.2.6.

Theorem 7.2.8. *Any $n \times n$ matrix A with n distinct eigenvalues is diagonalizable.*

Proof. By Theorem 7.2.2 it suffices to show that the corresponding eigenvectors are linearly independent. But this is guaranteed by Part 1 of Theorem 7.2.6. ☐

A diagonalizable matrix need not have distinct eigenvalues, as Example 7.2.5 shows.

Theorem 7.2.9. *A matrix A is diagonalizable if and only for each eigenvalue λ, the geometric and algebraic multiplicities of λ are equal.*

Proof. Exercise. ☐

7.2.2 Powers of diagonalizable matrices

Let A be a diagonalizable $n \times n$ matrix diagonalized by P and D, so $A = PDP^{-1}$. Squaring yields $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$. We iterate to get $A^k = PD^kP^{-1}$. So finding A^k becomes easy *given* P, P^{-1}, and D.

If A is invertible, then 0 is not an eigenvalue of A by Theorem 7.1.14. Therefore D^{-1} exists, and we have $A^{-1} = (PDP^{-1})^{-1} = PD^{-1}P^{-1}$. Again, we may iterate to get $A^{-k} = PD^{-k}P^{-1}$. We have proved the following theorem.

Theorem 7.2.10. *If A is diagonalized by P and D, then for $k = 0, 1, 2, \ldots$, we have*

$$A^k = PD^kP^{-1}. \tag{7.9}$$

If in addition, A is invertible, then equation (7.9) is also valid for $k = -1, -2, -3, \ldots$.

Example 7.2.11. Find a formula for A^k, $k = 0, 1, 2, \ldots$, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{bmatrix}.$$

Solution. The matrix A has eigenvalues $0, 2, 4$, and the corresponding basic eigenvectors $(-1, 0, 1)$, $(0, 1, 0)$, $(1, 0, 3)$ are linearly independent. Hence by Theorem 7.2.10

$$A^k = PD^kP^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 4^k \end{bmatrix} \begin{bmatrix} -\frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 4^{k-1} & 0 & 4^{k-1} \\ 0 & 2^k & 0 \\ 3 \cdot 4^{k-1} & 0 & 3 \cdot 4^{k-1} \end{bmatrix}.$$

7.2.3 An important change of variables

Let us now discuss an idea that is in the core of most applications of diagonalization. Let A be diagonalizable, diagonalized by P and D. Often a matrix–vector equation $f(A, \mathbf{x}) = \mathbf{0}$ can be substantially simplified if we replace $\mathbf{x}$ by the new vector $\mathbf{y}$ such that

$$\mathbf{x} = P\mathbf{y} \quad \text{or} \quad \mathbf{y} = P^{-1}\mathbf{x} \tag{7.10}$$

and replace A with PDP^{-1} to get an equation of the form $g(D, \mathbf{y}) = \mathbf{0}$, which involves the diagonal matrix D and the new vector $\mathbf{y}$.

To illustrate, suppose we have a linear system $A\mathbf{x} = \mathbf{b}$. Then we can convert this system into a diagonal system as follows. Let $\mathbf{y}$ be the new variable vector defined by $\mathbf{y} = P\mathbf{x}$. We have

$$\begin{aligned} A\mathbf{x} = \mathbf{b} &\Leftrightarrow PA\mathbf{x} = P\mathbf{b} \\ &\Leftrightarrow PAP^{-1}\mathbf{y} = P\mathbf{b} \\ &\Leftrightarrow D\mathbf{y} = P\mathbf{b}. \end{aligned}$$

The last equation defines a diagonal system.

Exercises 7.2

In Exercises 1–5, diagonalize the matrix A if it is diagonalizable, that is, if possible, find P invertible and D diagonal such that $P^{-1}AP = D$.

1. (a) $A = \begin{bmatrix} -2 & 5 \\ 5 & -2 \end{bmatrix}$, (b) $A = \begin{bmatrix} -2 & 0 \\ 5 & -2 \end{bmatrix}$

2. (a) $A = \begin{bmatrix} 11 & 1 & 1 \\ 1 & 11 & 1 \\ 1 & 1 & 11 \end{bmatrix}$, (b) $A = \begin{bmatrix} 8 & 1 & 1 \\ 1 & 8 & 1 \\ 1 & 1 & 8 \end{bmatrix}$.

3. (a) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -2 & 5 \\ 0 & 5 & -2 \end{bmatrix}$, (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 5 \\ 2 & 0 & -2 \end{bmatrix}$.

4. $A = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}$, $b \neq 0$.

5. $A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

In Exercises 6–8, verify that S is a linearly independent set of eigenvectors of matrix A. Diagonalize A by using S.

6. $S = \left\{ \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\}$ and $A = \begin{bmatrix} -3 & 6 \\ 6 & -3 \end{bmatrix}$.

7. $S = \left\{ \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \end{bmatrix} \right\}$ and $A = \begin{bmatrix} -1 & 6 \\ 0 & 4 \end{bmatrix}$.

8. $S = \left\{ \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$ and $A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & 6 & 0 \end{bmatrix}$.

In Exercises 9–11, S is a linearly independent set of eigenvectors of some matrix A, and E is the set of the corresponding eigenvalues. Find A.

9. $S = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $E = \{-10, 12\}$.

10. $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $E = \{1, 2, 3\}$.

11. $S = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $E = \{-2, 2, 3\}$.

12. Suppose that a 3×3 matrix A has eigenvalues $3, 0, -7$. Is A diagonalizable? Why, or why not.

13. Suppose that a 3×3 matrix A is upper triangular with diagonal entries $2, 1, -5$. Prove that A is diagonalizable. What is D?

14. Find a basis of $\mathbf{R}^3$ that consists of eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

15. Find a basis of $\mathbf{R}^3$ that consists of eigenvectors of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 2 \end{bmatrix}.$$

16. Prove that $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix}$ is diagonalizable.

17. Prove that $A = \begin{bmatrix} 2 & 3 & -5 \\ 2 & 3 & -5 \\ 2 & 3 & -5 \end{bmatrix}$ is not diagonalizable.

18. Prove that $A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$ is not diagonalizable.

19. Prove that

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

is not diagonalizable.

20. Prove that

$$A = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

is diagonalizable if and only if $a + b + c \neq 0$. We assume that at least one of a, b, c is nonzero.

21. Let A be diagonalizable. Prove that $\operatorname{tr}(A^2) \geq 0$.

22. For which value(s) of a is the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & a \end{bmatrix}$$

diagonalizable?

23. What restrictions on the real number a make the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & a & 0 \end{bmatrix}$$

diagonalizable with D and P real matrices?

24. Prove that the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ a & 0 & 0 \end{bmatrix}$$

is diagonalizable for all real a.

25. Use diagonalization to compute A^6 and A^9, where

$$A = \begin{bmatrix} 0 & 8 \\ 2 & 0 \end{bmatrix}.$$

26. Find a formula for A^n, if A is as in Exercise 25.

27. Use diagonalization to compute

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}^7.$$

28. Use diagonalization to compute

$$\begin{bmatrix} -2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix}^{-n}.$$

29. Use diagonalization to prove that for $k = 0, 1, 2, \dots$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 2^k & 2^k \\ 2^k & 2^k \end{bmatrix}.$$

30. Use diagonalization to prove that for $k = 0, 1, 2, \dots$,

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 2^k & -2^k \\ -2^k & 2^k \end{bmatrix}.$$

31. Use diagonalization to prove that for $k = 0, 1, 2, \dots$,

$$\begin{bmatrix} 7 & 8 & 9 \\ 7 & 8 & 9 \\ 7 & 8 & 9 \end{bmatrix}^{k+1} = (7 + 8 + 9)^k \begin{bmatrix} 7 & 8 & 9 \\ 7 & 8 & 9 \\ 7 & 8 & 9 \end{bmatrix}.$$

32. Prove the identity

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} r_1^{k+1} - r_2^{k+1} \\ r_1^k - r_2^k \end{bmatrix},$$

where $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$, and k is a positive integer.

33. Prove Theorem 7.2.9.

7.3 Applications: Discrete dynamical systems

One of the important applications of eigenvalues and eigenvectors is in the theory of discrete dynamical systems. These were first introduced in Section 1.3.[1]

7.3.1 Basic Concepts

Definition 7.3.1. A *dynamical system* or *difference equation* is an equation involving a time-dependent vector quantity $\mathbf{x}(t)$. In a *discrete dynamical system* the time variable is an integer k, and we write $\mathbf{x}_k$ for $\mathbf{x}(k)$. A *first-order discrete homogeneous dynamical system* is a vector equation of the form

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \tag{7.11}$$

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[1] A recommended book on dynamical systems is James T. Sandefur's *Discrete Dynamical Systems, Theory and Applications*, Clarendon Press, Oxford, 1990.

where A is a fixed square matrix of size matching that of the vector $\mathbf{x}_k$. We only consider matrices A with *real* entries that do *not* depend on k.

Equation (7.11) gives the next value of $\mathbf{x}$ in terms of the current value. We can compute $\mathbf{x}_k$ by repeated applications of (7.11):

$$\mathbf{x}_k = A\mathbf{x}_{k-1} = A^2\mathbf{x}_{k-2} = \cdots .$$

So the kth vector is given by

$$\mathbf{x}_k = A^k\mathbf{x}_0. \tag{7.12}$$

Equation (7.12) is called the *solution* of the dynamical system. It gives $\mathbf{x}_k$ in terms of the initial vector $\mathbf{x}_0$.

The calculation of $\mathbf{x}_k$ by (7.12) has a practical flow: the computation of A^k may be expensive. Besides, we are often interested in the *long-term behavior* of the system, that is, in the limit vector

$$\lim_{k\to\infty} \mathbf{x}_k = \lim_{k\to\infty} A^k\mathbf{x}_0$$

if it exists. Here we take the limit of a sequence of vectors by taking the limits of the individual coordinates.

Suppose we can write the vector $\mathbf{x}_0$ as a linear combination of eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ of A, say,

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues. By Exercise 22, Section 7.1, $A^k\mathbf{v}_i = \lambda^k\mathbf{v}_i$. Hence

$$\begin{aligned} A^k\mathbf{x}_0 &= A^k\left(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n\right) \\ &= c_1A^k\mathbf{v}_1 + \cdots + c_nA^k\mathbf{v}_n \\ &= c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_n\lambda_n^k\mathbf{v}_n. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{x}_0 &= c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n \\ &\Rightarrow A^k\mathbf{x}_0 = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_k\lambda_n^k\mathbf{v}_n. \end{aligned} \tag{7.13}$$

So the solution of the system simplifies to

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_k\lambda_n^k\mathbf{v}_n. \tag{7.14}$$

Equation (7.14) involves *no matrix powers*, and its right-hand side is easy to compute, provided that we know the coefficients c_i, eigenvectors $\mathbf{v}_i$, and eigenvalues λ_i.

In the special case where the matrix A is *diagonalizable*, this method applies to *any* initial n-vector $\mathbf{x}_0$. Because then $\mathbf{R}^n$ has a basis of eigenvectors of A, any n-vector is a linear combination of the eigenvectors. We have proved the following theorem.

Theorem 7.3.2. *Let A be an $n \times n$ diagonalizable matrix with linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ and corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Then the solution of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ with initial vector $\mathbf{x}_0$ is given by*

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_n\lambda_n^k\mathbf{v}_n,$$

where the coefficients $c_1, \dots, c_n$ are such that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$

7.3.2 Long-term behavior of dynamical systems

Definition 7.3.3. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. An eigenvalue, say, λ_1, is called the *dominant eigenvalue* if

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n| .$$

For example, if A has eigenvalues $1, 4, -5$, then -5 is the dominant eigenvalue. If A has eigenvalues $1, 4, -4$, then there is *no* dominant eigenvalue, because there are two eigenvalues with maximum absolute value $|4| = |-4|$.

Note that a dominant eigenvalue cannot be zero. Also, it must be *real* if the matrix is real. This is because complex eigenvalues of a real matrix come in conjugate pairs and conjugate numbers have the same absolute values. The following theorem explains our interest in matrices with a dominant eigenvalue.

Theorem 7.3.4 (The power method). *Let A be an $n \times n$ diagonalizable matrix with basic eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, and a dominant eigenvalue, say, λ_1. Let $\mathbf{x}$ be a vector that is a linear combination of $\mathbf{v}_i$s,*

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n,$$

so that $c_1 \neq 0$. Then as k grows, a scalar multiple of $A^k\mathbf{x}$ approaches a scalar multiple of $\mathbf{v}_1$. In particular, the direction of $A^k\mathbf{x}$ approaches that of $\mathbf{v}_1$.

Proof. By using (7.13) we have

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_n\lambda_n^k\mathbf{v}_n.$$

Because $\lambda_1 \neq 0$, we can divide by λ_1^k:

$$(1/\lambda_1^k)A^k\mathbf{x} = c_1\mathbf{v}_1 + c_2(\lambda_2/\lambda_1)^k\mathbf{v}_2 + \cdots + c_n(\lambda_n/\lambda_1)^k\mathbf{v}_n. \tag{7.15}$$

Since λ_1 is dominant, we have $|\lambda_i/\lambda_1| < 1$, $i = 2, \ldots, n$. Since $r^k \to 0$ as $k \to \infty$ if $|r| < 1$, taking $r = \lambda_i/\lambda_1$, we conclude that

$$(1/\lambda_1^k)A^k\mathbf{x} \to c_1\mathbf{v}_1 \quad \text{as } k \to \infty.$$

So $A^k\mathbf{x}$ scaled by $1/\lambda_1^k$ becomes parallel to $\mathbf{v}_1$ for large k, because $c_1 \neq 0$. □

Numerical note

As we mentioned in Section 7.1, the characteristic equation is an inefficient way of computing eigenvalues of large matrices. The power method theorem is the backbone for many different approximation schemes of eigenvectors and eigenvalues. We start with an initial guess $\mathbf{x}_0$ for an eigenvector. Then we compute the vectors $A^k\mathbf{x}_0$ for large k. These vectors approximate an eigenvector $\mathbf{v}$ of the dominant eigenvalue. Once we have an approximation for $\mathbf{v}$, we find $A\mathbf{v}$, which should approximately be a scalar product of $\mathbf{v}$. The scaling factor is an approximation of the dominant eigenvalue. Approximations of eigenvalues are studied in detail in Section 7.6.

Example 7.3.5. Let

$$A = \begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 5 \\ 15 \end{bmatrix}.$$

(a) Find the eigenvalues and basic eigenvectors.
(b) Use the power method theorem to approximate an eigenvector of A.
(c) Use Part (b) to approximate the dominant eigenvalue.

Solution.

(a) The eigenvalues of A are 2 and -3 with corresponding basic eigenvectors $\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$. The dominant eigenvalue is -3.

(b) By the power method theorem we know that $A^k\mathbf{x}_0 = \mathbf{x}_k$ will have direction approaching the direction of $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$, the eigenvector corresponding to $\lambda = -3$. For example, for $k = 11$, we have

$$\mathbf{x}_{11} = A^{11}\mathbf{x}_0 = \begin{bmatrix} -1781\,710 \\ -865\,255 \end{bmatrix} \quad \text{with component ratio} \simeq 2.0592.$$

This is approximately a scalar multiple of the true eigenvector $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$.

(c) The dominant eigenvalue is found from the fact that the product $A\mathbf{x}_{11}$ must be approximately a scalar product of $\mathbf{x}_{11}$. So

$$\begin{bmatrix} -2 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1781\,710 \\ -865\,255 \end{bmatrix} = \begin{bmatrix} 5293\,930 \\ 2698\,165 \end{bmatrix},$$

and the ratio $5293\,930/(-1781\,710.) = -2.971\,3$ yields an approximation of the dominant eigenvalue. The ratio $2698\,165/(-865\,255.) = -3.118\,3$ is another approximation. In Figure 7.9, we sketch the points $\mathbf{x}_0, \ldots, \mathbf{x}_k$ for $k = 2, 3, 6$. Consecutive points are joined by a straight line segment.

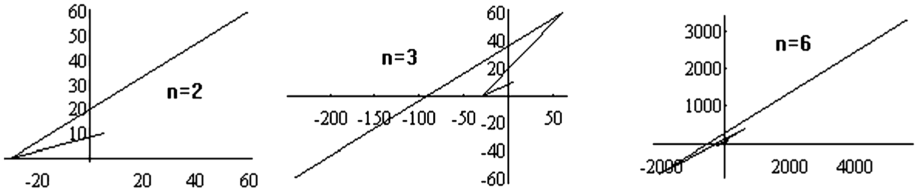

Figure 7.9: Iterative approximation of the dominant eigenvalue eigenvector.

Attractors, repellers, and saddle points

In the following examples, we discuss the long-term behavior and plot some of the solutions of the dynamical system for the given matrices A. All plots include $(\pm 1, \pm 1)$, $(\pm 1, \pm 2)$, $(\pm 2, \pm 1)$ as starting points. Consecutive points $\mathbf{x}_k$, $\mathbf{x}_{k+1}$ are joined by straight line segments. The resulting polygonal lines are the *trajectories* of the solutions.

Example 7.3.6. Consider the diagonal matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

Discuss the long-term behavior of the dynamical systems

(a) $\mathbf{x}_{k+1} = A^k \mathbf{x}_0$,

(b) $\mathbf{x}_{k+1} = B^k \mathbf{x}_0$.

Solution.

(a) The matrix A has eigenvalues 3 and 2 and corresponding eigenvectors $(0, 1)$ and $(1, 0)$. Hence, if $\mathbf{x}_0 = c_1(0, 1) + c_2(1, 0)$, then

$$\mathbf{x}_k = c_1 3^k (0, 1) + c_2 2^k (1, 0) = (2^k c_2, 3^k c_1). \tag{7.16}$$

Therefore the components of $\mathbf{x}_k$ go to $\pm\infty$ depending on the signs of c_1 and c_2. By Theorem 7.3.4 the direction of $\mathbf{x}_k$ approaches that of $(0, 1)$ if $c_1 \neq 0$. In the long run, all trajectories tend to become parallel to the y-axis and move *away* from the origin, except for the points of the x-axis (where $c_1 = 0$). These points remain on the

x-axis and move toward $\pm\infty$, depending on the sign of c_1. In Figure 7.10a, we see trajectories up to $k = 2$.

(b) The matrix B has eigenvalues are 0.3 and 0.2 and corresponding eigenvectors $(0, 1)$ and $(1, 0)$. So, just as in Part (a), we have

$$\mathbf{x}_k = (0.2^k c_2, 0.3^k c_1) \quad \text{with } \mathbf{x}_0 = (c_2, c_1).$$

Both components of $\mathbf{x}_k$ go to zero. By Theorem 7.3.4 the direction of $\mathbf{x}_k$ approaches that of $(0, 1)$ for large k. So the trajectories become parallel to the y-axis and move toward the origin, except for the points of the x-axis (where $c_1 = 0$). These points remain on the x-axis and move toward the origin. (Figure 7.10b, $k = 2$).

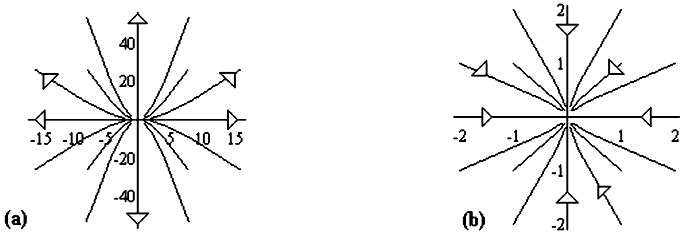

Figure 7.10: Trajectories for the dynamical systems of Example 7.3.6.

Example 7.3.7. Let

$$A = \begin{bmatrix} 2.5 & 0.5 \\ 0.5 & 2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}.$$

Discuss the long-term behavior of the dynamical systems

(a) $\mathbf{x}_{k+1} = A\mathbf{x}_k$,

(b) $\mathbf{x}_{k+1} = B\mathbf{x}_k$.

Solution.

(a) The eigenvalues of A are 3 and 2 with corresponding eigenvectors $(1, 1)$ and $(-1, 1)$. Hence, for $\mathbf{x}_0 = c_1(1, 1) + c_2(-1, 1)$, we have

$$\mathbf{x}_k = c_1 3^k (1, 1) + c_2 2^k (-1, 1) = (c_1 3^k - c_2 2^k, c_1 3^k + c_2 2^k). \tag{7.17}$$

As k becomes large, $\mathbf{x}_k$ approaches (∞, ∞) if $c_1 > 0$ and approaches $(-\infty, -\infty)$ if $c_1 < 0$. So if $c_1 \neq 0$, then eventually all the trajectories end up in the *first* or *third* quadrants. By Theorem 7.3.4 the long-term direction of $\mathbf{x}_k$ is parallel to $(1, 1)$, with the exception of the points $\mathbf{x}_0$ with $c_1 = 0$. These are the points of the line l through the origin and $(-1, 1)$, which is the eigenspace E_2. They remain on l, and their distance from the origin increases with k, because

$$\mathbf{x}_k = (-c_2 2^k, c_2 2^k) = 2^k c_2(-1, 1)$$

for $c_1 = 0$ (Figure 7.11a, $k = 2$).

(b) For the matrix B, we have the same eigenvectors and eigenvalues 0.6 and 0.4. The trajectories again tend to become parallel to $(1, 1)$ but move toward the origin, except for the points of the line l (where $c_1 = 0$). These points remain on l and also move toward the origin (Figure 7.11b, $k = 4$).

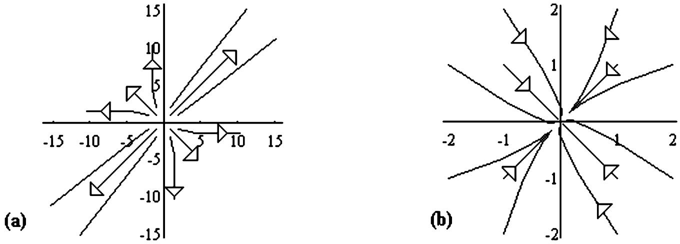

Figure 7.11: Trajectories for the dynamical systems of Example 7.3.7.

In Examples 7.3.6 and 7.3.7, we saw that if all eigenvalues have absolute value less than 1, then all trajectories approach the origin. In such a case, we say that the origin is an *attractor*. This is true in general, because each term of $\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_n\lambda_n^k\mathbf{v}_n$ would approach zero if all $|\lambda_i| < 1$. If, on the other hand, all eigenvalues have absolute value greater than 1, then all trajectories move away from the origin. We then say that the origin is a *repeller*. If, finally, some trajectories move toward the origin and some move away from it, then we say that the origin is a *saddle point*.

Note that the trajectories become parallel to the eigenvector with the largest absolute value eigenvalue, with the exception of the points along the line of the other eigenvector, which remain on that line. The same is true (with zig-zagging) if one or both eigenvalues are negative. The reader can see this by drawing a few trajectories of

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & -2 \\ 1 & -4 \end{bmatrix}.$$

Let us now look at the case where one eigenvalue has the absolute value less than 1 and one greater that 1.

Example 7.3.8. Answer the questions of Example 7.3.7 for the matrices

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$

Solution. The eigenvalues for both matrices are 0.5 and 1.5. The corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are $(0,1)$ and $(1,0)$ for the first matrix and $(1,1)$ and $(-1,1)$ for the second matrix. Hence

$$\mathbf{x}_k = c_1(1.5)^k(0,1) + c_2(0.5)^k(1,0), \quad \text{where } \mathbf{x}_0 = c_1(0,1) + c_2(1,0),$$

for the first matrix, and

$$\mathbf{x}_k = c_1(1.5)^k(1,1) + c_2(0.5)^k(-1,1), \quad \text{where } \mathbf{x}_0 = c_1(1,1) + c_2(-1,1),$$

for the second matrix. In each case, $(0.5)^k \to 0$ and $(1.5)^k \to \infty$ as $k \to \infty$, so if c_1 is not zero, then all trajectories become parallel to $\mathbf{v}_1$. If c_1 is zero, then we have vectors along the direction of $\mathbf{v}_2$, and their trajectories go to zero. Figure 7.12a displays trajectories for the first system ($k = 4$), and Figure 7.12b for the second ($k = 3$). This time the origin is a *saddle* point.

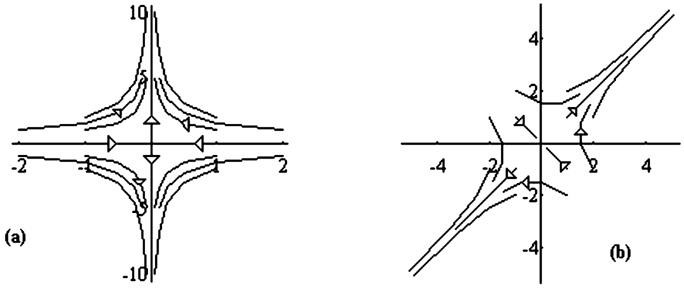

Figure 7.12: Trajectories for the dynamical systems of Example 7.3.8.

Repeated eigenvalue

If a 2×2 matrix has only one eigenvalue λ with two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then for $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, we have

$$\begin{aligned}\mathbf{x}_k &= c_1\lambda^k\mathbf{v}_1 + c_2\lambda^k\mathbf{v}_2 \\ &= \lambda^k(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) \\ &= \lambda^k\mathbf{x}_0.\end{aligned}$$

Hence $\mathbf{x}_k$ and $\mathbf{x}_0$ are on the same line.

Figure 7.13 illustrates this observation for the matrices

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

For the first matrix, the origin is a repeller and for the second, an attractor.

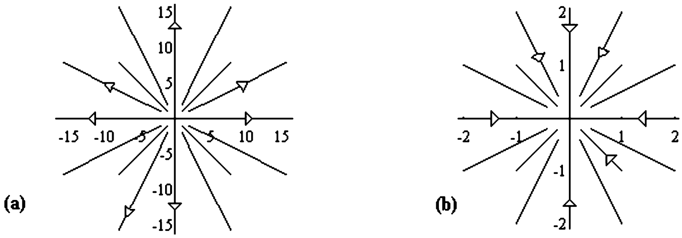

Figure 7.13: Cases of a repeated eigenvalue.

7.3.3 Uncoupling dynamical systems

The graphs of Example 7.3.7 are similar to those of Example 7.3.6, in which the matrices were diagonal. The roles of the axes in Example 7.3.7 are played by the *eigenspaces*. The graphs for the nondiagonal matrices can be obtained from the those of the diagonal ones by the *change of variables* discussed in Section 7.2.

Let A be an $n \times n$ diagonalizable matrix, and let P be the matrix with columns the elements of a basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of eigenvectors of A. We define new variables $\mathbf{y}_k$ by

$$\mathbf{x}_k = P\mathbf{y}_k. \tag{7.18}$$

So $\mathbf{y}_k = [\mathbf{x}_k]_{\mathcal{B}}$. In this case the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ can be rewritten as

$$P\mathbf{y}_{k+1} = AP\mathbf{y}_k \Rightarrow \mathbf{y}_{k+1} = P^{-1}AP\mathbf{y}_k.$$

Therefore

$$\mathbf{y}_{k+1} = D\mathbf{y}_k, \tag{7.19}$$

where D is the diagonal matrix with diagonal entries the eigenvalues of A. The change of variables (7.18) transforms $\mathbf{x}_{k+1} = A\mathbf{x}_k$ into a *diagonal* or *uncoupled* dynamical system (7.19). In uncoupled dynamical systems the components of the unknowns vectors are not mixed during iterations. This means that the ith component of $\mathbf{y}_{k+1}$ depends only on the ith component of $\mathbf{y}_k$. It is always advantageous to work with uncoupled systems.

To illustrate these observations, let

$$A = \begin{bmatrix} 2.5 & 0.5 \\ 0.5 & 2.5 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

and let $\mathbf{x}_k$ be as in equation (7.17). We have

$$\mathbf{y}_k = P^{-1}\mathbf{x}_k = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} c_1 3^k - c_2 2^k \\ c_1 3^k + c_2 2^k \end{bmatrix} = \begin{bmatrix} c_2 2^k \\ c_1 3^k \end{bmatrix},$$

which is formula (7.16) of Example 7.3.6.

Exercises 7.3

In Exercises 1–4, consider the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ with given matrix A and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find $\mathbf{x}_5$ by using

(a) $A^5\mathbf{x}_0$,
(b) eigenvalues.

1. $A = \begin{bmatrix} 7 & 6 \\ 4 & 5 \end{bmatrix}$.

2. $A = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$.

3. $A = \begin{bmatrix} -6 & 5 \\ 2 & -3 \end{bmatrix}$.

4. $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$.

In Exercises 5–11, suppose that a matrix A has eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with corresponding given eigenvalues λ_1 and λ_2. Consider the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ with initial vector $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

(a) Find a formula for $\mathbf{x}_k$.
(b) Compute $A\mathbf{x}_0$ and $A^2\mathbf{x}_0$.
(c) Indicate whether the origin is an attractor, repeller, or neither.

5. $\lambda_1 = 1, \lambda_2 = 5$.

6. $\lambda_1 = 2, \lambda_2 = 10$.

7. $\lambda_1 = -7, \lambda_2 = -1$.

8. $\lambda_1 = 1, \lambda_2 = 9$.

9. $\lambda_1 = 2, \lambda_2 = 14$.

10. $\lambda_1 = -13, \lambda_2 = -1$.

11. $\lambda_1 = -1/10, \lambda_2 = 1/2$.

In Exercises 12–16, indicate whether the origin is an attractor, repeller, or neither for the dynamical system.

12. $\mathbf{x}_{k+1} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}\mathbf{x}_k$.

13. $\mathbf{x}_{k+1} = \begin{bmatrix} 12 & 0 & -10 \\ 0 & 8 & 0 \\ -4 & 0 & 6 \end{bmatrix}\mathbf{x}_k$.

14. $\mathbf{x}_{k+1} = \begin{bmatrix} -\frac{3}{4} & 0 & \frac{5}{8} \\ 0 & -\frac{1}{2} & 0 \\ \frac{1}{4} & 0 & -\frac{3}{8} \end{bmatrix}\mathbf{x}_k$.

15. $\mathbf{x}_{k+1} = \begin{bmatrix} 6 & 0 & -4 \\ 0 & 8 & 0 \\ -4 & 0 & 6 \end{bmatrix} \mathbf{x}_k.$

16. $\mathbf{x}_{k+1} = \begin{bmatrix} 3 & 0 & -2 \\ 0 & -3 & 0 \\ -2 & 0 & 3 \end{bmatrix} \mathbf{x}_k.$

In Exercises 17–21, perform a change of variables to uncouple the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, that is, transform the dynamical system into one of the form $\mathbf{y}_{k+1} = D\mathbf{y}_k$, where D is a diagonal matrix.

17. $\mathbf{x}_{k+1} = \begin{bmatrix} -3 & -2 \\ -2 & -3 \end{bmatrix} \mathbf{x}_k.$

18. $\mathbf{x}_{k+1} = \begin{bmatrix} 10 & -8 \\ -8 & 10 \end{bmatrix} \mathbf{x}_k.$

19. $\mathbf{x}_{k+1} = \begin{bmatrix} -3 & 0 & -5 \\ 0 & -2 & 0 \\ -2 & 0 & -6 \end{bmatrix} \mathbf{x}_k.$

20. $\mathbf{x}_{k+1} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \mathbf{x}_k.$

21. $\mathbf{x}_{k+1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x}_k.$

7.4 Applications: Dynamical systems (2) and Markov chains

In this section, we continue our study of discrete dynamical systems from Section 7.2.3. We also continue the study of Markov chains and stochastic matrices introduced in Section 3.6.

7.4.1 Dynamical systems with complex eigenvalues

If A has complex eigenvalues, then the trajectories typically spiral around the origin toward or away from it depending on whether the magnitudes of the eigenvalues are less than 1 or greater than 1. Sometimes, the trajectories circle around the origin.

Example 7.4.1. Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Discuss the long-term behavior of the dynamical systems

(a) $\mathbf{x}_{k+1} = A\mathbf{x}_k$,

(b) $\mathbf{x}_{k+1} = B\mathbf{x}_k$.

Solution.

(a) The eigenvalues of A are $1+i$ and $1-i$ with eigenvectors $\mathbf{v}_1 = (i, 1)$ and $\mathbf{v}_2 = (-i, 1)$. Hence, if $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then $\mathbf{x}_k$ is of the form

$$\mathbf{x}_k = c_1\,(1+i)^k \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2\,(1-i)^k \begin{bmatrix} -i \\ 1 \end{bmatrix}. \tag{7.20}$$

The components of $\mathbf{x}_k$ are real, so the right-hand side of (7.20) must have only real entries. Let us look at the trajectory that starts at $(1, 1)$. The relation

$$(1,1) = c_1 \begin{bmatrix} i \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

implies that $c_1 = \frac{1}{2} - \frac{1}{2}i$ and $c_2 = \frac{1}{2} + \frac{1}{2}i$, so that we have

$$\mathbf{x}_k = \left(\frac{1}{2} - \frac{1}{2}i\right)(1+i)^k \begin{bmatrix} i \\ 1 \end{bmatrix} + \left(\frac{1}{2} + \frac{1}{2}i\right)(1-i)^k \begin{bmatrix} -i \\ 1 \end{bmatrix}. \tag{7.21}$$

Hence, for $k = 1, 2, 3, 4, \ldots$, we get

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ -4 \end{bmatrix}, \quad \ldots .$$

These vectors are of increasing magnitude and spiral away form the origin (Figure 7.14a). This spiral behavior can be in fact predicted from (7.21), but we skip the details.

(b) For matrix B, a similar calculation yields

$$\begin{aligned} \mathbf{x}_k = {} & c_1\left(\frac{1}{2} + \frac{1}{2}i\sqrt{3}\right)^k \begin{bmatrix} \frac{1}{2} - \frac{1}{2}i\sqrt{3} \\ 1 \end{bmatrix} \\ & + c_2\left(\frac{1}{2} - \frac{1}{2}i\sqrt{3}\right)^k \begin{bmatrix} \frac{1}{2} + \frac{1}{2}i\sqrt{3} \\ 1 \end{bmatrix}, \end{aligned} \tag{7.22}$$

and for $\mathbf{x}_0 = (1, 1)$, it is easy to show that $c_1 = \frac{1}{2} + \frac{1}{6}i\sqrt{3}$ and $c_2 = \frac{1}{2} - \frac{1}{6}i\sqrt{3}$. In this case, for $k = 0, \ldots, 6$, we get

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \ldots .$$

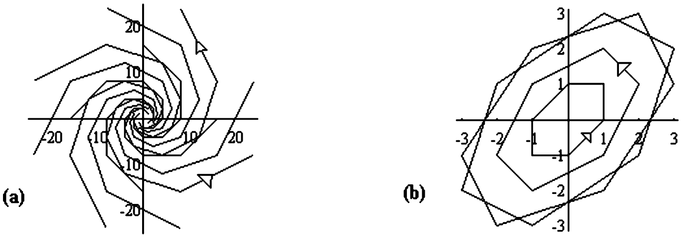

Figure 7.14: Trajectories for the dynamical systems of Example 7.4.1.

Notice that $\mathbf{x}_6$ is the same as $\mathbf{x}_0$. Hence $\mathbf{x}_7$ is the same as $\mathbf{x}_1$, and so on. This time we have a 6-*cycle*, i. e., the vectors are repeated every 6 time units (Figure 7.14b). So, for $k = 0, 1, 2, \ldots,$

$$\mathbf{x}_{k+6} = \mathbf{x}_k.$$

Also, we may write

$$\mathbf{x}_k = \mathbf{x}_r,$$

where r is the remainder of the division of k by 6. For example,

$$\mathbf{x}_{44} = \mathbf{x}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

This cyclical behavior is due to the fact that the eigenvalues are sixth roots of 1. This means that $(\frac{1}{2} \pm \frac{1}{2} i\sqrt{3})^6 = 1$. So we see from (7.22) that the values of $\mathbf{x}_k$ are repeated when k is incremented by 6. This is true for all choices of c_1 and c_2. Note that there is nothing special about 6-cycles. We can also have cases with 2-, 3-, 4-, … cycles.

7.4.2 Application to a population growth problem

Let us apply our new knowledge of the long-term behavior of dynamical systems to the insect population problem introduced in Section 2.7.

Example 7.4.2. If there are initially 1,000 insects in each age group, what is the population distribution in the long run?

Solution. We have the dynamical system

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

with

$$A = \begin{bmatrix} \frac{2}{5} & 4 & 5 \\ \frac{1}{10} & 0 & 0 \\ 0 & \frac{2}{5} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_k = \begin{bmatrix} A_k \\ B_k \\ C_k \end{bmatrix}, \quad k = 0, 1, 2, \dots .$$

Therefore

$$\mathbf{x}_{k+1} = A^k \mathbf{x}_0.$$

The eigenvalues of A are found to be $\lambda = 1, r/10$ and $\bar{r}/10$, where $r = -3 - i\sqrt{11}$ and $\bar{r} = -3 + i\sqrt{11}$ (the conjugate of r), and the corresponding eigenvectors $(50, 5, 2)$, $(r^2, r, 4)$, and $(\bar{r}^2, \bar{r}, 4)$. Hence, if

$$\mathbf{x}_0 = c_1 \begin{bmatrix} 50 \\ 5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} r^2 \\ r \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} \bar{r}^2 \\ \bar{r} \\ 4 \end{bmatrix}, \tag{7.23}$$

then

$$\mathbf{x}_k = c_1 1^k \begin{bmatrix} 50 \\ 5 \\ 2 \end{bmatrix} + c_2 \left(\frac{r}{10} \right)^k \begin{bmatrix} r^2 \\ r \\ 4 \end{bmatrix} + c_3 \left(\frac{\bar{r}}{10} \right)^k \begin{bmatrix} \bar{r}^2 \\ \bar{r} \\ 4 \end{bmatrix}. \tag{7.24}$$

Note that $|r/10| = |\bar{r}/10| = 1/\sqrt{5} < 1$. Hence the positive numbers $|r/10|^k$ and $|\bar{r}/10|^k$ approach 0 as $k \to \infty$. Thus the complex numbers $(r/10)^k$ and $(\bar{r}/10)^k$ approach 0 as $k \to \infty$. Therefore, for large k, (7.24) reduces to

$$\mathbf{x}_k \simeq c_1 \begin{bmatrix} 50 \\ 5 \\ 2 \end{bmatrix}. \tag{7.25}$$

So for any given initial vector $\mathbf{x}_0 = (A_0, B_0, C_0)$, it suffices to compute c_1 from (7.23) and substitute into (7.25) to find $\mathbf{x}_k$ (for large k). In (7.23), we can solve for c_1 by Cramer's rule:

$$c_1 = \begin{vmatrix} A_0 & r^2 & \bar{r}^2 \\ B_0 & r & \bar{r} \\ C_0 & 4 & 4 \end{vmatrix} \begin{vmatrix} 50 & r^2 & \bar{r}^2 \\ 5 & r & \bar{r} \\ 2 & 4 & 4 \end{vmatrix}^{-1} = \frac{1}{90} A_0 + \frac{1}{15} B_0 + \frac{1}{18} C_0.$$

Hence, for the initial vector $\mathbf{x}_0 = (1000, 1000, 1000)$, we have

$$\mathbf{x}_k \simeq 1000 \left(\frac{1}{90} + \frac{1}{15} + \frac{1}{18} \right) \begin{bmatrix} 50 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 6666.\overline{6} \\ 666.\overline{6} \\ 266.\overline{6} \end{bmatrix}.$$

Therefore,under the given survival and birth rates, the numbers of insects in age groups A, B, and C approach $6666.\overline{6}$, $666.\overline{6}$, and $266.\overline{6}$, respectively. In Figure 7.15, we see that the trajectory spirals to the point with coordinates these numbers.

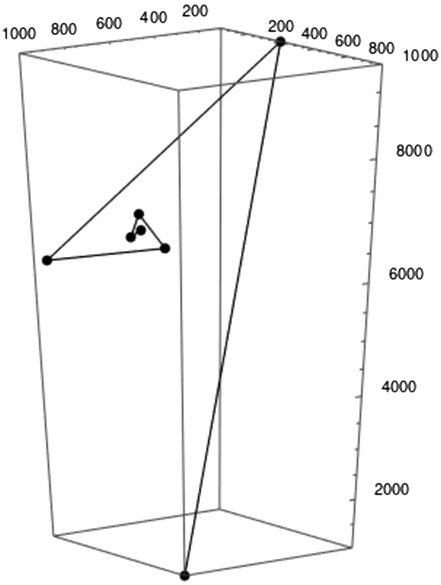

Figure 7.15: The long-term population vector approaches $(6666.\overline{6}, 666.\overline{6}, 266.\overline{6})$ in Example 7.4.2.

7.4.3 Markov chains and stochastic matrices

One of the most interesting applications of eigenvalues is in computing advanced stages of Markov chains. Markov chains were studied in Section 3.6. Recall that in a Markov chain the next state of a system depends only on its current state. To illustrate, we revisit the study of smokers versus nonsmokers of that section.

Suppose that the probability of a smoker to continue smoking a year later is 65 %, whereas the probability of a nonsmoker to continue nonsmoking is 85 %. This information was tabulated in Section 3.6 by using the stochastic matrix of transition probabilities

$$A = \begin{bmatrix} 0.65 & 0.15 \\ 0.35 & 0.85 \end{bmatrix}.$$

For example, the entry 0.35 means that a smoker has 35 % chance of quitting a year later, whereas 0.15 means that a nonsmoker has a 15 % chance of picking up smoking.

Example 7.4.3. What are the percentages of smokers and nonsmokers in the long run if $100p$ percent are initially smokers and $100q$ percent nonsmokers?

Solution. First, we note that $p + q = 1$. Recall from Section 3.6 that in k years the new percentages can be computed as

$$\begin{bmatrix} 0.65 & 0.15 \\ 0.35 & 0.85 \end{bmatrix}^k \begin{bmatrix} p \\ q \end{bmatrix}.$$

So we need the value of this vector as k approaches ∞. Diagonalization of A yields

$$A^k = \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{1}{2}\right)^k \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix}^{-1}.$$

Now $(\frac{1}{2})^k$ approaches 0 as $k \to \infty$, so we have

$$\lim_{k\to\infty} A^k = \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 7 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.3 & 0.3 \\ 0.7 & 0.7 \end{bmatrix}.$$

Therefore

$$\lim_{k\to\infty} A^k \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0.3 & 0.3 \\ 0.7 & 0.7 \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0.3p + 0.3q \\ 0.7p + 0.7q \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix},$$

because $p+q = 1$. So, in the long run the smokers will be 30 % versus 70 % of nonsmokers. This is true for any starting percentage vector (p, q) with $p + q = 1$.

A vector whose components are all nonnegative and add up to 1 is called a *probability* vector. For example,

$$\begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0.2 \\ 0.4 \\ 0.4 \end{bmatrix}$$

are probability vectors.

In Example 7.4.3, we showed that for any probability vector $\mathbf{v}$, the limit of $A^k\mathbf{v}$ is $(0.3, 0.7)$ as $k \to \infty$.

7.4.4 Limits of stochastic matrices

We have just seen how to use diagonalization to find power limits of stochastic matrices, but is it clear that these limits always exist?

For example, consider the stochastic matrix $B = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Then

$$B^2 = I, \quad B^3 = B, \quad B^4 = I, \quad B^5 = B, \quad \ldots,$$

and clearly $\lim_{k\to\infty} B^k$ does not exist even though B is diagonalizable.

Question. *When are we guaranteed that the limit* $\lim A^k$ *of a stochastic matrix A exists?*

The answer is in Theorem 7.4.4, for which we need the following key definition. A stochastic matrix A is called *regular* if some power A^k (k positive integer) consists of strictly positive entries. For example, $A = \left[\begin{smallmatrix} 0.5 & 1 \\ 0.5 & 0 \end{smallmatrix}\right]$ is regular because

$$A^2 = \begin{bmatrix} 0.75 & 0.5 \\ 0.25 & 0.5 \end{bmatrix}$$

has only positive entries. On the other hand, B above is not regular, because all its powers have *some* zero entries.

The following theorem answers the question for regular matrices. Its proof can be found in the book *Finite Markov Chains*, by Kemeny and Snell [21].

Theorem 7.4.4. *Let A be a regular n×n stochastic matrix. Then as* $k \to \infty$, A^k *approaches an* $n \times n$ *matrix L of the form*

$$L = [\mathbf{v}\,\mathbf{v}\,\cdots\,\mathbf{v}],$$

where $\mathbf{v}$ *is a probability n-vector with only positive entries.*

So, for any regular stochastic matrix, the limit of powers L exists. However, computing L using limits is inefficient. A better way is a consequence of the next theorem.

Theorem 7.4.5. *Let A be a regular stochastic matrix, and let L and* $\mathbf{v}$ *be as in Theorem* 7.4.4. *Then*

1. *For any initial probability vector* $\mathbf{x}_0$, $A^k\mathbf{x}_0$ *approaches* $\mathbf{v}$ *as* $k \to \infty$. *So,*

$$\lim_{k\to\infty} (A^k\mathbf{x}_0) = \mathbf{v};$$

2. $\mathbf{v}$ *is the only probability vector that satisfies*

$$A\mathbf{v} = \mathbf{v}.$$

So $\mathbf{v}$ *is an eigenvector of A with eigenvalue* $\lambda = 1$.

Proof. 1. Let $\mathbf{x}_0 = (x_1, \dots, x_n)$. By Theorem 7.4.4 we have

$$\begin{aligned}\lim_{k\to\infty} (A^k\mathbf{x}_0) = \Big(\lim_{k\to\infty} A^k\Big)\mathbf{x}_0 &= L\mathbf{x}_0 \\ &= x_1\mathbf{v} + \cdots + x_n\mathbf{v} \\ &= (x_1 + \cdots + x_n)\mathbf{v} = \mathbf{v},\end{aligned}$$

because, $x_1 + \cdots + x_n = 1$.

2. We have

$$\mathbf{v} = \lim_{k\to\infty} (A^k \mathbf{x}_0) = \lim_{k\to\infty} (A^{k+1} \mathbf{x}_0) = A \lim_{k\to\infty} (A^k \mathbf{x}_0) = A\mathbf{v}.$$

The proof of uniqueness of $\mathbf{v}$ is left as an exercise. □

> Theorems 7.4.4 and 7.4.5 can be viewed as particular cases of the important *Perron–Frobenius theorem*, which examines properties of eigenvalues of square matrices with nonnegative entries (see "Matrix Analysis and Applied Linear Algebra" by Meyer and Stewart [22]).

Definition 7.4.6. A nonzero vector $\mathbf{v}$ that satisfies $A\mathbf{v} = \mathbf{v}$ is called a *steady-state vector* or an *equilibrium* of A.

Let us now see how to compute the equilibrium $\mathbf{v}$ without using limits: Because $\mathbf{v}$ is an eigenvector of A with eigenvalues 1, we just solve the system

$$(A - I)\mathbf{x} = \mathbf{0}$$

and pick out the solution whose entries add up to 1.

Example 7.4.7. Find $\mathbf{v}$ and L for $A = \left[\begin{smallmatrix} 0.5 & 1 \\ 0.5 & 0 \end{smallmatrix}\right]$.

Solution. We have

$$[A - I : \mathbf{0}] = \begin{bmatrix} -0.5 & 1 & 0 \\ 0.5 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the solution is $(2r, r)$, $r \in \mathbf{R}$. We want $2r + r = 1$. Hence $r = \frac{1}{3}$. So

$$\mathbf{v} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

> The proof of Part 1 of Theorem 7.4.5 shows that if A is regular, then for *any* initial vector $\mathbf{x}_0$ (not necessarily probability vector),
>
> $$A^k \mathbf{x}_0 \to r\mathbf{v} \quad \text{as } k \to \infty,$$
>
> where $r = x_1 + \cdots + x_n$. Thus for any initial vector $\mathbf{x}_0$, the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ has a limit, namely $r\mathbf{v}$, which is a steady-state vector of A and can be easily computed.

Exercises 7.4

In Exercises 1–2, use eigenvalues and eigenvectors to compute $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ for the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ with the given matrix A and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Draw the trajectory through $\mathbf{x}_0, \ldots, \mathbf{x}_3$ and determine whether the origin is an attractor, repeller, or neither.

1. $A = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$.

2. $A = \begin{bmatrix} 0 & 0.5 \\ -2 & 1 \end{bmatrix}$.

3. Prove that all the solutions of the dynamical system $\mathbf{x}_{k+1} = \begin{bmatrix} 0 & 2 \\ -0.5 & 1 \end{bmatrix} \mathbf{x}_k$ are 6-cycles.

4. Prove that the stochastic matrices are regular.

$$\text{(a)} \quad \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix}; \quad \text{(b)} \quad \begin{bmatrix} 0 & 0.5 \\ 1 & 0.5 \end{bmatrix}$$

5. Prove that the following matrices are not regular:

$$\text{(a)} \quad \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}; \quad \text{(b)} \quad \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix}.$$

6. Which of the following matrices are regular?

$$\text{(a)} \quad \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.2 & 0.5 & 1 \\ 0.6 & 0 & 0 \end{bmatrix}; \quad \text{(b)} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix};$$

$$\text{(c)} \quad \begin{bmatrix} 0 & 0.5 & 1 \\ 0 & 0.5 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad \text{(d)} \quad \begin{bmatrix} 0 & 0.5 & 1 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}.$$

7. Find the steady-state vectors of the matrices of Exercise 4.

8. Find the steady-state vectors of the following matrices:

$$\text{(a)} \quad \begin{bmatrix} 0.6 & 0.5 \\ 0.4 & 0.5 \end{bmatrix}; \quad \text{(b)} \quad \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}.$$

9. Find the steady-state vector of the matrix

$$\begin{bmatrix} 0.2 & 0.5 & 0.5 \\ 0.2 & 0.5 & 0 \\ 0.6 & 0 & 0.5 \end{bmatrix}.$$

10. Prove the uniqueness of the steady-state vector claimed in Theorem 7.4.5.

11. (*Psychology*) A psychologist places 40 rats in a box with 3 colored compartments: blue (B), green (G), and red (R). Each compartment has doors that lead to the other ones as shown in the figure below. The rats constantly move toward a door, so that the probability that they will stay in one compartment is 0. A rat in B has probability 3/4 to go to G and probability 1/4 to go to R, according to the distribution of doors. Likewise, a rat in R has probability 1/2 to go to G and probability 1/2 to go to B. So the transition of probabilities matrix is of the form

$$A = \begin{bmatrix} 0 & * & 1/2 \\ 3/4 & 0 & 1/2 \\ 1/4 & * & 0 \end{bmatrix}.$$

Replace the asterisks in A with the correct probabilities. Prove that A is regular and compute its steady-state vector. What is the long distribution of rats? What is the probability that a given rat will be in G in the long run? (Figure 7.16)

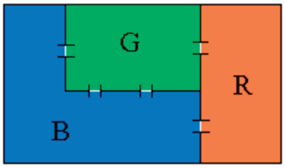

Figure 7.16: Three compartment rat experiment.

12. The kth generation of an animal population consists of A_k females and B_k males. Suppose that the next generation depends on the current one according to

$$\begin{aligned} A_{k+1} &= 0.8A_k + 0.7B_k, \\ B_{k+1} &= 0.2A_k + 0.3B_k. \end{aligned}$$

Write this dynamical system in matrix notation. If initially there were 300 females and 100 males, then what is approximately the population (a) right after the third generation? (b) in the long run? Which gender will eventually dominate?

13. Repeat Exercise 12 for the following dependencies:

$$\begin{aligned} A_{k+1} &= 0.7B_k, \\ B_{k+1} &= A_k + 0.3B_k. \end{aligned}$$

14. (*Demographics*) Statistical data for a city and its surrounding suburban area show the following yearly residential moving trends. For example, the probability that a person living in the city will move to the suburban area in a year is 55 %.

	Initial	**State**
	City	**Suburb**
City	45 %	35 %
Suburb	55 %	65 %

What is the long-run distribution of a population living in this city and its surrounding suburban areas?

15. (*Economics*) Currently there are three investment plans available for the employees of a company: A, B, and C. An employee can only use one plan at a time and may switch from one plan to another only at the end of each year. The probability that someone in Plan A will continue with A is 20 %, switch to Plan B is 50 %, switch to Plan C is 50 %, and so on. The transition of probabilities matrix M for the employees that participate is given below.

		This	Year	
		A	**B**	**C**
Next	A	0.2	0.5	0.5
Year	B	0.2	0.5	0
	C	0.6	0	0.5

Prove that M is regular and find its equilibrium. Find the most and the least popular plans in the long run.

16. (*Election voting*) Voting trends of successive elections are given in the following matrix. For example, the probability that a Democrat will vote Republican next elections is 20 %.

	Initial	**State**	
	Dem.	**Rep.**	**Ind.**
Democrat	70 %	20 %	40 %
Republican	20 %	70 %	40 %
Independent	10 %	10 %	20 %

What is the long-run voter distribution likely to be?

7.5 Special topic: The PageRank algorithm of Google

Search engines receive a key word and try to find as fast as possible web pages that contain this key word. They can be millions of such pages, so a decision needs to be made on a quick ranking of these pages in the order of relevance. Initially, engines tried to rank as high the web pages that had many occurrences of the keyword. This idea is not ideal: there can be a site that repeats the key word thousands of times and does not provide any other information. Clearly, the usefulness of a search engine depends on the *relevance* of the result it returns.

One of the most important algorithms for computing the relevance of web pages is the PageRank algorithm that is used by the Google search engine. This algorithm was proposed by Larry Page and Sergey Brin, and it became a Google trademark in 1998.

The idea of the algorithm is that a web page is important if many web pages link to it. If a web page i has a hyperlink to a web page j, this means that j is considered important by i. If there are many pages that link to j, this means that j is important overall. In the special case that j has only one back-link, but it comes from highly ranked page k (like www.jhu.edu or www.google.com), we say that k *transfers its authority* to j. This means that k views j as important.

To illustrate these ideas, suppose we have an internet of four pages A, B, C, and D. We assume that Page A links to C and D, Page B links to A, C, and D, Page C links to D and B, and Page D links to B (Figure 7.17).

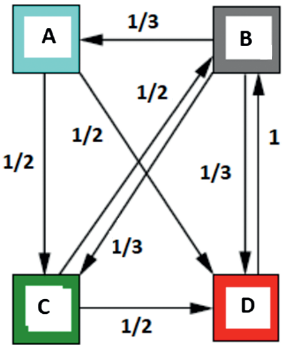

Figure 7.17: Internet with four pages.

We also assume that each page weights equally its links. So, for example, in Page A the links to C and D are equally weighted, with probability $\frac{1}{2}$. This means that there is a 50 % chance to use one link or the other. Likewise, we assign probabilities $\frac{1}{3}$ to each link of Page B, probabilities $\frac{1}{2}$ to each link of Page C, and probability 1 to the only link in Page D. Since Page D has only one link and it is to Page B, we see that D transferred its authority to B.

Next, we may form a probability transition matrix A of the graph in Figure 7.17 that will transition from each one page to the next in one step. The columns of the matrix represent in order A, B, C, and D. Page A has no link to itself, so $a_{11} = 0$. It has no link to B, so $a_{21} = 0$. It has links to C and D, so $a_{31} = a_{41} = \frac{1}{2}$. The remaining entries are obtained similarly. Thus the probability transition matrix is

$$A = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & 0 \end{bmatrix}.$$

Suppose that initially the importance is uniformly distributed among the 4 nodes, each getting $\frac{1}{4}$. If $\mathbf{v}_0$ the initial rank vector, which we may assume that has all entries equal to $\frac{1}{4}$, then each incoming link increases the importance of a page. So at Step 1, we update the rank of each page by adding to the current value the importance of the incoming links. The effect is the vector $A\mathbf{v}_0$. We then iterate the process, so at Step 2, the updated rank vector is $A^2\mathbf{v}_0$. If we continue like this and compute the limit of $A^n\mathbf{v}_0$ as n gets large, then we find the limit vector $\mathbf{v} = \lim_{n\to\infty} A^n\mathbf{v}_0$, which will rank importance to each web page. This vector is called the *PageRank vector*.

Now A is stochastic. In fact, it is regular, because all entries of A^4 are strictly positive (check). So by Theorem 7.4.5, A has a steady-state eigenvector $\mathbf{v}$ with eigenvalue 1. This vector is exactly the limit vector that we need according to same theorem. Hence $A\mathbf{v} = \mathbf{v}$.

This eigenvector $\mathbf{v}$ is computed to be $\mathbf{v} = [\,\frac{4}{9}\ \frac{4}{3}\ \frac{2}{3}\ 1\,]^T$. It is usually desirable that $\mathbf{v}$ is a *probability vector*, so that the sum of its entries equals 1. So we see that the PageRank vector for our matrix becomes

$$\mathbf{v} = \frac{1}{31}\begin{bmatrix} \frac{4}{9} \\ \frac{4}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{31} \\ \frac{12}{31} \\ \frac{6}{31} \\ \frac{9}{31} \end{bmatrix} \approx \begin{bmatrix} 0.129032 \\ 0.387097 \\ 0.193548 \\ 0.290323 \end{bmatrix}.$$

We conclude that Page B has the highest ranking. This may be surprising because Page B has two back-links, whereas Page D has three back-links. However, Page D has only one link, which is to Page B, so it transferred its authority to Page B.

We examined a simple version of the PageRank algorithm. In practice, there are complications. For example, there may be places where the internet graph is disconnected, with a cluster of pages that are not linked to any other pages. Or there may be pages that have no links, in which case we may get PageRank vectors where every page is ranked as zero. Page and Brin found a solution: instead of using directly A, they use a modification of it. If A is of size $n \times n$ and p is a number in between 0 and 1, then the following matrix M is used instead of A:

$$M = (1-p)A + pB, \quad \text{where } B = \frac{1}{n}\begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

The matrix M is known as the *PageRank matrix* or the *Google matrix*. This matrix is stochastic with positive entries, and hence it has a limit vector $\mathbf{v}$. This vector is scaled so that the sum of its entries is 1 and is the final PageRank vector.

7.6 Approximations of eigenvalues

Up to now we have computed eigenvalues by solving the characteristic equation and then finding the corresponding eigenvectors. For large matrices, this is not practical even for approximate solutions, because the determinant of a large characteristic matrix is computationally expensive and so is the solution of the high-degree characteristic equation. More efficient methods are used that first approximate an eigenvector and then the corresponding eigenvalue.

7.6.1 The power method

The approximate methods we study are based on the power method, expressed in Theorem 7.3.4. Recall that for an $n \times n$ diagonalizable matrix A with basic eigenvectors $\mathbf{v}_i$, eigenvalues λ_i, and a dominant eigenvalue λ_1, if a vector $\mathbf{x}$ is a linear combination

$$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

such that $c_1 \neq 0$, then as k grows, a scalar multiple of $A^k\mathbf{x}$ approaches a scalar multiple of $\mathbf{v}_1$. In particular, the direction of $A^k\mathbf{x}$ approaches that of $\mathbf{v}_1$.

So, to find an approximation of an eigenvector of the dominant eigenvalue starting with an initial vector $\mathbf{x}$, we compute $A^k\mathbf{x}$ by the following iteration. Let $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ be defined by

$$\mathbf{x} = \mathbf{x}_0, \quad \mathbf{x}_1 = A\mathbf{x}_0, \quad \mathbf{x}_2 = A^2\mathbf{x}_0 = A\mathbf{x}_1, \quad \ldots .$$

So $\mathbf{x}_k = A^k\mathbf{x}$, and

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \quad k = 0, 1, \quad \ldots . \tag{7.26}$$

To approximate the corresponding eigenvalue λ_1, we observe that if $\mathbf{x}_k$ is an approximate eigenvector, then $\mathbf{x}_{k+1} = A\mathbf{x}_k \simeq \lambda_1\mathbf{x}_k$. Hence λ_1 can be computed by taking the quotient of two corresponding components of $\mathbf{x}_{k+1}$ and $\mathbf{x}_k$.

Example 7.6.1. Approximate the dominant eigenvalue of A and a corresponding eigenvector by using iteration (7.26) with initial vector $\mathbf{x}_0$, where

$$A = \begin{bmatrix} 8 & 7 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution. We compute $\mathbf{x}_k$ for several values of k. The table below displays the values of k, $\mathbf{x}_k$, the quotients d_k between the components of each $\mathbf{x}_k$, and the quotients l_k between the first components of $\mathbf{x}_k$ and $\mathbf{x}_{k-1}$.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 22 \\ 5 \end{bmatrix}$	$\begin{bmatrix} 211 \\ 32 \end{bmatrix}$	$\begin{bmatrix} 1912 \\ 275 \end{bmatrix}$	$\begin{bmatrix} 17221 \\ 2462 \end{bmatrix}$	$\begin{bmatrix} 155002 \\ 22145 \end{bmatrix}$
d_k	0.5	4.4	6.5938	6.9527	6.9947	6.9994
l_k		22.0	9.5909	9.0616	9.0068	9.0008

The ratio d_5 of the components of $\mathbf{x}_5$ is 6.9994. So $(6.9994, 1)$ is an approximate eigenvector. Also, $l_5 = 9.0008$. So 9.0008 is an approximate eigenvalue. Actually, $(7, 1)$ is an exact eigenvector with eigenvalue 9. In Figure 7.18, we display $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ and the line l through $(0, 0)$ and $(7, 1)$.

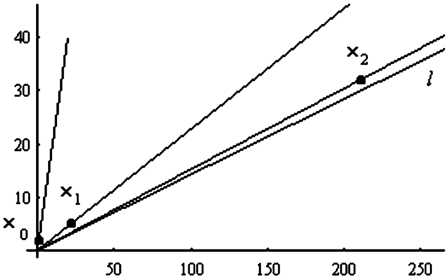

Figure 7.18: Graphical illustration of the power method.

There is a problem with the calculation of Example 7.6.1. The components of $\mathbf{x}_k$ grow very fast, almost out of control, while all we need is either $(7, 1)$ or some manageable scalar multiple of it. Since we are mainly interested in the direction of $(7, 1)$, we can scale each $\mathbf{x}_k$ to keep the numbers small. One way is to make $\mathbf{x}_k$ a unit vector by multiplying by $1/\|\mathbf{x}_k\|$. An easier way is to scale $\mathbf{x}_k$ so that its largest entry is 1. In Example 7.6.1, we start with $\mathbf{x}_0$ and scale it:

$$\mathbf{y}_0 = \frac{1}{2}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix}.$$

Then we compute

$$\mathbf{x}_1 = \begin{bmatrix} 8 & 7 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 11.0 \\ 2.5 \end{bmatrix}$$

and scale it:

$$\mathbf{y}_1 = \frac{1}{11.0}\begin{bmatrix} 11.0 \\ 2.5 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.22727 \end{bmatrix},$$

and so on. In this setting, we may approximate the eigenvalue λ_1 as follows. Suppose we have reached a scaled $\mathbf{y}_k$. Then we compute $\mathbf{x}_{k+1} = A\mathbf{y}_k$, and before we scale it, we save the component that corresponds to the component with 1 in $\mathbf{y}_k$. This component is an approximation for λ_1 (why?). In the following table, we display the $\mathbf{x}_k$s, $\mathbf{y}_k$s, and the components l_k that approximate λ_1 for A and $\mathbf{x}_0$ of Example 7.6.1.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\2\end{bmatrix}$	$\begin{bmatrix}11.0\\2.5\end{bmatrix}$	$\begin{bmatrix}9.5909\\1.4545\end{bmatrix}$	$\begin{bmatrix}9.0616\\1.3033\end{bmatrix}$	$\begin{bmatrix}9.0068\\1.2877\end{bmatrix}$	$\begin{bmatrix}9.0008\\1.2859\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}0.5\\1.0\end{bmatrix}$	$\begin{bmatrix}1.0\\0.22727\end{bmatrix}$	$\begin{bmatrix}1.0\\0.15165\end{bmatrix}$	$\begin{bmatrix}1.0\\0.14383\end{bmatrix}$	$\begin{bmatrix}1.0\\0.14297\end{bmatrix}$	$\begin{bmatrix}1.0\\0.14287\end{bmatrix}$
l_k	2	11.0	9.5909	9.0616	9.0068	9.0008

In Figure 7.19, we display $\mathbf{y}_0$, $\mathbf{y}_1$, $\mathbf{y}_7$. Higher $\mathbf{y}$ are too close to $\mathbf{y}_7$ to be displayed.

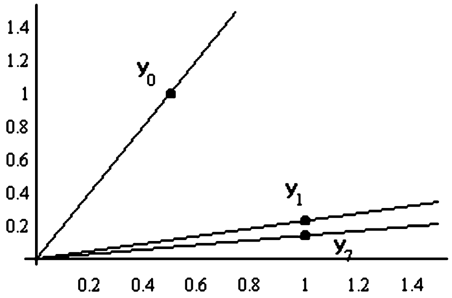

Figure 7.19: Graphical illustration of the modified power method.

Algorithm 1 (Power method – Maximum entry 1). Let A be an $n \times n$ diagonalizable matrix with a dominant eigenvalue. Let $\mathbf{x}_0$ be any vector. Let k be the desired number of iterations.

```
Input: $A$, $\mathbf{x}_0$, $k$.
1. Let $l_0$ be the component of $\mathbf{x}_0$ of largest absolute value.
2. Set $\mathbf{y}_0 = (1/l_0)\mathbf{x}_0$.
3. For $i = 1, \dots, k$,
    (a) Compute $A\mathbf{y}_{i-1}$. Let $\mathbf{x}_i = A\mathbf{y}_{i-1}$.
    (b) Let $l_i$ be the component of $\mathbf{x}_i$ of the largest absolute value.
    (c) Let $\mathbf{y}_i = (1/l_i)\mathbf{x}_i$.
Output: $\mathbf{y}_k$, and $l_k$.
```

For most $\mathbf{x}_0$, l_k approximates the dominant eigenvalue of A, and $\mathbf{y}_k$ approximates the corresponding eigenvector.

1. The power method applies to almost all $\mathbf{x}_0$, because we do not know in advance whether the coefficient c_1 of $\mathbf{v}_1$ in terms of the eigenvector is nonzero as Theorem 7.3.4 in Section 7.3 requires. However, in practice the method works for any initial vector, because computer rounding errors will likely replace zeros with small floating point numbers.

2. The power method is a *self-correcting method*. If at any point, our $\mathbf{x}_i$ has been miscalculated, then we can still go on, as if we were starting at this vector for the first time.
3. How slow or fast the iteration converges depends on the ratio $|\lambda_2/\lambda_1|$, where λ_2 is the eigenvalue with the second largest absolute value in equation (7.15), Section 7.3. If $|\lambda_2/\lambda_1|$ is close to 0, then the convergence is very fast. This was the case in Example 7.6.1, where $|\lambda_2/\lambda_1| = 1/9 \simeq 0.11111$. If $|\lambda_2/\lambda_1|$ is close to 1, then the convergence is slow.
4. The power method works even when we have a *repeated dominant eigenvalue*:

$$\lambda_1 = \cdots = \lambda_r \quad \text{and} \quad |\lambda_1| > |\lambda_{r+1}| \geq \cdots \geq |\lambda_n| \,.$$

For example, say, A has eigenvalues $4, 1, -5, -5$ (so $\lambda_1 = \lambda_2 = -5, \lambda_3 = 4, \lambda_4 = 1$). This is easily verified since equation (7.15) of Theorem 7.3.4, Section 7.3, would then be

$$\left(1/\lambda_1^k\right)A^k\mathbf{x} = \sum_{i=1}^{r} c_i\mathbf{v}_i + \sum_{i=r+1}^{n} c_i(\lambda_i/\lambda_1)^k\mathbf{v}_i,$$

so that $(1/\lambda_1^k)A^k\mathbf{x}$ approaches $\sum_{i=1}^{r} c_i\mathbf{v}_i$, which is an eigenvector of λ_1.
5. The power method works even when A is nondiagonalizable.[2] The convergence is usually slow in this case.

7.6.2 Rayleigh quotients (the Rayleigh–Ritz method)

There is a common variation of the power method, where we normalize by dividing by the *norm* of the vector and then use the *Rayleigh quotient*

$$r(x) = \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$$

to approximate the eigenvalues. This works because if $\mathbf{x}$ is an eigenvector, then $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T\lambda\mathbf{x} = \lambda(\mathbf{x}^T\mathbf{x})$. To illustrate using the data of Example 7.6.1, we let

$$\mathbf{y}_0 = \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.44721 \\ 0.89443 \end{bmatrix}$$

and then compute

$$\mathbf{x}_1 = A\mathbf{y}_0 = \begin{bmatrix} 9.8387 \\ 2.2361 \end{bmatrix}.$$

Next, we compute the Rayleigh quotient

$$r(\mathbf{y}_0) = \frac{\mathbf{y}_0^T A\mathbf{y}_0}{\mathbf{y}_0^T\mathbf{y}_0} = \mathbf{y}_0 \cdot \mathbf{x}_1,$$

because $\mathbf{y}_0^T\mathbf{y}_0 = \mathbf{y}_0 \cdot \mathbf{y}_0 = \|\mathbf{y}_0\|^2 = 1$. We continue the same way:

---

[2] A nondiagonalizable matrix is also called *defective*.

$$\mathbf{y}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|}, \quad \mathbf{x}_2 = A\mathbf{y}_1, \quad r(\mathbf{y}_1) = \mathbf{y}_1 \cdot \mathbf{x}_2, \quad \dots .$$

The Rayleigh quotients

$$r(\mathbf{y}_0), \quad r(\mathbf{y}_1), \quad r(\mathbf{y}_2), \quad \dots,$$

approach the dominant eigenvalue. In our case, we have

k	**0**	**1**	**2**	**3**	**4**	**5**
$\mathbf{y}_k$	$\begin{bmatrix} 0.44721 \\ 0.89443 \end{bmatrix}$	$\begin{bmatrix} 0.97509 \\ 0.22162 \end{bmatrix}$	$\begin{bmatrix} 0.9887 \\ 0.14994 \end{bmatrix}$	$\begin{bmatrix} 0.98981 \\ 0.14236 \end{bmatrix}$	$\begin{bmatrix} 0.98994 \\ 0.14152 \end{bmatrix}$	$\begin{bmatrix} 0.98995 \\ 0.14143 \end{bmatrix}$
$\mathbf{x}_{k+1}$	$\begin{bmatrix} 9.8387 \\ 2.2361 \end{bmatrix}$	$\begin{bmatrix} 9.3521 \\ 1.4183 \end{bmatrix}$	$\begin{bmatrix} 8.9592 \\ 1.2886 \end{bmatrix}$	$\begin{bmatrix} 8.915 \\ 1.2745 \end{bmatrix}$	$\begin{bmatrix} 8.9102 \\ 1.273 \end{bmatrix}$	$\begin{bmatrix} 8.9096 \\ 1.2728 \end{bmatrix}$
$r(y_k)$	6.4	9.4335	9.0512	9.0056	9.0007	9.0001

Algorithm 2 (**Rayleigh quotients** or the **Rayleigh–Ritz method**). Let A be an $n \times n$ diagonalizable matrix with a dominant eigenvalue. Let $\mathbf{x}_0$ be any vector. Let k be the desired number of iterations.

```
Input: $A$, $\mathbf{x}_0$, $k$.

For $i = 0, \dots, k-1$
1.  Let $\mathbf{y}_i = (1/\|\mathbf{x}_i\|)\mathbf{x}_i$.
2.  Let $\mathbf{x}_{i+1} = A\mathbf{y}_i$.
3.  Let $r_i = \mathbf{y}_i \cdot \mathbf{x}_{i+1}$.

Output: $\mathbf{y}_{k-1}$ and $r_{k-1}$.
```

For most $\mathbf{x}_0$, r_k approximates the dominant eigenvalue of A, and $\mathbf{y}_k$ approximates the corresponding eigenvector.

For *symmetric* matrices, this method is very efficient and requires fewer iterations to achieve the same accuracy.[3] For example, consider the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}.$$

Its eigenvalues are 3 and −2. The power method is slow, because $|(-2)/3|$ is close to 1. To get $\lambda = 3.0$ to five decimal places, it takes 33 iterations by the first power method and only 18 iterations by using Rayleigh quotients if we start at $\mathbf{x}_0 = (1, 2)$.

---

[3] Approximately, half the number of iterations.

7.6.3 Origin shifts

Now that we know how to compute the dominant eigenvalue, we can use a simple trick to *find the eigenvalue farthest from the dominant one* (if there is one).

If λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then for any scalar c, the matrix $A - cI$ has the eigenvalue $\lambda - c$ and corresponding eigenvector $\mathbf{v}$. This is because

$$(A - cI)\mathbf{v} = A\mathbf{v} - c\mathbf{v} = (\lambda - c)\mathbf{v}.$$

So if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then $0, \lambda_2 - \lambda_1, \ldots, \lambda_n - \lambda_1$ are the eigenvalues of $B = A - \lambda_1 I$.

We can combine this observation with the power method to compute the eigenvalue λ_r farthest from the dominant eigenvalue, say λ_1. First, we compute λ_1. Now $\lambda_r - \lambda_1$ is the dominant eigenvalue of B, so we may use any power method to compute it. Finally, we add λ_1 to get λ_r.

To illustrate, let us compute the smallest eigenvalue of A in Example 7.6.1, given that we have computed the dominant eigenvalue $\lambda_1 = 9$. We form the matrix

$$B = A - 9I = \begin{bmatrix} 8-9 & 7 \\ 1 & 2-9 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 1 & -7 \end{bmatrix}.$$

Then we compute the dominant eigenvalue of B by, say, the power method with Rayleigh quotients to get a fast convergence, starting at $\mathbf{x}_0 = (1, 2)$, to get

k	0	1	2
y_k	$\begin{bmatrix} 0.44721 \\ 0.89443 \end{bmatrix}$	$\begin{bmatrix} 0.7071 \\ -0.7071 \end{bmatrix}$	$\begin{bmatrix} -0.70711 \\ 0.70711 \end{bmatrix}$
$r(y_k)$	−2.6	−7.9998	−8.0001

Hence the dominant eigenvalue of B is approximately −8.0001. Therefore the eigenvalue of A farthest from $\lambda_1 = 9$ is $9 - 8.0001 = 0.9999$. The corresponding eigenvector is $\left[\begin{smallmatrix} -0.70711 \\ 0.70711 \end{smallmatrix}\right]$. Actually, the exact eigenvalue is 1, and a unit eigenvector is $\left[\begin{smallmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{smallmatrix}\right]$.

7.6.4 Inverse power method

In this section, we show how to use the power method to find *the eigenvalue closest to the origin* (if there is one).

The inverse power method is based on the observation that if λ is an eigenvalue of A with corresponding eigenvector $\mathbf{v}$, then λ^{-1} is an eigenvalue of A^{-1} with the same eigenvector, because

$$A\mathbf{x} = \lambda\mathbf{x} \Rightarrow \mathbf{x} = A^{-1}\lambda\mathbf{x} \Rightarrow A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}.$$

Thus, to find the eigenvalue of A closest to the origin, we only need to compute the dominant eigenvalue of A^{-1}. The computation of A^{-1} is expensive, but we can avoid this by just solving the system

$$A\mathbf{x}_{k+1} = \mathbf{x}_k$$

for $\mathbf{x}_{k+1}$. Here is a version of this method that uses Rayleigh quotients.

Algorithm 3 (Inverse power method).

```
Input: $A$, $\mathbf{x}_0$, $k$.

For $i = 0, \dots, k-1$
1.  Let $\mathbf{y}_i = (1/\|\mathbf{x}_i\|)\mathbf{x}_i$.
2.  Solve the system $A\mathbf{z} = \mathbf{y}_i$ for $\mathbf{z}$.
3.  Let $\mathbf{x}_{i+1} = \mathbf{z}$.
4.  Let $r_i = \mathbf{y}_i \cdot \mathbf{x}_{i+1}$.

Output: $\mathbf{y}_{k-1}$ and $r_{k-1}$.
```

For most $\mathbf{x}_0$, r_k^{-1} approximates the eigenvalue of A closest to the origin, and $\mathbf{y}_k$ is the corresponding eigenvector.

To illustrate, suppose we look for the eigenvalue of A in Example 7.6.1 closest to the origin. Then for $\mathbf{x}_0 = (1, 2)$, we have

$$\mathbf{y}_0 = \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.44721 \\ 0.89443 \end{bmatrix}.$$

Then we row reduce the system

$$\begin{bmatrix} 8 & 7 & 0.44721 \\ 1 & 2 & 0.89443 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -0.59629 \\ 0 & 1 & 0.74536 \end{bmatrix}$$

and set

$$\mathbf{x}_1 = \begin{bmatrix} -0.59629 \\ 0.74536 \end{bmatrix} \quad \text{and} \quad r_0 = \mathbf{x}_1 \cdot \mathbf{y}_0 = .40001.$$

We continue the same way to get

$$\mathbf{y}_1 = \begin{bmatrix} -0.62469 \\ 0.78087 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -0.69893 \\ 0.71519 \end{bmatrix}$$

and

$$r_1 = 1.0623, \qquad r_2 = 1.0075.$$

If we stop now, then the eigenvalue closest to the origin is approximately $1/r_2 = 1/1.0075 = .99256$, which is already close to 1, the true eigenvalue. The corresponding eigenvector is $\mathbf{y}_2$, which is parallel to $\begin{bmatrix}-1\\1.0233\end{bmatrix}$. This is close to the true eigenvector $\begin{bmatrix}-1\\1\end{bmatrix}$.

Algorithm 3 is more efficiently used with an LU factorization of A. In this case, Step 2 is replaced by 2a. Solve $L\mathbf{y} = \mathbf{y}_i$ and 2b. Solve $L\mathbf{z} = \mathbf{y}$.

7.6.5 Shifted inverse power method

Finally, we combine the inverse iteration with an origin shift to compute *the eigenvalue closest to a given number*. For example, if a number μ is closest to the eigenvalue λ than to any other eigenvalue, then $1/(\lambda - \mu)$ is a dominant eigenvalue of $(A - \mu I)^{-1}$, which we can compute by the power method. Just as with the inverse power method, it is more efficient to solve the system

$$(A - \mu I)\mathbf{x}_{k+1} = \mathbf{x}_k$$

rather than using the inverse matrix. We have the following.

Algorithm 4 (Shifted inverse power method).

```
Input: $A$, $\mathbf{x}_0$, $k$. Initial guess $\mu$ for eigenvalue.

For $i = 0, \dots, k-1$
1. Let $\mathbf{y}_i = (1/\|\mathbf{x}_i\|)\mathbf{x}_i$.
2. Solve the system $(A - \mu I)\mathbf{z} = \mathbf{y}_i$ for $\mathbf{z}$.
3. Let $\mathbf{x}_{i+1} = \mathbf{z}$.
4. Let $r_i = \mathbf{y}_i \cdot \mathbf{x}_{i+1}$.

Output: $\mathbf{y}_{k-1}$ and $r_{k-1}$.
```

For most $\mathbf{x}_0$, $\mu + r_k^{-1}$ approximates the eigenvalue of A closest to μ, and $\mathbf{y}_k$ is the corresponding eigenvector.

This iteration works extremely well if the given number is an initial guess for an eigenvalue that we want to compute. The better the guess, the faster the convergence of the iteration.

Back to Example 7.6.1, suppose we have an initial guess $\mu = 2$ for one of the eigenvalues of A. Then for $\mathbf{x}_0 = (1, 2)$, we have

$$\mathbf{y}_0 = \frac{\mathbf{x}_0}{\|\mathbf{x}_0\|} = \begin{bmatrix} 0.44721 \\ 0.89443 \end{bmatrix}.$$

Then we row reduce the system $[A - 2I : \mathbf{y}_0]$,

$$\begin{bmatrix} 6 & 7 & 0.44721 \\ 1 & 0 & 0.89443 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0.89443 \\ 0 & 1 & -0.70277 \end{bmatrix},$$

and set

$$\mathbf{x}_1 = \begin{bmatrix} 0.89443 \\ -0.70277 \end{bmatrix} \quad \text{and} \quad r_0 = \mathbf{x}_1 \cdot \mathbf{y}_0 = -0.22858.$$

We continue the same way to get

$$\mathbf{y}_1 = \begin{bmatrix} 0.78631 \\ -0.61782 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -0.69347 \\ 0.72049 \end{bmatrix}$$

and

$$r_1 = -0.88237, \quad r_2 = -1.016.$$

If we stop at this point, then the eigenvalue closest to 2 is $2+1/r_2 = 2+1/(-1.016) = 1.0157$, which is already pretty close to 1, the true eigenvalue.

Again, Algorithm 4 is typically used with an LU factorization of A.

7.6.6 Application to roots of polynomials

Numerical approximations of eigenvalues can be so efficient that instead of computing the eigenvalues using the characteristic equation, we compute the roots of the characteristic equation or *any* polynomial equation by approximating the eigenvalues of the *companion matrix*. In Exercises 38–44, Section 7.1, we defined the companion matrix $C(p)$ of a monic polynomial $p(x)$ and saw that its eigenvalues are the roots of $p(x)$.

Example 7.6.2 (Roots as eigenvalues). Approximate one root of $p(x) = x^3 - 15x^2 + 59x - 45$ using the initial guess $x = 10$.

Solution. We apply the shifted inverse power method to find the eigenvalue of

$$C(p) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 45 & -59 & 15 \end{bmatrix}$$

closest to 10. Starting with a unit initial vector, say, $\mathbf{y}_0 = (2/3, 2/3, 1/3)$, we reduce $[C(p) - 10I : \mathbf{y}_0]$ to get $\mathbf{x}_1$, then compute $\mathbf{y}_0 \cdot \mathbf{x}_1$ to get $r(y_0)$, and so on. After four iterations, we have

$r(y_0)$	$r(y_1)$	$r(y_2)$	$r(y_3)$
0.17778	−1.2099	−1.0303	−1.0051

Hence an approximate root of $p(x)$ is $10 + (-1.0051)^{-1} = 9.0051$, which is already close to the true root $x = 9$. In a similar manner, we may obtain the other 2 roots, 1 and 5.

Computing roots of polynomials as approximate eigenvalues of the companion matrix is often the method of choice for numerical software packages. For example, MATLAB command roots is based on this method.

Exercises 7.6

In Exercises 1–2, use the information on the unspecified 2×2 matrix A and 2-vector $\mathbf{x}_0$ to
(a) estimate an eigenvalue of A,
(b) estimate an eigenvector with maximum entry 1.

1. $A^4\mathbf{x}_0 = \begin{bmatrix} 937 \\ 938 \end{bmatrix}$, $A^5\mathbf{x}_0 = \begin{bmatrix} 4687 \\ 4688 \end{bmatrix}$.

2. $A^5\mathbf{x}_0 = \begin{bmatrix} 1561 \\ -1564 \end{bmatrix}$, $A^6\mathbf{x}_0 = \begin{bmatrix} -7811 \\ 7814 \end{bmatrix}$.

Let $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$. In Exercises 3–5, find an eigenvector and an eigenvalue of some (unspecified) 2×2 matrix A if for some (unspecified) 2-vector $\mathbf{x}$,

3. $A^6\mathbf{x} = \mathbf{u}$ and $A^7\mathbf{x} = \mathbf{v}$,

4. $A^7\mathbf{x} = \mathbf{u}$ and $A^8\mathbf{x} = \mathbf{v}$,

5. $A^{-6}\mathbf{x} = \mathbf{u}$ and $A^{-7}\mathbf{x} = \mathbf{v}$.

In Exercises 6–13, apply the power method with $k = 4$ and $\mathbf{x}_0 = (1, 2)$ to
(a) approximate the dominant eigenvalue and a corresponding eigenvector for the given matrix,
(b) compare the approximate eigenvalue with the true one.

6. $\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

7. $\begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$.

8. $\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.

9. $\begin{bmatrix} -4 & 3 \\ 3 & -4 \end{bmatrix}$.

10. $\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$.

11. $\begin{bmatrix} -5 & 4 \\ 4 & -5 \end{bmatrix}$.

12. $\begin{bmatrix} -8 & 7 \\ 1 & -2 \end{bmatrix}$.

13. $\begin{bmatrix} -6 & 5 \\ 2 & -3 \end{bmatrix}$.

In Exercises 14–18, use the Rayleigh–Ritz method with $k = 4$ and $\mathbf{x}_0 = (1, 2)$ to approximate the dominant eigenvalue of the matrix.

14. The matrix of Exercise 6.

15. The matrix of Exercise 7.

16. The matrix of Exercise 8.

17. The matrix of Exercise 9.

18. The matrix of Exercise 10.

In Exercises 19–25, use the inverse power method with $k = 4$ and $\mathbf{x}_0 = (1, 2)$ to approximate the eigenvalue closest to the origin and compare your answer with the true eigenvalue of the matrix.

19. The matrix of Exercise 6.

20. The matrix of Exercise 7.

21. The matrix of Exercise 8.

22. The matrix of Exercise 9.

23. The matrix of Exercise 10.

24. The matrix of Exercise 11.

25. The matrix of Exercise 12.

In Exercises 26–28, use the inverse power method on the companion matrix to approximate the root of each polynomial $p(x)$ closest to r.

26. $p(x) = x^2 - 5x + 4$, $r = 5$.

27. $p(x) = x^2 - 3x - 4$, $r = 5$.

28. $p(x) = x^2 - 8x - 9$, $r = 10$.

7.7 Miniprojects

7.7.1 The Cayley–Hamilton theorem

If A is a square matrix of scalar entries and $p(x)$ is a polynomial, say,

$$p(x) = a_0 + a_1 x + \cdots + a_k x^k,$$

then we denote by $p(A)$ the matrix

$$p(A) = a_0 I + a_1 A + \cdots + a_k A^k.$$

For example, if $A = \left[\begin{smallmatrix} 2 & -2 \\ 1 & 4 \end{smallmatrix}\right]$ and $p(x) = 1 - 3x + x^2$, then

$$p(A) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}^2 = \begin{bmatrix} -3 & -6 \\ 3 & 3 \end{bmatrix}.$$

In this project, you are to verify and prove the following important theorem.

Theorem 7.7.1 (Cayley–Hamilton). *Every square matrix satisfies its characteristic equation. So, if $p(x)$ is the characteristic polynomial of A, then*

$$p(A) = \mathbf{0}.$$

To illustrate, it is easy to see that $p(x) = 10 - 6x + x^2$ is the characteristic polynomial of the above matrix A. Then

$$p(A) = 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 6\begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 1 & 4 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Problem A. Verify the Cayley–Hamilton theorem for the matrices

$$\text{(a)} \quad \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, \quad \text{(b)} \quad \begin{bmatrix} -5 & 6 \\ 8 & 1 \end{bmatrix}, \quad \text{(c)} \quad \begin{bmatrix} -1 & -1 & 0 \\ 1 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -1 \end{bmatrix}.$$

Next, you are going to prove the Cayley–Hamilton theorem for the particular case where A is diagonalizable. Just follow the instructions.

Problem B.

1. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ span $\mathbf{R}^n$, and let B be an $n \times n$ matrix such that

$$B\mathbf{v}_1 = \mathbf{0}, \quad \ldots, \quad B\mathbf{v}_k = \mathbf{0}.$$

Prove that B is the zero matrix.

2. Let λ be an eigenvalue of a square matrix A with corresponding eigenvector $\mathbf{v}$. Prove that for any positive integer k,

$$A^k\mathbf{v} = \lambda^k\mathbf{v}.$$

3. Let A be a diagonalizable matrix with characteristic polynomial $p(x)$. Prove the Cayley–Hamilton theorem for A as follows: Prove that for any eigenvector $\mathbf{v}$ of A, the vector $p(A)\mathbf{v}$ is zero (using Part 2). Then use Part 1 to conclude that $p(A) = \mathbf{0}$.

Next, you will be led to proving the Cayley–Hamilton theorem for any square matrix.

If B is an $n \times n$ matrix with entries polynomials in x, then there are unique matrices $B_0, B_1, \dots, B_k$ with scalar entries such that

$$B = B_0 + B_1x + \cdots + B_kx^k.$$

For example,

$$\begin{bmatrix} 1+x-3x^2 & -1+x \\ 2+5x & -6x+x^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 5 & -6 \end{bmatrix} x + \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix} x^2.$$

Much of the matrix arithmetic for ordinary matrices extends to matrices with polynomial entries. For example, for matrices B with polynomial entries, we have

$$\operatorname{Adj}(B)B = \det(B)I_n.$$

In the next few lines, we show how to prove the Cayley–Hamilton theorem for any 2×2 matrix A. If the characteristic polynomial of A is

$$p(\lambda) = a + b\lambda + \lambda^2,$$

then consider the matrix

$$B = A - \lambda I.$$

Since the maximum degree in λ of the elements of $\operatorname{Adj}(B)$ is 1. There are unique matrices B_0 and B_1 with scalar entries such that

$$\operatorname{Adj}(B) = B_0 + B_1\lambda.$$

Hence

$$\operatorname{Adj}(B)B = (B_0 + B_1\lambda)(A - \lambda I) = B_0A + (-B_0 + B_1A)\lambda - B_1\lambda^2.$$

On the other hand,

$$\mathrm{Adj}(B)B = \det(B)I = p(\lambda)I = aI + bI\lambda + I\lambda^2.$$

Therefore, by uniqueness,

$$aI = B_0A, \quad -B_0 + B_1A = bI, \quad -B_1 = I.$$

So

$$p(A) = aI + bA + A^2 = B_0A + (-B_0 + B_1A)A + A^2 = \mathbf{0}.$$

Problem C. Following the above steps, prove the Cayley–Hamilton theorem for any square matrix.

7.7.2 Gerschgorin circles

Gerschgorin discovered and proved in 1931 two important theorems that yield information on the location of eigenvalues on the complex plane, without the need of computing the eigenvalues. Here we study the first of Gerschgorin's theorems.

Theorem 7.7.2 (Gerschgorin circles). *Let $A = [a_{ij}]$ be an $n \times n$ matrix. Each eigenvalue of A lies in at least one of the circles in the complex plane centered at $c_i = a_{ii}$ with radii r_i given by*

$$r_i = \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, n.$$

The Gerschgorin circles theorem allows the entries of A to be complex numbers. The absolute values $|a_{ij}|$ then are those of complex numbers. Recall that the absolute value of $z = a + ib$ is $|z| = \sqrt{a^2 + b^2}$.

As an example, the matrix

$$A = \begin{bmatrix} 2 & 0 & -5/2 \\ 0 & -1 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

has eigenvalues -1, i, $-i$. The Gerschgorin circles are

(a) C_1, centered at $a_{11} = 2$ with radius $|a_{12}| + |a_{13}| = 0 + |-5/2| = 2.5$;
(b) C_2, centered at $a_{22} = -1$ with radius $|a_{21}| + |a_{23}| = 0 + 1 = 1$;
(c) C_3, centered at $a_{33} = -2$ with radius $|a_{31}| + |a_{32}| = 2 + 0 = 2$.

It is clear that each eigenvalue is in one of the circles C_1, C_2, and C_3 (Figure 7.20).

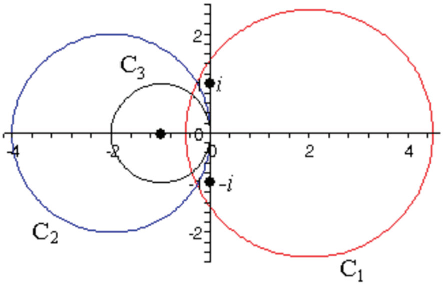

Figure 7.20: Gerschgorin circles for *A*.

Problem A. For each of the following matrices, compute the eigenvalues and find the center and radius of each Gerschgorin circle. Plot each Gerschgorin circle and verify graphically the Gerschgorin circles theorem.

$$A = \begin{bmatrix} -2 & -2 & 0 \\ 2 & -3 & 3 \\ 0 & 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} \sqrt{3} & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 0 & 5/2 \\ -2 & -1 & 2 \\ -2 & 0 & 2 \end{bmatrix}.$$

Problem B. Prove the Gerschgorin circles theorem as follows: Let λ be an eigenvalue of A, and let $\mathbf{v} = (v_1, \dots, v_n)$ be a corresponding eigenvector.

1. Prove that for $i = 1, \dots, n$, we have

$$(\lambda - a_{ii})\, v_i = \sum_{j \neq i} a_{ij} v_j$$

2. Let i be such that $|v_i|$ is the largest absolute value of $|v_j|$. So $|v_i| = \max_{1 \le j \le n} |v_j|$. Use Part 1 to prove that

$$|\lambda - a_{ii}| \le \sum_{j \neq i} |a_{ij}|.$$

3. Complete the proof of the theorem.

7.7.3 Transition of probabilities (Part 2)

We now return to Project 2 of Section 3.8 to answer a few more questions.

Problem A. A group of people buys cars every four years from one of three automobile manufacturers, A, B and C. The transition of probabilities of switching from one manufacturer to another is given by the matrix

$$R = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.1 \end{bmatrix}.$$

1. Use eigenvalues to compute $\lim_{n\to\infty} R^n$.
2. Will one of the manufacturers eventually dominate the market no matter what the initial sales are?

Problem B. Consider the stochastic matrix of transition probabilities

$$T = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{2} \end{bmatrix},$$

expressing the flow of customers from and to markets A and B after one purchase. Recall that a *market equilibrium* is a vector of shares (a, b) that remains the same from one purchase to the next.

1. Prove that a market equilibrium is an eigenvector of the transition probabilities matrix. What is the corresponding eigenvalue?
2. Prove that T has a market equilibrium.
3. Compute $\lim_{n\to\infty} T^n$.
4. Will one of the markets eventually dominate the other?

7.8 Technology-aided problems and answers

Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 4 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0.2 & 0.3 & 0.8 \\ 0.2 & 0.3 & 0.1 \\ 0.6 & 0.4 & 0.1 \end{bmatrix}, \quad R = \begin{bmatrix} 0.2 & 0 & 0.8 \\ 0 & 0 & 0.2 \\ 0.8 & 1 & 0 \end{bmatrix}.$$

1. Without computing, find one eigenvalue of A. Then use your program to compute all the eigenvalues and basic eigenvectors numerically and, if possible, exactly. Confirm your answer by showing that the computed eigenvalues satisfy the characteristic equation and that the computed basic eigenvectors are indeed eigenvectors of A.

2. Diagonalize A by finding D and P. Then verify your answer by showing that $A = PDP^{-1}$.

3. Verify the Hamilton–Cayley theorem (discussed in Project 1) for A. So if $p(x)$ is the characteristic polynomial, then prove that $p(A)$ (the expression where the variable x has been replaced by the matrix A) equals the zero matrix.

4. Find the roots of the polynomial $p(x) = x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120$ directly and by computing the eigenvalues of the companion matrix.

5. Find an approximation to 4 decimal places for the limit of the stochastic matrix S, $\lim_{n\to\infty} S^n$ (a) directly, by computing S^n for large n, and (b) by using eigenvalues.

6. Prove that R is a regular matrix.

7. Let A_n be the $n \times n$ matrix with entries 1. Find a formula for its eigenvalues and basic eigenvectors.

$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \dots .$$

8. Let B_n be the $n \times n$ matrix with diagonal entries n and remaining entries 1. Find a formula for its eigenvalues and basic eigenvectors.

$$B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \dots .$$

9. Let K_n be the 5×5 matrix with diagonal entries n and remaining entries 1. Find a formula for its eigenvalues and basic eigenvectors. If your software does not have symbolic computation, then use several values for n to come up with a guess for the formula.

$$K_n = \begin{bmatrix} n & 2 & 2 & 2 & 2 \\ 2 & n & 2 & 2 & 2 \\ 2 & 2 & n & 2 & 2 \\ 2 & 2 & 2 & n & 2 \\ 2 & 2 & 2 & 2 & n \end{bmatrix}.$$

10. Find the eigenvalues of L_2, L_3, L_4. Generalize to the case of an $n \times n$ matrix with as on the diagonal and bs elsewhere. If your software does not have symbolic computation, then use several values for n to come up with a guess for the formula.

$$L_2 = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, \quad L_3 = \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}, \quad L_4 = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}, \quad \dots .$$

11. This exercise is modeled after a known example in population dynamics attributed to H. Bernadelli, P. H. Leslie, and E. G. Leslie. A species of beetles lives 3 years. Let A, B, and C be the 0–1 year-old, 1–2 year-old, and 2–3 year-old females, respectively. No female from group A produces offspring. Each female in group B produces 8 females, and each female in group C produces 24 females. Suppose that only 1/4 from group A survive to group B and only 1/6 of group B survive to group C. If A_k, B_k, and C_k are the numbers of females in A, B, and C after k years, find a matrix M such that $M(A_k, B_k, C_k)$ is $(A_{k+1}, B_{k+1}, C_{k+1})$. If $A_0 = 100$, $B_0 = 40$, and $C_0 = 20$, then use eigenvalues and eigenvectors to determine whether the species will become extinct.

12. Use the Rayleigh–Ritz method with $k = 20$ and the given $\mathbf{x}_0$ to approximate the dominant eigenvalue of the matrix.

$$A = \begin{bmatrix} 17 & 1 & 1 \\ 1 & 17 & 1 \\ 1 & 1 & 17 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

13. Use the inverse power method with $k = 10$ and the given $\mathbf{x}_0$ to approximate the eigenvalue closest to the origin.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

7.8.1 Selected solutions with Mathematica

```
(* Exercise 1 *)
(* Non-invertible (repeated rows), so 0 is an eigenvalue.*)
A = {{1, 1, 0, 0}, {1, 1, 0, 0}, {0, 0, 1, 4}, {0, 0, 4, 1}}
EE=Eigensystem[A]  (* Both eigenvalues and eigenvectors.  *)
evas = EE[[1]]         (*The first vector are the eigenvalues.*)
P = EE[[2]]         (* the second vector are the eigenvectors. *)
p=CharacteristicPolynomial[A, x] (* Characteristic polynomial.*)
p /. x->evas[[1]]          (* Substituting, for example, the 4th*)
e = evas[[2]]                  (* Pick an eigenvalue and the    *)
v = P[[2]]                     (* corresponding eigenvector.    *)
A . v - e v                     (* Compute Av-ev and simplify to *)
(* Also checkout the commands Eigenvalues, Eigenvectors *)
(* Exercise 2 *)
DD = DiagonalMatrix[evas]     (* DD eigenvalues on the diagonal*)
PP = Transpose[P]  (* Make the eigenvectors column vectors for P. *)
PP. DD . Inverse[PP] - A   (* The zero matrix as expected. *)
(* Exercise 3 *)
p /. {Power->MatrixPower,x->A} (* Substitute A into the char. poly. and *)
(* convert Power to  MatrixPower to a get the zero matrix, as expected. *)
(* Exercise 4  *)
p1 =x^5-15 x^4+85 x^3-225 x^2+274 x-120
Solve[p1==0,x]
Eigenvalues[{{0, 1, 0, 0, 0}, {0, 0, 1, 0, 0},{0, 0, 0, 1, 0},
{0, 0, 0, 0, 1}, {120, -274, 225, -85, 15}}]
(* the eigenvalues of the companion matrix.*)
(* Exercise 5 *)
S = {{.2,.3,.8},{.2,.3,.1},{.6,.4,.1}}
MatrixPower[S,80]    (* Identical columns. Limit to displayed accuracy.*)
v=Eigensystem[S]                              (* Next form D and P.    *)
DD=DiagonalMatrix[{v[[1,1]],v[[1,2]],v[[1,3]]}] (* D is differentiation.*)
P=Transpose[v[[2]]]                            (* P.                   *)
P . MatrixPower[DD,80] . Inverse[P] (* Same answer to >15 decimal places. *)
(* Exercise 6  *)
R = {{.2,0,.8},{0,0,.2},{.8,1,0}}
For[i=1,i<=4,i++,Print[MatrixPower[R,i]]]
(*All entries of R^4 are not 0 so R is regular.*)
(* Exercise 7 , partial *)
An[n_]:= Table[1, {i,1,n},{j,1,n}]
Eigensystem[An[2]]                    (* Then Eigensystem[An[[3]]], etc.. *)
(* Exercise 8 - Partial  *)
Bn[n_]:= Table[If[i==j, n, 1], {i,1,n},{j,1,n}]
Eigensystem[Bn[2]]                    (* Then Eigensystem[Bn[[3]]], etc.. *)
(* Exercise 9 - Partial     *)
K = Table[If[i==j, n, 2], {i,1,5},{j,1,5}]
Eigensystem[K]                    (* Then Eigensystem[K[[3]]], etc.. *)
(* Exercise 10 - Partial  *)
L[a_,b_,n_] := Table[If[i==j, a, b], {i,1,n},{j,1,n}] (*Define L_n *)
```

```
L[a, b, 3]    (* as a function of a, b, and n and test if it works *)
L[a, b, 4]     (* etc. *)
Eigenvalues[L[a, b, 3]]      (* The pattern after a few trials *)
Eigenvalues[L[a, b, 4]]      (* emerges *)
(* Exercise 11 - Partial. *)
M = {{0,8,24},{1/4,0,0},{0,1/6,0}}     (* The correct matrix. Etc.. *)
(* Exercise 12. *)
RayleighRitz[A_, x0_, k_] :=
Module[{i, x, y, r}, x = N[ArrayReshape[x0, {Length[x0], 1}]];
For[i = 0, i < k, i++, y = (1/(Norm[x]))*x;
x = A . y;
r = Transpose[y] . x;
Print[{r, y}] ];
Return[{r, y}]]
(* Then type *)
A = {{17, 1, 1}, {1, 17, 1}, {1, 1, 17}};
x0 = {1, 2, 3};
k = 20;
RayleighRitz[A, x0, k]
```

7.8.2 Selected solutions with MATLAB

```
% Exercise 1
% Non-invertible (repeated rows) so 0 is an eigenvalue.
A = [1 1 0 0; 1 1 0 0; 0 0 1 4; 0 0 4 1]
eig(A) % Numerical eigenvalues. Also may
[eves,evas]=eig(A)      % use the [,] format. eves is the matrix with
% columns the eigenvectors and evas is the diagonal matrix
% with diagonal the eigenvalues.
p = poly(A)                % The characteristic polynomial as a vector.
roots(p)                   % All the eigenvalues are obtained as roots of p.
polyval(p,evas(1,1))       % Also evaluate p at an eigenvalue.
e = evas(1,1),v = eves(:,1) % Pick an eigenvalue and its eigenvector.
A * v - e * v                % The answer is zero.
% Exerise 2
eves*evas*eves^(-1)-A % The zero matrix as expected.
% Exercise 3
polyvalm(poly(A),A) % Substitute A in the char poly poly(A). Zero.
% Exercise 4
p1 =x^5-15 x^4+85 x^3-225 x^2+274 x-120
p1=[1 -15 85 -225 274 -120], roots(p1) % roots finds the
% evals of the companion!
eig([0 1 0 0 0; 0 0 1 0 0; 0 0 0 1 0;
0 0 0 0 1; 120 -274 225 -85 15]) % The same.
% Exercise 5
S = [.2 .3 .8; .2 .3 .1; .6 .4 .1]; S^80  % Yields identical columns.
[P,D] = eig(S)                            % P and D.
```

```
P * D^80 * P^(-1)                         % Same answer by diagonalization.
% Exercise 6
R = [.2 0 .8; 0 0 .2; .8 1 0]
R^2,R^3,R^4             % R^4 has all entries nonzero so R is regular.
% Exercise 7 - Patrial
[P2,D2] = eig(ones(2)),[P3,D3] = eig(ones(3))  % An is just ones(n).
% Exercise 8 - Partial
% To define Bn create a script file named B having the following lines:
function a = B(n)
for i=1:n,
for j=1:n,
if i==j
a(i,j) = n;
else a(i,j) =1;
end
end
end                      % Now type
clear
[P2,D2] = eig(B(2)),[P3,D3] = eig(B(3))          % Etc..
% Exercise 9 - Partial
% To define Kn create a script file named Bn.m having the following lines:
function a = K(n)
for i=1:5,
for j=1:5,
if i==j
a(i,j) = n;
else a(i,j) =2;
end
end
end                      % Now type
[P2,D2] = eig(K(2)),[P3,D3] = eig(K(3))          % Etc..
% Exercise 10
% We define the matrices L as functions of a, b, and n.
% To define L create a script file named L.m having the following:
function r = L(a,b,n)
for i=1:n,
for j=1:n,
if i==j
r(i,j) = a;
else r(i,j) =b;
end
end
end
L(3,5,4)                % Try a few matrics to
L(3,5,6)                % test the program and try to find a pattern.
[P2,D2] = eig(L(3,5,4)),[P3,D3] = eig(L(3,5,5)) % Try as many as needed.
% Exercise 11 - Partial
M = [0 8 24; 1/4 0 0; 0 1/6 0]                % The correct matrix. Etc..
% Exercise 12.
```

```
function [B] = RayleighRitz(A, x0, k) % Function file.
x=x0;
for i = 1:(k-1)
  y = (1/(norm(x)))*x;
  x=A*y;
  r=y'*x;
end
r,y    % end file.
% Then type
A=[17 1 1; 1 17 1; 1 1 17];
x0=[1;2;3]
k=20;
RayleighRitz(A, x0, k)
```

7.8.3 Selected solutions with Maple

```
# Exercise 1
# Non-invertible (repeated rows) so 0 is an eigenvalue.
with(LinearAlgebra);
A := Matrix([[1,1,0,0],[1,1,0,0],[0,0,1,4],[0,0,4,1]]);
Eigenvalues(A);            # Eigenvalues
evas, P := Eigenvectors(A); # evas is the vector of eigenvalues
# and P is the matrix of eigenvectors.
p:=CharacteristicPolynomial(A, x);  # The characteristic polynomial.
subs(x=evas[1],p); # Substitute the first eigenvalue into p to get 0.
e := evas[1];      # Let e be the first eigenvalue.
v := SubMatrix(P, [1 .. 4], [1 .. 1]);   # Let v be the first eigenvector.
A.v-e.v;  # Compute Av-ev and simplify to get the zero vector.
# Exercise 2
DD:=DiagonalMatrix(evas); # Diagonal D, eigenvalues on the diagonal
P.DD.P^(-1)-A;                   #The zero matrix as expected.
# Exercise 3
eval(subs(x = A, p)); # Substitute A in the char poly to get zero.
# Exercise 4
p1 :=x^5-15*x^4+85*x^3-225*x^2+274*x-120;
solve(p1=0,x);
Eigenvalues(CompanionMatrix(p1)); # The same answer as above.
# Exercise 5
S := Matrix([[0.2, 0.3, 0.8], [0.2, 0.3, 0.1], [0.6, 0.4, 0.1]]);
S^80;      # Yields identical columns. Limit to displayed accuracy.
e, P := Eigenvectors(S);
DD := DiagonalMatrix(e);  # DD since D is already used by Maple.
(P . (DD^80)) . (1/P);  # Same answer to high accuracy.
# Exercise 6
R := Matrix([[.2,0,.8],[0,0,.2],[.8,1,0]]);
R^2;R^3:R^4; # R^4 has all entries nonzero, so R is regular.
# Exercise 7 - Patrial
```

```
An := proc(n) local i, j; Matrix(n,n, (i,j) -> 1) end:   # Define A_n.
Eigenvectors(An(2));Eigenvectors(An(3));                # Etc..
# Exercise 8 - Partial
Bn:=proc(n) subs(nn=n,Matrix(n,n,(i,j)->if i<>j
then 1 else nn fi))end:
Eigenvectors(Bn(2));Eigenvectors(Bn(3));                # Etc..
# Exercise 9 - Partial
K:= proc(n) local i, j; Matrix(5,5,    # Define the 5x5 as a
(i,j) -> if i<>j then 2 else n fi) end: # function of the diagonal entries.
Eigenvectors(K(2));Eigenvectors(K(3));  # Etc..
# Exercise 10 - Partial
L:=proc(a,b,n) local i, j; Matrix(n,n,(i,j)->
if i<>j then b else a fi) end:  # Define L_n as a function in a,b,n.
L(a, b, 3);                     # and test if it works,
L(a, b, 4);
Eigenvalues(L(a, b, 3));  # The pattern after a few trials
Eigenvalues(L(a, b, 4));  # emerges
# Exercise 11 - Partial
M := Matrix([[0,8,24],[1/4,0,0],[0,1/6,0]]); # The correct matrix.
# Exercise 12.
with(LinearAlgebra):
RayleighRitz := proc(A,x0,k)
local i, x, y, r;
x:=evalf(x0);
for i to (k-1) do
y := ( 1/(norm(x,2)) ) *x;
x := A.y;
r := Transpose(y).x;
end do:
print(r,y);
end proc
# Then type
A := Matrix([[17, 1, 1], [1, 17, 1], [1, 1, 17]]);
x0 := Vector([1, 2, 3]);
k:=20;
RayleighRitz(A, x0, k);
```

8 Orthogonality and least squares

> It matters little who first arrives at an idea, rather what is significant is how far that idea can go.
>
> Sophie Germain, French Mathematician (Figure 8.1).

Figure 8.1: Marie-Sophie Germain.
1880 illustration of a young Germain (circa 1790)
Public Domain,
https://commons.wikimedia.org/w/index.php?curid=19953423.
Marie-Sophie Germain (1776–1831) was a French mathematician. Despite the societal barriers she faced as a woman, Germain made significant contributions to mathematics. She is best known for her fundamental work on Fermat’s Last Theorem. Her research laid the foundation for later efforts to prove this famous theorem. Germain became the first woman to win a prize from the Paris Academy of Sciences for her work on the vibrations of elastic surfaces.

Introduction

In this chapter, we study more in depth the dot product and also the *inner product*, which is a generalization of the dot product for abstract vectors. This important material applies to heat conduction, mechanical vibrations, electrostatic potential, wavelets, signal processing, trend analysis, and many other areas.

The generalized concept of orthogonality is very useful. Some of the applications are listed below.

Signal Processing. Orthogonal functions and waveforms play a crucial role in signal processing. In applications like telecommunications, orthogonal frequency-division multiplexing (OFDM) uses orthogonal subcarriers to transmit data efficiently.

Quantum Mechanics. In quantum mechanics, orthogonal wavefunctions represent states with different quantum numbers and are used to describe different energy levels and angular momentum states of particles.

Statistics. In statistics, orthogonal regression helps in modeling relationships between variables by minimizing the sum of squared perpendicular (orthogonal) distances from data points to the regression line. This is especially useful when you want to examine the relationship between variables that may not be linear.

Antenna Design. In telecommunications and radar systems, antenna arrays often use orthogonal radiating elements to minimize interference and improve signal reception and transmission.

https://doi.org/10.1515/9783111331850-008

Quantum Computing. Quantum computing relies on quantum bits or *qubits* that can be in superpositions of states. Making these states orthogonal and manipulating them is crucial for quantum algorithms and quantum information processing.

8.1 Orthogonal sets and matrices

In this section, we study sets of pairwise orthogonal vectors. Such sets are called orthogonal and share interesting properties that make them useful in computations. First, we prove identity (8.1), which is used several times in the chapter.

Theorem 8.1.1. *For an $m \times n$ matrix A, an n-vector $\mathbf{u}$ and an m-vector $\mathbf{v}$, we have*

$$(A\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (A^T\mathbf{v}). \tag{8.1}$$

Proof. $(A\mathbf{u}) \cdot \mathbf{v} = (A\mathbf{u})^T\mathbf{v} = (\mathbf{u}^TA^T)\mathbf{v} = \mathbf{u}^T(A^T\mathbf{v}) = \mathbf{u} \cdot (A^T\mathbf{v})$ □

8.1.1 Orthogonal sets

Definition 8.1.2. A set of n-vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is *orthogonal* if any two distinct vectors in it are orthogonal. This means that

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \quad \text{if} \quad i \neq j.$$

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbf{R}^3$ is orthogonal, then all possible pairs of distinct vectors $\{\mathbf{v}_1, \mathbf{v}_2\}, \{\mathbf{v}_1, \mathbf{v}_3\}, \{\mathbf{v}_2, \mathbf{v}_3\}$ must be orthogonal. Hence S forms a right-angled coordinate frame (Figure 8.2).

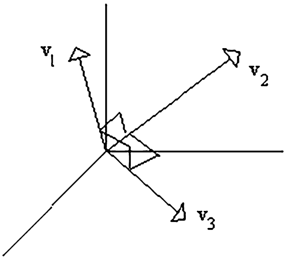

Figure 8.2: Orthogonal vectors in $\mathbf{R}^3$.

Example 8.1.3. Prove that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbf{R}^4$ is orthogonal, where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution. This is true, because

$$\begin{aligned} \mathbf{v}_1 \cdot \mathbf{v}_2 &= 2 \cdot 0 + 2 \cdot 2 + 4 \cdot (-1) + 0 \cdot 1 &&= 0, \\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= 2 \cdot (-2) + 2 \cdot 0 + 4 \cdot 1 + 0 \cdot 1 &&= 0, \\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= 0 \cdot (-2) + 2 \cdot 0 + (-1) \cdot 1 + 1 \cdot 1 &&= 0. \end{aligned}$$

An important property of orthogonal sets is summarized in the following theorem.

Theorem 8.1.4. *Let* $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *be an orthogonal set of nonzero vectors. If* $\mathbf{u}$ *is in* Span(S) *with*

$$\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k, \tag{8.2}$$

then

$$c_i = \frac{\mathbf{u} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}, \quad i = 1, \dots, k. \tag{8.3}$$

Proof. We form the dot product of each side of (8.2) with $\mathbf{v}_i$. We have

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v}_i &= (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) \cdot \mathbf{v}_i \\ &= c_1(\mathbf{v}_1 \cdot \mathbf{v}_i) + \cdots + c_k(\mathbf{v}_k \cdot \mathbf{v}_i) \\ &= c_i(\mathbf{v}_i \cdot \mathbf{v}_i), \end{aligned}$$

because $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$ by orthogonality. Hence $c_i = (\mathbf{u} \cdot \mathbf{v}_i)/(\mathbf{v}_i \cdot \mathbf{v}_i)$. Note that $\mathbf{v}_i \cdot \mathbf{v}_i = \|\mathbf{v}_i\|^2 \neq 0$ due to $\mathbf{v}_i \neq \mathbf{0}$. ☐

The strength of Theorem 8.1.4 lies in that a vector can be easily written as a linear combination in an orthogonal set by using (8.3) *without row reduction*.

Definition 8.1.5. The scalars $c_i = \frac{\mathbf{u} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i}$ of equation (8.3) can be defined for *any* n-vector $\mathbf{u}$, and they are called the *Fourier coefficients* of $\mathbf{u}$ with respect to S.

Theorem 8.1.6. *Any orthogonal set* $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *of nonzero* n*-vectors is linearly independent.*

Proof. We set a linear combination equal to **0**,

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

By Theorem 8.1.4 with $\mathbf{u} = \mathbf{0}$ we have

$$c_i = \frac{\mathbf{0}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i} = 0\,, \quad i = 1,\dots,k.$$

Hence S is linearly independent. □

We see that an orthogonal set of nonzero vectors is a *basis* for its span and that the coefficients c_i of equation (8.2) are *uniquely* determined by equation (8.3).

Definition 8.1.7. If a basis of a subspace V of $\mathbf{R}^n$ is an orthogonal set, then we call it an *orthogonal basis*.

Orthogonal bases are very useful, because the coordinates of vectors can be computed easily by using (8.3).

Example 8.1.8. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5/7 \\ 4/7 \\ 1/7 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 12 \\ -6 \\ 6 \end{bmatrix}.$$

Prove that the set $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis of $\mathbf{R}^3$ and write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

Solution. It is left as an exercise the verification that $\mathcal{B}$ is orthogonal. $\mathcal{B}$ is a linearly independent set of three 3-vectors by Theorem 8.1.6. So it is an orthogonal basis of $\mathbf{R}^3$. Let $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Then by (8.3) we have

$$\begin{aligned} \mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{u}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{u}\cdot\mathbf{v}_3}{\mathbf{v}_3\cdot\mathbf{v}_3}\mathbf{v}_3 \\ &= \frac{42}{14}\mathbf{v}_1 + \frac{-24}{12}\mathbf{v}_2 + \frac{6}{6/7}\mathbf{v}_3 \\ &= 3\mathbf{v}_1 - 2\mathbf{v}_2 + 7\mathbf{v}_3. \end{aligned}$$

This calculation is easier than the row reduction of the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{u}]$.

Theorem 8.1.9. *Let $\mathcal{B}$ be an orthogonal basis of a subspace V of $\mathbf{R}^n$. If a vector $\mathbf{u}$ of V is orthogonal to each vector of $\mathcal{B}$, then $\mathbf{u} = \mathbf{0}$.*

Proof. Exercise. □

Theorem 8.1.10. *If the columns of an $m \times n$ matrix A form an orthogonal set, then $A^T A$ is an $n \times n$ diagonal matrix. More precisely, if $A = [\mathbf{v}_1 \cdots \mathbf{v}_n]$, then*

$$A^T A = \begin{bmatrix} \|\mathbf{v}_1\|^2 & 0 & \cdots & 0 \\ 0 & \|\mathbf{v}_2\|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|\mathbf{v}_n\|^2 \end{bmatrix}. \tag{8.4}$$

Conversely, if (8.4) *is valid, then the columns of A form an orthogonal set.*

Proof. Let c_{ij} be the (i, j) entry of $A^T A$, and let $\mathbf{r}_i$ be the ith row of A^T. Hence $\mathbf{r}_i = \mathbf{v}_i$ as n-vectors. By the definition of matrix multiplication we have

$$c_{ij} = \mathbf{r}_i \cdot \mathbf{v}_j = \mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} \|\mathbf{v}_i\|^2 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

since $\mathbf{v}_i \cdot \mathbf{v}_i = \|\mathbf{v}_i\|^2$ and $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ by orthogonality. The proof of the converse is left as an exercise. ☐

To illustrate Theorem 8.1.10, let A have columns the vectors of Example 8.1.3. Then

$$A^T A = \begin{bmatrix} 2 & 2 & 4 & 0 \\ 0 & 2 & -1 & 1 \\ -2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -2 \\ 2 & 2 & 0 \\ 4 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

The diagonal entries are $\|\mathbf{v}_1\|^2$, $\|\mathbf{v}_2\|^2$, and $\|\mathbf{v}_3\|^2$, respectively.

8.1.2 Orthonormal sets

Definition 8.1.11. A set of vectors is *orthonormal* if it is orthogonal and consists of *unit* vectors (Figure 8.3). Thus $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is orthonormal if

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0, \quad i \neq j \quad \text{and} \quad \|\mathbf{v}_i\| = 1, \quad i = 1, \dots, k.$$

Because $\|\mathbf{v}_i\| = 1 \Leftrightarrow \|\mathbf{v}_i\|^2 = \mathbf{v}_i \cdot \mathbf{v}_i = 1$, we have

$$\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \text{ orthonormal} \Leftrightarrow \mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{8.5}$$

Example 8.1.12. The standard basis of $\mathbf{R}^n$ is orthonormal.

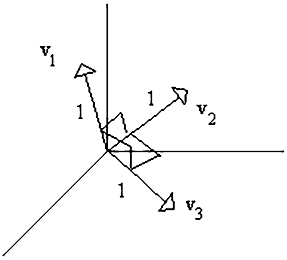

Figure 8.3: Orthonormal vectors in $\mathbf{R}^3$.

Example 8.1.13. The set $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ with

$$\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

is orthonormal, because $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ and $\|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1$.

We may normalize any orthogonal set of nonzero vectors to get an orthonormal set:

$$\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \text{ orthogonal} \Rightarrow \left\{ \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \dots, \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|} \right\} \text{ orthonormal.}$$

This is because each vector $\mathbf{v}_i / \|\mathbf{v}_i\|$ is unit, and for $i \neq j$,

$$\frac{\mathbf{v}_i}{\|\mathbf{v}_i\|} \cdot \frac{\mathbf{v}_j}{\|\mathbf{v}_j\|} = \frac{\mathbf{v}_i \cdot \mathbf{v}_j}{\|\mathbf{v}_i\| \|\mathbf{v}_j\|} = 0.$$

Example 8.1.14. Normalizing $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of Example 8.1.8 yields the orthonormal $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{14} \\ -2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 5/\sqrt{42} \\ 4/\sqrt{42} \\ 1/\sqrt{42} \end{bmatrix}. \tag{8.6}$$

Definition 8.1.15. An orthonormal set that is a basis of a subspace V of $\mathbf{R}^n$ is called an *orthonormal basis* of V.

Example 8.1.16. U of Example 8.1.14 is an orthonormal basis of $\mathbf{R}^3$.

Written for orthonormal bases, Theorem 8.1.4 takes the following special form.

Theorem 8.1.17. *If $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is an orthonormal basis of a subspace V of $\mathbf{R}^n$, then each n-vector $\mathbf{u}$ of V can be uniquely written as*

$$\mathbf{u} = (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_1 + \cdots + (\mathbf{u} \cdot \mathbf{v}_k)\mathbf{v}_k. \tag{8.7}$$

It is clear from Theorem 8.1.17 that computing the components of a vector with respect to an orthonormal basis is easy.

Along the same lines, we also have the following useful inequality.

Theorem 8.1.18 (Bessel's inequality). *Let* $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ *be an orthonormal subset of* $\mathbf{R}^n$ *(not necessarily a basis), and let* $\mathbf{u}$ *be any n-vector. Then*

$$\|\mathbf{u}\|^2 \geq (\mathbf{u} \cdot \mathbf{v}_1)^2 + \cdots + (\mathbf{u} \cdot \mathbf{v}_k)^2 . \tag{8.8}$$

Proof. See Exercise 25. ☐

Theorem 8.1.10 also has an important particular case. The matrix $A = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$ has orthonormal columns if and only if each diagonal entry $\|\mathbf{v}_i\|^2$ of $A^T A$ is 1. This is equivalent to $A^T A = I$. Thus we have the following theorem.

Theorem 8.1.19. *The columns of an* $m \times n$ *matrix* A *form an orthonormal set (hence* $m \geq n$*) if and only if*

$$A^T A = I_n.$$

8.1.3 Orthogonal matrices

Definition 8.1.20. A *square* matrix A is called *orthogonal* if it has *orthonormal* columns.

> A *nonsquare* matrix with orthonormal columns is *not* called orthogonal. Also, a square matrix with only orthogonal columns is *not* called orthogonal.

Orthogonal matrices are *invertible*, because they are square with linearly independent columns. In fact, Theorem 8.1.19 for $m = n$ implies the following important theorem.

Theorem 8.1.21. *Let* A *be a square matrix. The following are equivalent.*

1. *A is orthogonal.*
2. $A^T A = I$.
3. $A^{-1} = A^T$.

Example 8.1.22. Prove that the rotation matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is orthogonal.

Solution. This is true by Theorem 8.1.21, because

$$AA^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By Theorem 8.1.21 the inverse of an orthogonal matrix is its transpose.

Example 8.1.23. Compute the inverse of $A = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$, with $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ from Example 8.1.14.

Solution. A has orthonormal columns, so it is orthogonal. Thus by Theorem 8.1.21 we have

$$A^{-1} = A^T = \begin{bmatrix} 1/\sqrt{14} & -2/\sqrt{14} & 3/\sqrt{14} \\ -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 5/\sqrt{42} & 4/\sqrt{42} & 1/\sqrt{42} \end{bmatrix}.$$

Recall that a *permutation* matrix is a matrix obtained from the identity matrix I by permuting its columns.

Example 8.1.24. A permutation matrix is orthogonal.

Theorem 8.1.25. *For an $n \times n$ matrix A, the following statements are equivalent.*

1. *A is orthogonal.*
2. *$A\mathbf{u} \cdot A\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$ for any n-vectors $\mathbf{u}$ and $\mathbf{v}$* *(preservation of dot products).*
3. *$\|A\mathbf{v}\| = \|\mathbf{v}\|$ for any n-vector $\mathbf{v}$* *(preservation of norms).*

Proof.

$1 \Rightarrow 2$ If A is orthogonal, then $A^TA = I$. So by (8.1) we have

$$A\mathbf{u} \cdot A\mathbf{v} = \mathbf{u} \cdot (A^TA\mathbf{v}) = \mathbf{u} \cdot \mathbf{v}.$$

$2 \Leftarrow 1$ Suppose $A\mathbf{u} \cdot A\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$. In particular, $A\mathbf{e}_i \cdot A\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j$. Since the standard basis is orthonormal, we have

$$A\mathbf{e}_i \cdot A\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

which shows that A is orthogonal by (8.5), because $A\mathbf{e}_i$ is the ith column of A.

$2 \Leftrightarrow 3$ The proof of this equivalence is left as an exercise. □

ℹ The matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ defined by an orthogonal matrix A is also called orthogonal. By Theorem 8.1.25 we see that *orthogonal matrix transformations preserve dot products. Hence they preserve lengths and angles.*

Theorem 8.1.26. *Let A and B be $n \times n$ orthogonal matrices. Then*

1. *AB is orthogonal,*
2. *A^{-1} is orthogonal.*

Proof. By Theorem 8.1.25 it suffices to prove that AB preserves norms:

$$\|AB\mathbf{v}\| = \|A(B\mathbf{v})\| = \|B\mathbf{v}\| = \|\mathbf{v}\| . \qquad \square$$

Because the inverse and hence the transpose of an orthogonal matrix A is orthogonal, we conclude that *the rows of an orthogonal matrix are also orthonormal.* !

Another implication of Theorem 8.1.25 is the following.

Theorem 8.1.27. *If λ is an eigenvalue of an orthogonal matrix A, then* $|\lambda| = 1$.

Proof. If $\mathbf{v}$ is an eigenvector of A, then by Part 3 of Theorem 8.1.25

$$\|\mathbf{v}\| = \|A\mathbf{v}\| = \|\lambda\mathbf{v}\| = |\lambda| \, \|\mathbf{v}\| .$$

Hence $|\lambda| = 1$, since $\|\mathbf{v}\| \neq 0$. $\square$

Theorem 8.1.27 also holds for complex eigenvalues of a real matrix A.

8.1.4 What makes orthogonal matrices important

Orthogonal matrices are important in numerical approximations, because they preserve lengths by Part 3 of Theorem 8.1.25.

For example, suppose we were to compute a n-vector $\mathbf{v}$, but instead, we found an approximation $\mathbf{u}$ of $\mathbf{v}$ with error vector $\boldsymbol{\varepsilon}$. So

$$\mathbf{v} = \mathbf{u} + \boldsymbol{\varepsilon}.$$

Now if A is an orthogonal matrix, then

$$A\mathbf{v} = A\left(\mathbf{u} + \boldsymbol{\varepsilon}\right) = A\mathbf{u} + A\boldsymbol{\varepsilon}.$$

By Theorem 8.1.25 we have

$$\|A\boldsymbol{\varepsilon}\| = \|\boldsymbol{\varepsilon}\| .$$

So the magnitude of the error vector is preserved. We see that numerical errors do not grow out of control under orthogonal transformations.

Exercises 8.1

In Exercises 1–3, prove that the set of given n-vectors is orthogonal. Which of these sets form an orthogonal basis for $\mathbf{R}^n$?

1. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}.$

2. $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$

3. $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$

4. Give an example of a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ such that the pairs $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_2, \mathbf{v}_3$ are orthogonal but S is not orthogonal.

In Exercises 5–7, prove that each set of vectors forms an orthogonal basis for $\mathbf{R}^3$. Use Theorem 8.1.4 to express $\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ as a linear combination of these vectors.

5. $\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 20 \end{bmatrix}.$

6. $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$

7. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$

In Exercises 8–9, normalize the orthogonal set to get an orthonormal set.

8. $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$

9. $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$

10. Find all possible values of a and b such that $\left\{\begin{bmatrix} 1/3 \\ 2\sqrt{2}/3 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}\right\}$ is orthonormal.

In Exercises 11–12, use Theorem 8.1.17 to write $\mathbf{e}_1$ as a linear combination in the given orthonormal basis.

11. $= \left\{\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}\right\}.$

12. $\left\{\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}\right\}$.

13. *True or False?* Explain.

(a) Every orthogonal basis can be transformed into an orthonormal basis.

(b) Every orthogonal set of *n*-vectors is linearly independent.

(c) Every orthonormal set of *n*-vectors is linearly independent.

(d) Every orthonormal set of *n*-vectors is a basis of $\mathbf{R}^n$.

In Exercises 14–16, determine whether the given matrix is orthogonal. If the matrix is orthogonal, then find its inverse.

14. (a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

15. $\begin{bmatrix} 3/\sqrt{14} & 1/\sqrt{6} & -2/\sqrt{21} \\ -2/\sqrt{14} & 2/\sqrt{6} & -1/\sqrt{21} \\ 1/\sqrt{14} & 1/\sqrt{6} & 4/\sqrt{21} \end{bmatrix}$.

16. $\begin{bmatrix} 1/2 & 1/2 & 0 & 1/\sqrt{2} \\ 1/2 & 1/2 & 0 & -1/\sqrt{2} \\ -1/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 1/\sqrt{2} & 0 \end{bmatrix}$.

17. Prove that

$$A^T A = \left(a^2 + b^2 + c^2 + d^2\right) I_4,$$

where

$$A = \begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}.$$

What condition on a, b, c, d makes A orthogonal?

18. *True or False?* An orthogonal matrix is one

(a) whose columns are orthogonal,

(b) whose columns are orthonormal,

(c) that is square and its rows are orthonormal.

19. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Write explicitly all equations in a, b, c, d for A to be orthogonal.

20. Find an orthogonal basis of $\mathbf{R}^3$ that includes the vector $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.

21. Suppose that the columns of an $m \times n$ matrix A form an orthonormal set. Why should $m \geq n$?

22. Prove that the *rows* of an $n \times n$ orthogonal matrix form a basis for $\mathbf{R}^n$.

23. Prove Theorem 8.1.9.

24. Complete the proof of Theorem 8.1.10.

25. Prove *Bessel's inequality* (Theorem 8.1.18). (*Hint:* Let $\mathbf{v} = \sum_{i=1}^{k}(\mathbf{u}\cdot\mathbf{v}_i)\mathbf{v}_i$, and let $\mathbf{r} = \mathbf{u}-\mathbf{v}$. Prove that $\mathbf{r}\cdot\mathbf{v} = \mathbf{0}$. Then use the Pythagorean theorem.)

26. Complete the proof of Theorem 8.1.25.

27. Complete the proof of Theorem 8.1.26.

28. Let $T(\mathbf{x}) = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}\mathbf{x}$. Prove:

(a) T is a *reflection* about the line $\operatorname{Span}\left\{\begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix}\right\}$.

(b) T is an orthogonal transformation of the plane.

A square matrix H of size n is called *Hadamard* if its entries are $h_{ij} = \pm 1$ and if $H^T H = nI_n$. For example,

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

are Hadamard. These matrices are useful in error-correcting codes and signal processing. It is an *open problem* for which sizes n such matrices exist.

29. Prove that a Hadamard matrix H has orthogonal columns. Why is H not an orthogonal matrix? If n is the size of H, then what modification is needed to make H orthogonal?

30. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$, $T(\mathbf{x}) = A\mathbf{x}$, be an orthogonal transformation of the plane. Prove that T is either

(a) a rotation with $\det A = 1$ or

(b) a reflection about a line through $\mathbf{0}$ with $\det A = -1$.

31. (*General Pythagorean theorem*) Let $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ be an orthogonal set of $\mathbf{R}^n$. Prove that

$$\|\mathbf{v}_1 + \cdots + \mathbf{v}_k\|^2 = \|\mathbf{v}_1\|^2 + \cdots + \|\mathbf{v}_k\|^2 .$$

In Exercises 32–35 determine whether the transformation is orthogonal. Your arguments may be geometrical.

32. Reflection about the xy-bisector (Figure 8.4).

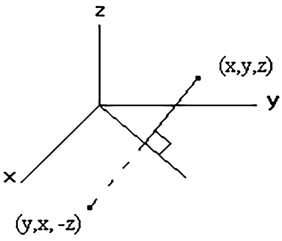

Figure 8.4: Reflection about a bisector.

33. Projection onto the x-axis (Figure 8.5).

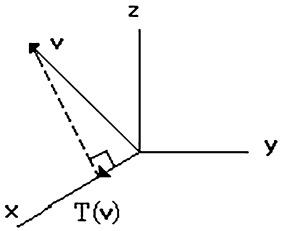

Figure 8.5: Projection onto line.

34. Reflection about the *bisector plane* $y = x$ (Figure 8.6).

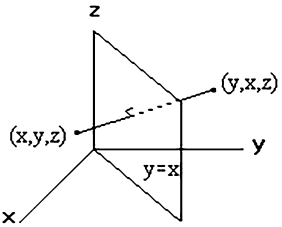

Figure 8.6: Reflection onto plane.

35. Rotation θ radians about the z-axis in the positive direction (Figure 8.7).

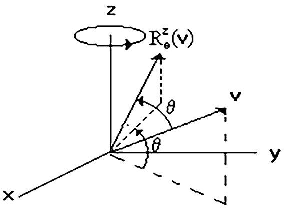

Figure 8.7: Rotation about axis.

8.2 The Gram–Schmidt process

In this section, we study orthogonal *projections* and *complements*. These generalize basic notions of Section 2.5 such as projection of a vector along another vector, projection of a vector onto a plane, and the normal vector to a plane. We then use our new tools to construct orthogonal bases of subspaces of $\mathbf{R}^n$.

On occasion, we use, again, the space-saving notation $(x_1, x_2, \ldots, x_n)$ to denote vectors, instead of writing them in matrix column form or in transposed matrix row form.

8.2.1 Orthogonal complements

Definition 8.2.1. Let V be a subspace of $\mathbf{R}^n$, and let $\mathbf{u}$ be an n-vector. We say that $\mathbf{u}$ is *orthogonal* to V if $\mathbf{u}$ is orthogonal to each vector of V.

Theorem 8.2.2. *The n-vector* $\mathbf{u}$ *is orthogonal to* $V = \mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbf{R}^n$ *if and only if*

$$\mathbf{u} \cdot \mathbf{v}_i = 0\,, \quad i = 1, \ldots, k. \tag{8.9}$$

Proof. If $\mathbf{u}$ is orthogonal to V, then (8.9) holds. Conversely, if we assume (8.9) and let $\mathbf{v}$ be any element of V, then there are scalars c_i such that

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k.$$

Forming the dot product with $\mathbf{u}$ yields

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \mathbf{u} \cdot (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k)\\ &= c_1(\mathbf{u} \cdot \mathbf{v}_1) + \cdots + c_k(\mathbf{u} \cdot \mathbf{v}_k)\\ &= c_1 0 + \cdots + c_k 0 = 0.\end{aligned}$$

So $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Hence $\mathbf{u}$ is orthogonal to V, as stated. □

By Theorem 8.2.2 we check that $\mathbf{u}$ is orthogonal to a subspace V by checking orthogonality only on a *spanning subset* of V.

Definition 8.2.3. The set of all n-vectors orthogonal to V is called the *orthogonal complement* of V and is denoted by $V^\perp$ (read “V perp.”).

Example 8.2.4. In $\mathbf{R}^3$ the orthogonal complement of a plane through $\mathbf{0}$ is the line through $\mathbf{0}$ perpendicular to the plane. Also, the orthogonal complement of a line through $\mathbf{0}$ is the plane through $\mathbf{0}$ perpendicular to the line (Figure 8.8).

Theorem 8.2.5. *Let* V *be a subspace of* $\mathbf{R}^n$. *Then*

1. $V^\perp$ *is a subspace of* $\mathbf{R}^n$;
2. $(V^\perp)^\perp = V$.

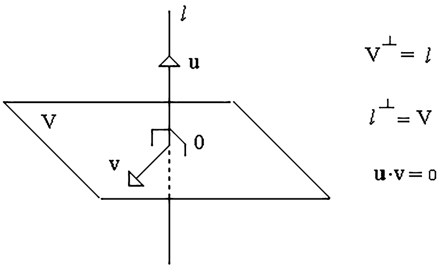

Figure 8.8: Orthogonal complement of a line and of a plane.

Proof of 1. $V^\perp$ is nonempty, because it contains $\mathbf{0}$. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be in $V^\perp$, and let $\mathbf{v}$ be any vector of V. Then $\mathbf{u}_1 \cdot \mathbf{v} = 0$ and $\mathbf{u}_2 \cdot \mathbf{v} = 0$. For any scalars c_1 and c_2, we have

$$(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) \cdot \mathbf{v} = c_1\mathbf{u}_1 \cdot \mathbf{v} + c_2\mathbf{u}_2 \cdot \mathbf{v} = c_1 0 + c_2 0 = 0.$$

Therefore $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ is orthogonal to every vector of V, and hence it is in $V^\perp$. So $V^\perp$ is a subspace of $\mathbf{R}^n$. ☐

The notion of orthogonal complement allows us to express an important relation between the column space of a matrix and the null space of its transpose.

Theorem 8.2.6 (The fundamental theorem of linear algebra). *Let A be any $m \times n$ matrix. Then the orthogonal complement of the column space of A equals the null space of A^T, and the orthogonal complement of the row space of A equals the null space of A (Figure 8.9):*[1]

$$\mathrm{Col}(A)^\perp = \mathrm{Null}(A^T) \quad \text{and} \quad \mathrm{Row}(A)^\perp = \mathrm{Null}(A).$$

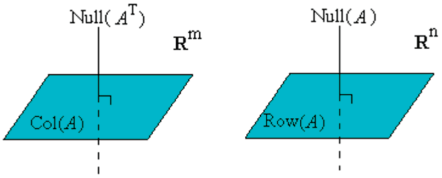

Figure 8.9: The fundamental theorem of linear algebra.

Proof. Let $A = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$. Then $\mathbf{u}$ is in $\mathrm{Col}(A)^\perp$ if and only if $\mathbf{u}$ is orthogonal to every vector of $\mathrm{Col}(A)$. Equivalently, $\mathbf{u}$ is orthogonal to the columns of A by Theorem 8.2.2. So we have

---

[1] See article [28].

$$\begin{aligned} \mathbf{u} \in \operatorname{Col}(A)^{\perp} &\Leftrightarrow \mathbf{u} \cdot \mathbf{v}_1 = 0, \ldots, \mathbf{u} \cdot \mathbf{v}_n = 0 \\ &\Leftrightarrow \mathbf{v}_1^T \mathbf{u} = 0, \ldots, \mathbf{v}_n^T \mathbf{u} = 0 \\ &\Leftrightarrow A^T \mathbf{u} = 0 \\ &\Leftrightarrow \mathbf{u} \in \operatorname{Null}(A^T). \end{aligned}$$

The second relation follows from the first by switching rows to columns. □

Example 8.2.7. Verify the relation $\operatorname{Col}(A)^{\perp} = \operatorname{Null}(A^T)$ of Theorem 8.2.6 for the matrix $A = \left[\begin{smallmatrix} 1 & 2 \\ -2 & 0 \\ 1 & 4 \end{smallmatrix}\right]$.

Solution. By Theorem 8.2.2 it suffices to prove that each vector of some basis of $\operatorname{Col}(A)$ is orthogonal to each vector of some basis of $\operatorname{Null}(A^T)$. Since the columns of A are linearly independent, they form a basis for $\operatorname{Col}(A)$. By reducing $[A^T : \mathbf{0}]$ it is easy to prove that $\{(4, 1, -2)\}$ is a basis for $\operatorname{Null}(A^T)$. We now check

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} = 0 = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix},$$

as predicted by Theorem 8.2.6.

8.2.2 Orthogonal projections and best approximation

In Section 2.5, we studied the orthogonal projection of a plane or of a space vector $\mathbf{u}$ onto another vector $\mathbf{v}$ ($\neq \mathbf{0}$). We decomposed $\mathbf{u}$ as a sum

$$\mathbf{u} = \mathbf{u}_{\text{pr}} + \mathbf{u}_{\text{c}},$$

where $\mathbf{u}_{\text{pr}}$ and $\mathbf{u}_{\text{c}}$ are *orthogonal* vectors with $\mathbf{u}_{\text{pr}}$ being in the direction of $\mathbf{v}$ (Figure 8.10). The vector $\mathbf{u}_{\text{pr}}$ is the *orthogonal projection* of $\mathbf{u}$ along $\mathbf{v}$, and $\mathbf{u}_{\text{c}}$ is the *component* of $\mathbf{u}$ *orthogonal* to $\mathbf{v}$. Furthermore, we found formulas for $\mathbf{u}_{\text{pr}}$ and $\mathbf{u}_{\text{c}}$ in terms of the dot product:

$$\mathbf{u}_{\text{pr}} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}, \quad \mathbf{u}_{\text{c}} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}. \tag{8.10}$$

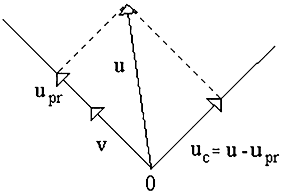

Figure 8.10: The orthogonal projection $\mathbf{u}_{\text{pr}}$ of $\mathbf{u}$ along $\mathbf{v}$.

Note that $\operatorname{Span}\{\mathbf{v},\mathbf{u}\} = \operatorname{Span}\{\mathbf{v},\mathbf{u}_c\}$. Also note that $\mathbf{u}_{\mathrm{pr}}$ and $\mathbf{u}_c$ remain the same if we replace $\mathbf{v}$ with $c\mathbf{v}$, $c \neq 0$, because

$$\frac{\mathbf{u}\cdot c\mathbf{v}}{c\mathbf{v}\cdot c\mathbf{v}}c\mathbf{v} = \frac{c(\mathbf{u}\cdot\mathbf{v})}{c^2(\mathbf{v}\cdot\mathbf{v})}c\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}.$$

Thus $\mathbf{u}_{\mathrm{pr}}$ and $\mathbf{u}_c$ depend *only* on $\operatorname{Span}\{\mathbf{v}\}$ and not on $\mathbf{v}$ itself.

The length of $\mathbf{u}_c$ is clearly the shortest distance from the tip of $\mathbf{u}$ to the line $l = \operatorname{Span}\{\mathbf{v}\}$.

Example 8.2.8. Find the shortest distance from $\mathbf{u} = (1,-2,3)$ to the line $\mathbf{p} = (1,1,1)t$, $t \in \mathbf{R}$ (Figure 8.11).

Solution. Let $\mathbf{v} = (1,1,1)$. Then $\mathbf{u}\cdot\mathbf{v} = 2$ and $\mathbf{v}\cdot\mathbf{v} = 3$. We need $\|\mathbf{u}_c\|$. We have

$$\mathbf{u}_{\mathrm{pr}} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \left(\frac{2}{3},\frac{2}{3},\frac{2}{3}\right) \Rightarrow$$

$$\|\mathbf{u}_c\| = \|\mathbf{u}-\mathbf{u}_{\mathrm{pr}}\| = \left\|\left(\frac{1}{3},-\frac{8}{3},\frac{7}{3}\right)\right\| = \frac{\sqrt{114}}{3}.$$

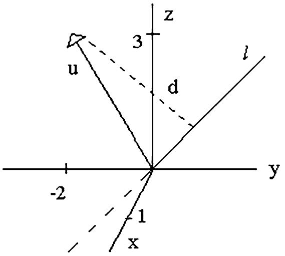

Figure 8.11: The length of the component of **u** orthogonal to *l* is the shortest distance from **u** to *l*.

We now generalize the notions of orthogonal projection and orthogonal complement of a 3-vector onto the line defined by another vector. We project any n-vector onto a subspace V of $\mathbf{R}^n$ so that the main properties of the plane or space projections are preserved. To extend equations (8.10) in an easy way, we assume that V has an orthogonal basis. This is not a restriction because, as we will see, an orthogonal basis always exists.

Definition 8.2.9. Let $\mathbf{u}$ be an n-vector, and let V be a subspace of $\mathbf{R}^n$ with an orthogonal basis $\mathcal{B} = \{\mathbf{v}_1,\dots,\mathbf{v}_k\}$. Then the *orthogonal projection* of $\mathbf{u}$ onto V is the vector

$$\mathbf{u}_{\mathrm{pr}} = \frac{\mathbf{u}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 + \cdots + \frac{\mathbf{u}\cdot\mathbf{v}_k}{\mathbf{v}_k\cdot\mathbf{v}_k}\mathbf{v}_k. \tag{8.11}$$

The difference $\mathbf{u}_c = \mathbf{u} - \mathbf{u}_{\mathrm{pr}}$ is called the *component of* $\mathbf{u}$ *orthogonal to* V,

$$\mathbf{u}_\mathrm{c} = \mathbf{u} - \frac{\mathbf{u}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 - \cdots - \frac{\mathbf{u}\cdot\mathbf{v}_k}{\mathbf{v}_k\cdot\mathbf{v}_k}\mathbf{v}_k, \tag{8.12}$$

$$\mathbf{u} = \mathbf{u}_\mathrm{pr} + \mathbf{u}_\mathrm{c}. \tag{8.13}$$

First, we note that $\mathbf{u}_\mathrm{c}$ is orthogonal to all vectors of $\mathcal{B}$:

$$\mathbf{u}_\mathrm{c}\cdot\mathbf{v}_1 = 0, \quad \dots, \quad \mathbf{u}_\mathrm{c}\cdot\mathbf{v}_k = 0. \tag{8.14}$$

This is because by the orthogonality of $\mathcal{B}$ we have

$$\begin{aligned}
\mathbf{u}_\mathrm{c}\cdot\mathbf{v}_i &= \mathbf{u}\cdot\mathbf{v}_i - \frac{\mathbf{u}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1\cdot\mathbf{v}_i - \cdots - \frac{\mathbf{u}\cdot\mathbf{v}_k}{\mathbf{v}_k\cdot\mathbf{v}_k}\mathbf{v}_k\cdot\mathbf{v}_i \\
&= \mathbf{u}\cdot\mathbf{v}_i - \frac{\mathbf{u}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}\mathbf{v}_i\cdot\mathbf{v}_i \\
&= \mathbf{u}\cdot\mathbf{v}_i - \mathbf{u}\cdot\mathbf{v}_i = 0.
\end{aligned}$$

So $\mathbf{u}_\mathrm{c}$ is orthogonal to V by Theorem 8.2.2. Hence

$$\mathbf{u}_\mathrm{pr} \in V \quad \text{and} \quad \mathbf{u}_\mathrm{c} \in V^\perp. \tag{8.15}$$

Equation (8.12) implies that

$$\mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}\} = \mathrm{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_\mathrm{c}\}. \tag{8.16}$$

Geometrically, for $n = 3$, $\mathbf{u}_\mathrm{pr}$ is the vector sum of the projections of $\mathbf{u}$ along $\mathbf{v}_1$ and $\mathbf{v}_2$ and lies in the plane $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. The vector $\mathbf{u}_\mathrm{c}$ is then the normal to this plane such that $\mathbf{u} = \mathbf{u}_\mathrm{pr} + \mathbf{u}_\mathrm{c}$ (Figure 8.12).

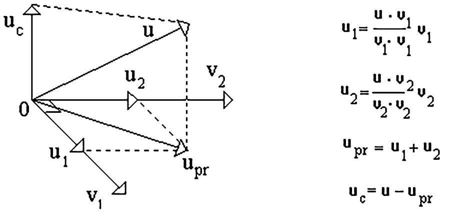
$\mathbf{u}_\mathrm{c}$
$\mathbf{u}$
$\mathbf{u}_2$
$\mathbf{v}_2$
0
$\mathbf{u}_1$
$\mathbf{u}_\mathrm{pr}$
$\mathbf{v}_1$
$\mathbf{u}_1 = \frac{\mathbf{u}\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1$
$\mathbf{u}_2 = \frac{\mathbf{u}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2$
$\mathbf{u}_\mathrm{pr} = \mathbf{u}_1 + \mathbf{u}_2$
$\mathbf{u}_\mathrm{c} = \mathbf{u} - \mathbf{u}_\mathrm{pr}$

Figure 8.12: The orthogonal projection $\mathbf{u}_\mathrm{pr}$ of $\mathbf{u}$ onto $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

The vector $\mathbf{u}_\mathrm{pr}$ satisfies an interesting property. It is the closest point in V to $\mathbf{u}$ or the *best approximation* of $\mathbf{u}$ by vectors of V. More precisely, we have the following theorem.

Theorem 8.2.10 (Best approximation). *With the above notation, we have*

$$\|\mathbf{u}_\mathrm{c}\| = \|\mathbf{u} - \mathbf{u}_\mathrm{pr}\| < \|\mathbf{u} - \mathbf{v}\|$$

for any vector $\mathbf{v}$ *of* V *other than* $\mathbf{u}_{\text{pr}}$. *In other words, the orthogonal projection* $\mathbf{u}_{\text{pr}}$ *is the only vector of* V *closest to* $\mathbf{u}$ *(Figure* 8.13*).*

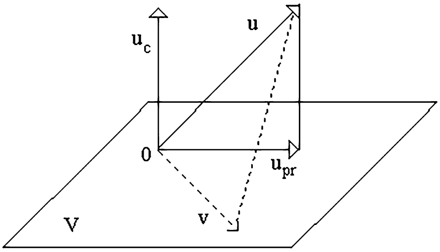

Figure 8.13: The orthogonal projection $\mathbf{u}_{\text{pr}}$ is the only vector of V closest to $\mathbf{u}$.

Proof. The vectors $\mathbf{u}_{\text{pr}} - \mathbf{v}$ and $\mathbf{u} - \mathbf{u}_{\text{pr}}$ are orthogonal, because the first is in V, and the second in $V^{\perp}$. Hence by the Pythagorean theorem for n-vectors (Section 2.5) we have

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}_{\text{pr}}\|^2 + \|\mathbf{u}_{\text{pr}} - \mathbf{v}\|^2 &= \left\|(\mathbf{u} - \mathbf{u}_{\text{pr}}) + (\mathbf{u}_{\text{pr}} - \mathbf{v})\right\|^2 \\ &= \|\mathbf{u} - \mathbf{v}\|^2.\end{aligned}$$

Therefore $\|\mathbf{u} - \mathbf{u}_{\text{pr}}\| < \|\mathbf{u} - \mathbf{v}\|$, because $\mathbf{u}_{\text{pr}} - \mathbf{v} \neq \mathbf{0}$. ☐

Example 8.2.11. Find the vector in the plane spanned by the orthogonal vectors $\mathbf{v}_1 = (-1, 4, 1)$ and $\mathbf{v}_2 = (5, 1, 1)$ that best approximates $\mathbf{u} = (1, -1, 2)$.

Solution. The vector we need is $\mathbf{u}_{\text{pr}}$. Because

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v}_1 &= -3, & \mathbf{u} \cdot \mathbf{v}_2 &= 6, \\ \mathbf{v}_1 \cdot \mathbf{v}_1 &= 18, & \mathbf{v}_2 \cdot \mathbf{v}_2 &= 27,\end{aligned}$$

we have

$$\begin{aligned}\mathbf{u}_{\text{pr}} &= \frac{\mathbf{u} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{u} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\ &= \frac{-3}{18}(-1, 4, 1) + \frac{6}{27}(5, 1, 1) \\ &= \left(\frac{23}{18}, -\frac{4}{9}, \frac{1}{18}\right).\end{aligned}$$

> ! The best approximation theorem (Theorem 8.2.10) implies that the orthogonal projection $\mathbf{u}_{\text{pr}}$ is unique and *does not depend on the orthogonal basis* of V that we use to compute it. Another orthogonal basis $\mathcal{B}'$ would produce the same $\mathbf{u}_{\text{pr}}$ and $\mathbf{u}_{\text{c}}$. In fact, we will see that both $\mathbf{u}_{\text{pr}}$ and $\mathbf{u}_{\text{c}}$ only depend on $\mathbf{u}$ and V.

8.2.3 The Gram–Schmidt process

We describe an important method, called the *Gram–Schmidt process*, that orthogonalizes a basis $\mathcal{B}$ of a subspace V of $\mathbf{R}^n$, i. e., it transforms $\mathcal{B}$ into an orthogonal basis.[2]

Let V be any subspace of $\mathbf{R}^n$, and let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be any basis of V. We want to gradually replace the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ by vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ that are orthogonal and still form a basis of V. First, we replace the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ by an orthogonal set $\{\mathbf{u}_1, \mathbf{u}_2\}$ such that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. We simply let $\mathbf{u}_1 = \mathbf{v}_1$ and let $\mathbf{u}_2$ be the component of $\mathbf{v}_2$ orthogonal to $\mathbf{v}_1$. By (8.14) $\{\mathbf{u}_1, \mathbf{u}_2\}$ is orthogonal. By (8.12) $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{u}_1, \mathbf{u}_2\}$ have the same span. Also, we have

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1.$$

We continue the same way orthogonalizing the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_3\}$. We replace $\mathbf{v}_3$ by $\mathbf{u}_3$, the component of $\mathbf{v}_2$ orthogonal to $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal and spans $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. In addition, we have

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2$$

(Figure 8.14). By induction we continue until all of $\mathcal{B}$ are replaced by $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ that is orthogonal and spans the span of $\mathcal{B}$, which is all of V. If we want to obtain an orthonormal basis, then we normalize $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$.

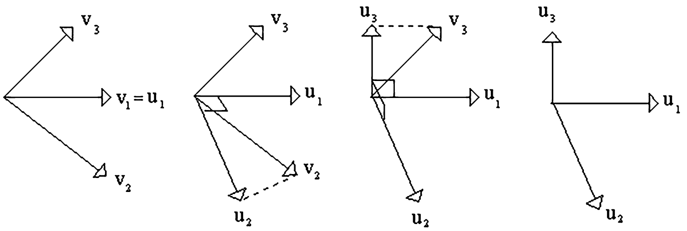

Figure 8.14: Stages of the Gram–Schmidt process.

Let us summarize the process in the following theorem.

Theorem 8.2.12 (Gram–Schmidt process). *Every subspace V of $\mathbf{R}^n$ has at least one orthogonal basis and at least one orthonormal basis. If $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis of V, then $\mathcal{B}' = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthogonal basis, where*

---

[2] The Gram–Schmidt process is named after the Danish mathematician and actuarian *Jorgen Pederson Gram* (1850–1916) and the German mathematician *Erhardt Schmidt* (1876–1959).

$$
\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1,\\
\mathbf{u}_2 &= \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1,\\
\mathbf{u}_3 &= \mathbf{v}_3 - \frac{\mathbf{v}_3\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2,\\
&\vdots\\
\mathbf{u}_k &= \mathbf{v}_k - \frac{\mathbf{v}_k\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_k\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2\cdots - \frac{\mathbf{v}_k\cdot\mathbf{u}_{k-1}}{\mathbf{u}_{k-1}\cdot\mathbf{u}_{k-1}}\mathbf{u}_{k-1},
\end{aligned}
$$

and

$$\mathrm{Span}\{\mathbf{v}_1,\dots,\mathbf{v}_i\} = \mathrm{Span}\{\mathbf{u}_1,\dots,\mathbf{u}_i\},\quad i = 1,\dots,k.$$

An orthonormal basis $\mathcal{B}''$ is obtained by normalizing $\mathcal{B}'$,

$$\mathcal{B}'' = \left\{\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|},\dots,\frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}\right\}.$$

Example 8.2.13. Find orthogonal and orthonormal bases of $\mathbf{R}^3$ by applying the Gram–Schmidt process to the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1\\ -1\\ 1 \end{bmatrix},\quad \mathbf{v}_2 = \begin{bmatrix} -2\\ 3\\ -1 \end{bmatrix},\quad \mathbf{v}_3 = \begin{bmatrix} 1\\ 2\\ -4 \end{bmatrix}.$$

Solution. Let $\mathbf{u}_1 = \mathbf{v}_1$. We have

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} -2\\ 3\\ -1 \end{bmatrix} - \frac{-6}{3}\begin{bmatrix} 1\\ -1\\ 1 \end{bmatrix} = \begin{bmatrix} 0\\ 1\\ 1 \end{bmatrix}.$$

Thus

$$
\begin{aligned}
\mathbf{u}_2 &= \mathbf{v}_3 - \frac{\mathbf{v}_3\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2\\
&= \begin{bmatrix} 1\\ 2\\ -4 \end{bmatrix} - \frac{-5}{3}\begin{bmatrix} 1\\ -1\\ 1 \end{bmatrix} - \frac{-2}{2}\begin{bmatrix} 0\\ 1\\ 1 \end{bmatrix} = \begin{bmatrix} 8/3\\ 4/3\\ -4/3 \end{bmatrix}.
\end{aligned}
$$

So the orthogonal basis is $\mathcal{B}' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1\\ -1\\ 1 \end{bmatrix},\quad \mathbf{u}_2 = \begin{bmatrix} 0\\ 1\\ 1 \end{bmatrix},\quad \mathbf{u}_3 = \begin{bmatrix} 8/3\\ 4/3\\ -4/3 \end{bmatrix}.$$

We normalize to get the orthonormal basis

$$\mathcal{B}'' = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\}.$$

The Gram–Schmidt process theorem states that orthogonal bases exist. This, combined with Theorem 8.1.9, yields the important observation that if $\mathbf{v}$ is in *both* V and $V^{\perp}$, then $\mathbf{v} = \mathbf{0}$.

Theorem 8.2.14. *Let V be any subspace of $\mathbf{R}^n$. Then*

$$V \cap V^{\perp} = \{\mathbf{0}\}.$$

The definition of the vectors $\mathbf{u}_{\mathrm{pr}}$ and $\mathbf{u}_{\mathrm{c}}$ initially assumed the existence of an orthogonal basis,but now that we know that orthogonal bases exist, we conclude that $\mathbf{u}_{\mathrm{pr}}$ and $\mathbf{u}_{\mathrm{c}}$ are *always* defined for any $\mathbf{u}$ and V. The best approximation theorem implies that $\mathbf{u}_{\mathrm{pr}}$ and $\mathbf{u}_{\mathrm{c}}$ do not depend on the particular orthogonal basis that constructed them. So they only depend on $\mathbf{u}$ and V.

Theorem 8.2.15 (Orthogonal decomposition). *Let $\mathbf{u}$ be an n-vector, and let V be a subspace of $\mathbf{R}^n$. Then $\mathbf{u}$ has always an orthogonal projection $\mathbf{u}_{\mathrm{pr}}$ onto V and a component $\mathbf{u}_{\mathrm{c}}$ orthogonal to V:*

$$\mathbf{u} = \mathbf{u}_{\mathrm{pr}} + \mathbf{u}_{\mathrm{c}} \quad \text{with } \mathbf{u}_{\mathrm{pr}} \in V,\ \mathbf{u}_{\mathrm{c}} \in V^{\perp}. \tag{8.17}$$

*The vectors $\mathbf{u}_{\mathrm{pr}}$ and $\mathbf{u}_{\mathrm{c}}$ can be computed by (*8.11*) and (*8.12*) if an orthogonal basis is known. Furthermore, decomposition (*8.17*) is unique. So, if*

$$\mathbf{u} = \mathbf{v} + \mathbf{v}^{\perp} \quad \text{with } \mathbf{v} \in V,\ \mathbf{v}^{\perp} \in V^{\perp},$$

then

$$\mathbf{v} = \mathbf{u}_{\mathrm{pr}} \quad \text{and} \quad \mathbf{v}^{\perp} = \mathbf{u}_{\mathrm{c}}.$$

Proof. The only remaining part of the proof is uniqueness. We have

$$\mathbf{u} = \mathbf{u}_{\mathrm{pr}} + \mathbf{u}_{\mathrm{c}} = \mathbf{v} + \mathbf{v}^{\perp} \Rightarrow \mathbf{u}_{\mathrm{pr}} - \mathbf{v} = \mathbf{v}^{\perp} - \mathbf{u}_{\mathrm{c}}.$$

This common vector is zero by Theorem 8.2.14, because $\mathbf{u}_{\mathrm{pr}} - \mathbf{v} \in V$ and $\mathbf{v}^{\perp} - \mathbf{u}_{\mathrm{c}} \in V^{\perp}$. Therefore $\mathbf{v} = \mathbf{u}_{\mathrm{pr}}$ and $\mathbf{v}^{\perp} = \mathbf{u}_{\mathrm{c}}$, as stated. □

Definition 8.2.16. The unique decomposition (8.17) of $\mathbf{u}$ into a summand in V and one in $V^{\perp}$ is called the *orthogonal decomposition* of $\mathbf{u}$ with respect to V.

Note that in the particular case where $\mathbf{u}$ is already in V, then $\mathbf{u}_{\mathrm{pr}} = \mathbf{u}$ and $\mathbf{u}_{\mathrm{c}} = \mathbf{0}$.

Example 8.2.17. Find the orthogonal decomposition of $\mathbf{u} = (1,1,1)$ with respect to $V = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are as in Example 8.2.13.

Solution. In Example 8.2.13, we orthogonalized $\{\mathbf{v}_1, \mathbf{v}_2\}$ and obtained $\{\mathbf{u}_1, \mathbf{u}_2\}$ with $\mathbf{u}_1 = (1,-1,1)$ and $\mathbf{u}_2 = (0,1,1)$. Therefore

$$\mathbf{u}_{\mathrm{pr}} = \frac{\mathbf{u}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{u}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 1/3 \\ 2/3 \\ 4/3 \end{bmatrix}, \quad \mathbf{u}_c = \begin{bmatrix} 2/3 \\ 1/3 \\ -1/3 \end{bmatrix}.$$

So

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 4/3 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 1/3 \\ -1/3 \end{bmatrix},$$

where $\mathbf{u}_{\mathrm{pr}} = (1/3, 2/3, 4/3) \in V$ and $\mathbf{u}_c = (2/3, 1/3, -1/3) \in V^{\perp}$.

8.2.4 Distance and angle from vector to subspace

Let $\mathbf{u}$ be an n-vector, and let V be a subspace of $\mathbf{R}^n$. If $\mathbf{u}_{\mathrm{pr}}$ is the orthogonal projection of $\mathbf{u}$ onto V, then by the best approximation theorem $\mathbf{u}_c = \mathbf{u} - \mathbf{u}_{\mathrm{pr}}$ has the shortest length among all the vectors $\mathbf{u} - \mathbf{v}$ with $\mathbf{v}$ in V. We call the length $\|\mathbf{u}_c\|$ the *distance* from $\mathbf{u}$ to V. Note that if the distance from $\mathbf{u}$ to V is zero, then $\mathbf{u}$ is in V.

The angle between $\mathbf{u}$ and $\mathbf{u}_{\mathrm{pr}}$ is called the *angle between the vector* $\mathbf{u}$ *and the subspace* V. If l is the line spanned by $\mathbf{u}$, then the above angle is the *angle between the line* l *and the subspace* V.

Since $(\mathbf{u} - \mathbf{u}_{\mathrm{pr}}) \cdot \mathbf{u}_{\mathrm{pr}} = 0$, we have that $\mathbf{u} \cdot \mathbf{u}_{\mathrm{pr}} = \mathbf{u}_{\mathrm{pr}} \cdot \mathbf{u}_{\mathrm{pr}} \geq 0$. So the angle between $\mathbf{u}$ and $\mathbf{u}_{\mathrm{pr}}$ is not obtuse.

Example 8.2.18. Compute the distance from $\mathbf{u}$ to V of Example 8.2.17. Then compute the cosine of the angle between $\mathbf{u}$ and V.

Solution. From the answers of Example 8.2.17 the distance d is

$$d = \|\mathbf{u}_c\| = \|(2/3, 1/3, -1/3)\| = \frac{\sqrt{6}}{3}.$$

The cosine of the angle is

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{u}_{\mathrm{pr}}}{\|\mathbf{u}\|\|\mathbf{u}_{\mathrm{pr}}\|} = \frac{(1,1,1)\cdot(1/3,2/3,4/3)}{\|(1,1,1)\|\|(1/3,2/3,4/3)\|} = \frac{\sqrt{7}}{3}.$$

Numerical note

The Gram–Schmidt process is not well suited for large numerical calculations. Often, there is loss of orthogonality in the computed $\mathbf{u}_i$ due to round-off errors. In practice, we use variants of the QR method of Section 8.3 and a process called *modified Gram–Schmidt* that has better numerical properties than the Gram–Schmidt process.

Exercises 8.2

Orthogonal projections

For Exercises 1–2, let

$$\mathbf{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

1. Find the projection of **u** onto the line l through **p** and the origin.

2. Find the shortest distance from **u** to the line l through **p** and the origin.

3. (a) Find the projection of the vector $\mathbf{u} = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^T$ onto the line

$$l = \left\{ t \begin{bmatrix} 1 & 1 & -3 \end{bmatrix}^T, \ t \in \mathbf{R} \right\}.$$

(b) Find the shortest distance from **u** to l.

4. Let l be the line

$$l = \left\{ t \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \ r \in \mathbf{R} \right\}.$$

Find the orthogonal decomposition of $\mathbf{u} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$ with respect to l (Figure 8.15).

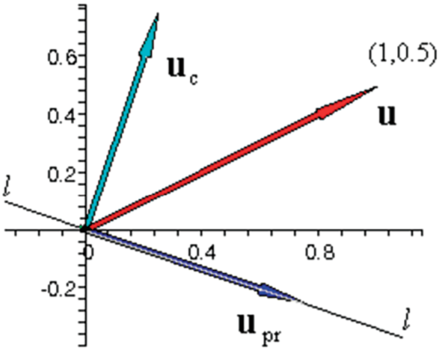

Figure 8.15: Orthogonal projection of **u** onto l.

In Exercises 5–6, find the orthogonal decomposition of **u** with respect to V.

5. $\mathbf{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $V = \text{Span}\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\}$.

6. $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $V = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\right\}$.

In Exercises 7–8, write **u** as a sum of two orthogonal vectors, one in V and one in $V^\perp$.

7. $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $V = \text{Span}\left\{\begin{bmatrix} -2 \\ 2 \end{bmatrix}\right\}$.

8. $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $V = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$.

In Exercises 9–10, find the vector in the plane V that best approximates **u**.

9. $V = \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}\right\}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

10. $V = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

11. Complete the proof of Theorem 8.2.5.

12. Verify Theorem 8.2.6 for $A = \begin{bmatrix} 4 & 2 \\ -2 & 2 \\ 1 & 2 \end{bmatrix}$.

Gram–Schmidt process

In Exercises 13–14, find orthogonal and orthonormal bases of $\mathbf{R}^3$ by applying the Gram–Schmidt process to the basis $\mathcal{B}$.

13. $\mathcal{B} = \left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right\}$.

14. $\mathcal{B} = \left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$.

In Exercises 15–16, apply the Gram–Schmidt process to find an orthogonal basis for V.

15. $V = \text{Span}\left\{\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$.

16. V is the span of the set of vectors

$$\left\{\begin{bmatrix} 3 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix}\right\}.$$

17. Find an orthonormal basis for the column space of

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

18. Find an orthonormal basis for the null space of

$$A = \begin{bmatrix} 1 & -1 & 2 & 2 \\ 1 & 0 & -4 & 0 \end{bmatrix}.$$

19. Find an orthonormal basis for the row space of

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In Exercises 20–21, find the orthogonal decomposition of **u** with respect to V. In each case, you first need to apply Gram–Schmidt to find an orthogonal basis of V.

20. $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and $V = \text{Span}\left\{\begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right\}$.

21. $\mathbf{u} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, and $V = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$.

22. Let $\mathcal{P}$ be the plane spanned by $\begin{bmatrix} 3/2 \\ 1 \\ 1/2 \end{bmatrix}$ and $\begin{bmatrix} -1/3 \\ 1 \\ -1/8 \end{bmatrix}$. Find the orthogonal decomposition of $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ with respect to $\mathcal{P}$ (Figure 8.16).

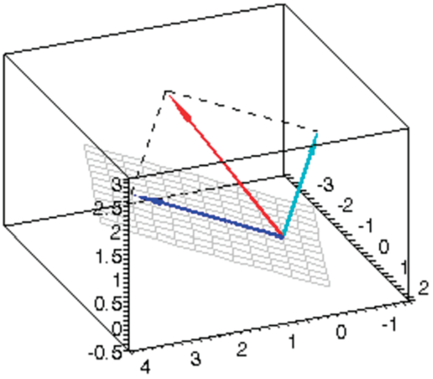

Figure 8.16: Find the projection.

23. *True or False?* Explain.

(a) In general, Gram–Schmidt changes the vectors that get orthogonalized.

(b) Gram–Schmidt changes the span of the vectors that get orthogonalized.

24. (*The Gram determinant*) Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of n-vectors orthogonalized by $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ using the Gram–Schmidt process. The Gram determinant of S is the determinant $\det(A)$ of the matrix A with (i,j) entry $\mathbf{v}_i \cdot \mathbf{v}_j$.

(a) For $n = 2$, prove that

$$\begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 \end{vmatrix} = (\mathbf{u}_1 \cdot \mathbf{u}_1)(\mathbf{u}_2 \cdot \mathbf{u}_2).$$

(b) For $n = 3$, prove that

$$\begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \mathbf{v}_1 \cdot \mathbf{v}_3 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \mathbf{v}_2 \cdot \mathbf{v}_3 \\ \mathbf{v}_3 \cdot \mathbf{v}_1 & \mathbf{v}_3 \cdot \mathbf{v}_2 & \mathbf{v}_3 \cdot \mathbf{v}_3 \end{vmatrix} = (\mathbf{u}_1 \cdot \mathbf{u}_1)(\mathbf{u}_2 \cdot \mathbf{u}_2)(\mathbf{u}_3 \cdot \mathbf{u}_3).$$

25. Find the distance from the vector $\mathbf{u} = \begin{bmatrix} 4 & -4 & 4 \end{bmatrix}^T$ to the subspace

$$\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Also, find the cosine of the angle between the vector and the subspace.

26. Find the distance from the vector $\mathbf{u} = \begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}^T$ to the subspace

$$\operatorname{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

27. Find the distance from $\mathbf{u}$ to the null space of A.

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

28. Find the distance from $\mathbf{u}$ to the column space of A.

$$\mathbf{u} = \begin{bmatrix} 0 \\ 7 \\ -3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -2 & 8 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

8.3 The QR factorization

In this section, we study the *QR factorization* or *QR decomposition* of any matrix A. This factorization is achieved by an orthogonalization of the columns of A. This factorization has many applications in numerical analysis. One such important application is a reliable numerical approximation of eigenvalues and eigenvectors.

Theorem 8.3.1 (QR factorization). *Let A be an $m \times n$ matrix with linearly independent columns (hence $m \geq n$). Then A can be factored as*

$$A = QR,$$

where Q is an $m \times n$ matrix with orthonormal columns, and R is an $n \times n$ invertible upper triangular matrix.

Proof. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the columns of A, and let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be the vectors obtained by orthonormalizing them in such a way that $\operatorname{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_i\} = \operatorname{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_i\}$, $i = 1, \dots, n$. For example, the Gram–Schmidt process will guarantee these conditions (see Section 8.2). Let

$$Q = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_n].$$

Each $\mathbf{v}_i$ is a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_i$ and hence a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_n$ of the form

$$\mathbf{v}_i = r_{1i}\mathbf{u}_1 + \cdots + r_{ni}\mathbf{u}_n = Q \begin{bmatrix} r_{1i} \\ \vdots \\ r_{ni} \end{bmatrix}, \quad i = 1, \dots, n, \tag{8.18}$$

with

$$r_{i+1,i} = \cdots = r_{ni} = 0, \quad i = 1, \dots, n. \tag{8.19}$$

Therefore

$$A = [\mathbf{v}_1\ \cdots\ \mathbf{v}_n] = \left[Q \begin{bmatrix} r_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ Q \begin{bmatrix} r_{12} \\ r_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ \cdots\ Q \begin{bmatrix} r_{1n} \\ r_{2n} \\ r_{3n} \\ \vdots \\ r_{nn} \end{bmatrix} \right] = QR,$$

where R is the matrix with (i,j)th entry r_{ij}, $i,j = 1, \dots, n$; Q and R are the matrices we want. Q has orthonormal columns, and R is upper triangular by (8.19). R is also *invertible*, because the homogeneous system $R\mathbf{x} = \mathbf{0}$ has only the trivial solution. This is because otherwise the system $QR\mathbf{x} = \mathbf{0}$ or $A\mathbf{x} = \mathbf{0}$ would have a nontrivial solution, so A would have linearly *dependent* columns. ☐

1. It is easy to give formulas for Q and R based on the equations of the Gram–Schmidt process, but it is not necessary. We simply orthonormalize the columns of A to get Q. Then compute R by
$$R = Q^T A.$$
This can be done because
$$Q^T A = Q^T(QR) = \left(Q^T Q\right)R = IR = R$$
since $Q^T Q = I$ by Theorem 8.1.19.
2. The matrix R can be so arranged that its diagonal entries are always *strictly positive*. If $r_{ii} < 0$ in (8.18), then we replace $\mathbf{u}_i$ by $-\mathbf{u}_i$. If we do this, then Q is unique, because when we orthonormalize, the $\mathbf{u}_i$s are unique up to sign.
3. The columns of Q form an *orthonormal basis* for Col(A). Furthermore, we have
$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_i\} = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_i\}$$
for $i = 1, \ldots, n$.
4. In the particular case where A is square, Q is an *orthogonal* matrix.

Example 8.3.2. Find the QR decomposition of

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution. First, we note that the columns $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of A are linearly independent, and hence a QR decomposition exists. Next, we need to orthonormalize $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. By the Gram–Schmidt process we have

$$\mathbf{u}_1 = \mathbf{v}_1 = (1, 1, 1, 1), \qquad \mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \left(\frac{1}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right),$$
$$\mathbf{u}_2 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \left(-\frac{2}{3}, 0, \frac{1}{3}, \frac{1}{3}\right),$$

and we form the matrix

$$Q = \left[\frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \ \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \ \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|}\right] = \begin{bmatrix} 1/2 & \sqrt{3}/6 & -\sqrt{6}/3 \\ 1/2 & -\sqrt{3}/2 & 0 \\ 1/2 & \sqrt{3}/6 & \sqrt{6}/6 \\ 1/2 & \sqrt{3}/6 & \sqrt{6}/6 \end{bmatrix}.$$

Because $R = Q^T A$, we have

$$R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ \sqrt{3}/6 & -\sqrt{3}/2 & \sqrt{3}/6 & \sqrt{3}/6 \\ -\sqrt{6}/3 & 0 & \sqrt{6}/6 & \sqrt{6}/6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 0 & \sqrt{3} & \sqrt{3}/3 \\ 0 & 0 & \sqrt{6}/3 \end{bmatrix}.$$

8.3.1 The QR method for eigenvalues

The QR decomposition can be used to approximate the eigenvalues of a square matrix.[3] The resulting algorithm, called the *QR method*, adds an important tool to the numerical approximations of eigenvalues, which were studied in Section 7.6. In contrast with the other methods, the QR method finds *all* eigenvalues of a matrix. It is also used to solve linear systems.

Let A be an invertible $n \times n$ matrix. We compute the QR factorization of $A = QR$ and then form the matrix $A_1 = RQ$. We observe that A and A_1 are similar, because

$$Q^{-1}AQ = Q^{-1}(QR)Q = RQ = A_1.$$

Hence A and A_1 have the same eigenvalues. Now we continue by finding the QR factorization R_1Q_1 of A_1 and forming the matrix $A_2 = Q_1R_1$ that has the same eigenvalues as A. We iterate to get a sequence of matrices

$$A,\ A_1,\ A_2,\ A_3,\ \dots .$$

It turns out that if A has n eigenvalues of different magnitudes, then this sequence approaches an upper triangular matrix $\widehat{R}$ similar to A. Hence the diagonal entries of $\widehat{R}$ are *all* the eigenvalues of A.

The following algorithm–theorem, whose proof we omit, is the core of the QR method just described.

Algorithm 8.3.3 (The QR method). The eigenvalues of an invertible matrix A are approximated as follows.

```
Input: The $n \times n$ invertible matrix $A$ with eigenvalues $\lambda_1,\dots,\lambda_n$, such that
```

$$|\lambda_1| < |\lambda_2| < \cdots < |\lambda_n|.$$

---

[3] The QR method as it stands today was introduced in 1961 by J. G. F. Francis and independently by V. N. Kublanovskaya. The original idea, however, is due to H. Rutishauser (1958), who used the LU factorization of a matrix to compute eigenvalues and called the iterations “LR transformations”.

```
1. Set $A_0 = A$.
2. For $i = 1, 2, \ldots, k-1$
    (a) Find the QR decomposition of $A_i$, say, $A_i = Q_i R_i$.
    (b) Let $A_{i+1} = R_i Q_i$.
Output: $A_k$ that approximates a triangular matrix $\widehat{R}$ with diagonal entries
all the eigenvalues of $A$.
```

The Gram–Schmidt process is numerically unstable, and as a result, the triangular matrix R in the QR decomposition is *not* exactly triangular after a numerical calculation. Entries that were supposed to be zero are often small numbers. One way to minimize such errors is to use a method known as the *modified Gram–Schmidt process* for the orthogonalizations.

8.3.2 Householder transformations and QR

Another very useful way to obtain a QR factorization of an invertible matrix is by using reflection matrices.

Let $\mathbf{u}$ be a unit vector in $\mathbf{R}^n$. The *Householder matrix* $H_{\mathbf{u}}$ is defined by

$$H_{\mathbf{u}} = I - 2\mathbf{u}\mathbf{u}^T.$$

The vector $\mathbf{u}$ is called the *Householder vector*. The corresponding transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$T(\mathbf{v}) = H_{\mathbf{u}}\mathbf{v} = \mathbf{v} - 2\mathbf{u}\mathbf{u}^T\mathbf{v}$$

is called the *Householder transformation*.

Example 8.3.4. The Householder matrix of $\mathbf{u} = (1/\sqrt{2}, -1/\sqrt{2})$ is

$$H_{\mathbf{u}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that $H_{\mathbf{u}}$ in this case represents reflection about the line $y = x$.

Theorem 8.3.5. *Let $H_{\mathbf{u}}$ be a Householder matrix. Then*

1. *$H_{\mathbf{u}}$ is orthogonal;*
2. *$H_{\mathbf{u}}$ is symmetric;*
3. *$H_{\mathbf{u}}\mathbf{u} = -\mathbf{u}$;*
4. *$H_{\mathbf{u}}\mathbf{v} = \mathbf{v}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.*

Proof. Exercise. □

Parts 3 and 4 of Theorem 8.3.5 imply that Householder matrices represent reflections. More precisely, we have the following:

1. In $\mathbf{R}^2$ a Householder matrix $H_{\mathbf{u}}$ represents a reflection about the line through the origin perpendicular to $\mathbf{u}$;
2. In $\mathbf{R}^3$ a Householder matrix $H_{\mathbf{u}}$ represents a reflection about the plane through the origin perpendicular to $\mathbf{u}$ (Figure 8.17);
3. In general, in $\mathbf{R}^n$ a Householder matrix $H_{\mathbf{u}}$ represents a reflection about the hyperplane $(\mathrm{Span}\{\mathbf{u}\})^{\perp}$.

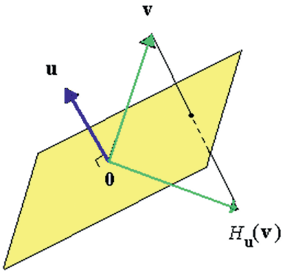

Figure 8.17: Householder transformations are reflections.

The following theorem whose proof is left as an exercise is the main idea behind an easy QR factorization of an invertible matrix.

Theorem 8.3.6. *Let* $\mathbf{v} = (v_1, \dots, v_n)$ *be a nonzero n-vector of length c. The vector* $\mathbf{u}$

$$\mathbf{u} = \frac{1}{\sqrt{2c\,(c - v_1)}}\,(v_1 - c, v_2, \dots, v_n) \tag{8.20}$$

is unit, and

$$H_{\mathbf{u}}\mathbf{v} = c\mathbf{e}_1$$

So the effect of $H_{\mathbf{u}}$ on $\mathbf{v}$ is that it makes its last $n-1$ components zero. This is very useful in practice.

Example 8.3.7. Find a Householder transformation that transforms $\mathbf{v} = (-4, 2, 4)$ into a vector whose last two components are zero.

Solution. Equation (8.20) with $c = \|(-4, 2, 4)\| = 6$ yields $\mathbf{u} = (-\frac{\sqrt{30}}{6}, \frac{\sqrt{30}}{30}, \frac{\sqrt{30}}{15})$. So

$$H_{\mathbf{u}} = I - 2\mathbf{u}\mathbf{u}^T = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{14}{15} & -\frac{2}{15} \\ \frac{2}{3} & -\frac{2}{15} & \frac{11}{15} \end{bmatrix}.$$

It is now easy to verify that $H_{\mathbf{u}}\mathbf{v} = 6\mathbf{e}_1$.

It is possible to zero the last $n - k$ entries of a vector $\mathbf{v} = (v_1, \dots, v_k, \dots, v_n)$. To see how, let $\mathbf{v}_1 = (v_1, \dots, v_{k-1})$ and $\mathbf{v}_2 = (v_k, \dots, v_n)$. We apply Theorem 8.3.6 to $\mathbf{v}_2$ to get a Householder matrix $H_{\mathbf{w}}$ such that $H_{\mathbf{w}}\mathbf{v}_2 = \|\mathbf{v}_2\|\mathbf{e}_1$. We consider the block matrix

$$H = \begin{bmatrix} I_{k-1} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & H_{\mathbf{w}} \end{bmatrix}. \tag{8.21}$$

We have

$$H\mathbf{v} = \begin{bmatrix} I_{k-1} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & H_{\mathbf{w}} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 \\ \cdots \\ \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} I_{k-1}\mathbf{v}_1 \\ \cdots \\ H_{\mathbf{w}}\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \cdots \\ \|\mathbf{v}_2\|\,\mathbf{e}_1 \end{bmatrix}.$$

So the last $n - k$ entries of $H\mathbf{v}$ are zero. In fact, $H = H_{\mathbf{u}}$ itself is a Householder matrix for the vector

$$\mathbf{u} = \begin{bmatrix} \mathbf{0} \\ \cdots \\ \mathbf{w} \end{bmatrix},$$

because $\mathbf{u}$ is unit, and

$$H = H_{\mathbf{u}} = \begin{bmatrix} I_{k-1} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & I_{n-k+1} - 2\mathbf{w}\mathbf{w}^T \end{bmatrix} = I_n - 2\mathbf{u}\mathbf{u}^T.$$

Example 8.3.8. Find a Householder transformation that transforms $\mathbf{v} = (7, -4, 2, 4)$ into a vector whose last two components are zero.

Solution. We break up $\mathbf{v} = (7, -4, 2, 4)$ into $\mathbf{v}_1 = (7)$ and $\mathbf{v}_2 = (-4, 2, 4)$. The Householder matrix for $\mathbf{v}_2$ found in Example 8.3.7 is used to form the block matrix of equation (8.21). We have

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{14}{15} & -\frac{2}{15} \\ 0 & \frac{2}{3} & -\frac{2}{15} & \frac{11}{15} \end{bmatrix}.$$

It is easy to check that $H\mathbf{v} = (7, 6, 0, 0)$.

We are now in a position to explain how to obtain a QR decomposition of an $n \times n$ invertible matrix A by using Householder matrices. First, we find a Householder H_1 matrix for the first column of A. Then the product H_1A is of the form

$$H_1A = \begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & & & & \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

Next, we find a Householder matrix H_2 that makes the last $n-2$ entries of the second column zero. The product H_2H_1A is of the form

$$H_2H_1A = \begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & & & & \\ 0 & 0 & * & \cdots & * \end{bmatrix}.$$

We continue this way until we end up with a triangular matrix R. We have

$$H_{n-1}\cdots H_2H_1A = R.$$

Hence

$$\begin{aligned} A &= (H_{n-1}\cdots H_2H_1)^{-1}R = H_1^{-1}H_2^{-1}\cdots H_{n-1}^{-1}R \\ &= H_1H_2\cdots H_{n-1}R, \end{aligned}$$

because $H_i = H_i^T = H_i^{-1}$. So a QR decomposition of A is

$$A = QR,$$

where Q is the orthogonal matrix $Q = H_1H_2\cdots H_{n-1}$.

Exercises 8.3

1. Find the matrix with QR decomposition, where

$$Q = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{3}/3 & \sqrt{6}/6 \\ 0 & \sqrt{3}/3 & \sqrt{6}/3 \\ -\sqrt{2}/2 & -\sqrt{3}/3 & \sqrt{6}/6 \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}.$$

2. Find the matrix with QR decomposition, where

$$Q = \begin{bmatrix} 1/2 & \sqrt{3}/6 & -\sqrt{6}/3 \\ 1/2 & -\sqrt{3}/2 & 0 \\ 1/2 & \sqrt{3}/6 & \sqrt{6}/6 \\ 1/2 & \sqrt{3}/6 & \sqrt{6}/6 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \sqrt{3} & \sqrt{3}/3 \\ 0 & 0 & \sqrt{6}/3 \end{bmatrix}.$$

In Exercises 3–6, find a QR factorization of A.

3. (a) $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$; (b) $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$.

4. (a) $A = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 1 & 2 \\ -1 & 0 & 2 \end{bmatrix}$; (b) $A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

5. (a) $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$; (b) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

6. $A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \\ -1 & 4 \end{bmatrix}$.

7. Find a QR factorization of $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ for nonzero (a, b).

In Exercises 8–10, find the a triangular matrix R such that $A = QR$, given that Q was obtained from A by orthonormalizing its columns.

8. $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ -1 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$.

9. $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & -1 \\ -1 & -2 \end{bmatrix}$, $Q = \begin{bmatrix} 1/2 & -1/\sqrt{6} \\ 1/2 & 0 \\ 1/2 & -1/\sqrt{6} \\ -1/2 & -\sqrt{6}/3 \end{bmatrix}$.

10. $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$.

11. Find a QR factorization of an orthogonal matrix.

12. Let A be a square matrix with nonzero columns that form an orthogonal set. If $A = QR$ is a QR factorization, prove that R is a diagonal matrix.

13. Find a QR factorization for the matrix A, if (a, b, c, d) is a nonzero vector.

$$A = \begin{bmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{bmatrix}$$

14. Prove that A is invertible if and only if $A = QR$ for some orthogonal matrix Q and some upper triangular matrix R with nonzero main diagonal entries.

15. For $A = QR$ with linearly independent columns, prove that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $Q\mathbf{y} = \mathbf{b}$ is consistent. What is the relation between the column spaces of A and Q?

In Exercises 16–18 find A_1, A_2, and A_3 of the QR method. Use A_3 to estimate the eigenvalues of A. What is the error in each case?

16. $A = \begin{bmatrix} 8 & 7 \\ 1 & 2 \end{bmatrix}$

17. $A = \begin{bmatrix} 10 & 8 \\ 1 & 3 \end{bmatrix}.$

18. $A = \begin{bmatrix} 12 & 11 \\ 2 & 3 \end{bmatrix}.$

19. Prove Theorem 8.3.5.

20. Prove Theorem 8.3.6.

21. Find the Householder matrix that makes the vector $\mathbf{v} = \begin{bmatrix} 0 & 3 & 4 \end{bmatrix}^T$ a multiple of $\mathbf{e}_1$.

22. Find the Householder matrix that makes the last two entries of the vector $\mathbf{v} = \begin{bmatrix} -7 & 0 & 3 & 4 \end{bmatrix}^T$ zero.

23. Given that H_1 is a Householder matrix for the first column of A, use H_1 to find R in a QR factorization of A, where

$$A = \begin{bmatrix} -2 & 0 & -8 \\ 0 & -2 & 0 \\ 2 & 0 & -4 \end{bmatrix}, \quad H_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

24. Find the Householder matrix that represents reflection about the line $y = 2x$.

8.4 Least squares

In this section, we study a topic of considerable interest in applications, the method of least squares that was discussed in the introduction.

8.4.1 A least squares problem

In practice, we often have data points and need to find a function whose graph passes through these points. Usually, the nature of the problem dictates the kind of function we need.

Suppose our problem suggests that a straight line is appropriate and that we are given the data points $(1,2), (2,4), (3,3)$. Let $y = b + mx$ be the equation of this line. We want to find the slope m and the y-intercept b. Because the line should pass through the three points, we have

$$2 = b + m \cdot 1, \quad 4 = b + m \cdot 2, \quad 3 = b + m \cdot 3.$$

Unfortunately, the resulting linear system

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

in unknowns m and b is easily seen to be *inconsistent*. So our problem cannot be solved exactly.[4] The next best thing then is to try to find the straight line that best "fits" these points.

"Best fitting" may have different meanings depending on what aspects of the solution need to be emphasized. In this case, suppose we assume that our best line is in the following sense: If δ_1, δ_2, and δ_3 are the errors in the y-direction

$$\delta_1 = 2 - b - m \cdot 1, \quad \delta_2 = 4 - b - m \cdot 2, \quad \delta_3 = 3 - b - m \cdot 3,$$

then the number

$$\delta_1^2 + \delta_2^2 + \delta_3^2$$

is minimum (Figure 8.18). A solution for m and b that minimizes this sum of the squares of the errors is called a *least squares solution*.

We may express all this in vector notation. If Δ is the error vector

$$\Delta = (\delta_1, \delta_2, \delta_3),$$

then we want to minimize $\delta_1^2 + \delta_2^2 + \delta_3^2 = \|\Delta\|^2$ or, equivalently, minimize $\|\Delta\|$.

Example 8.4.1. Find which of the lines yields the smallest least squares error for the points $(1,2), (2,4), (3,3)$:

$$\text{(a)} \quad y = 2x, \quad \text{(b)} \quad y = 3, \quad \text{(c)} \quad y = 0.5x + 2.$$

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[4] Note that the quadratic $(-3/2)x^2 + (13/2)x - 3$ passes through these points but that is not what we need.

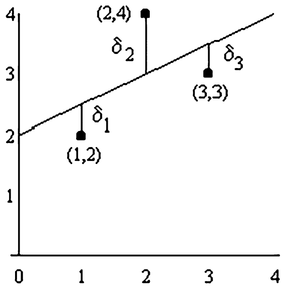

Figure 8.18: The method of least squares minimizing $\delta_1^2 + \delta_2^2 + \delta_3^2$.

Solution. We have

	$y = 2x$ $(m = 2,\ b = 0)$	$y = 3$ $(m = 0,\ b = 3)$	$y = 0.5x + 2$ $(m = .5,\ b = 2)$
δ_1	$2 - 0 - 1 \cdot 2 = 0$	$2 - 3 - 1 \cdot 0 = -1$	$2 - 2 - 1 \cdot .5 = -0.5$
δ_2	$4 - 0 - 2 \cdot 2 = 0$	$4 - 3 - 2 \cdot 0 = 1$	$4 - 2 - 2 \cdot .5 = 1$
δ_2	$3 - 0 - 3 \cdot 2 = -3$	$3 - 3 - 3 \cdot 0 = 0$	$3 - 2 - 3 \cdot .5 = -0.5$
$\|\Delta\|^2$	$2 \cdot 0^2 + (-3)^2 = 9$	$(-1)^2 + 1^2 + 0^2 = 2$	$2 \cdot (-0.5)^2 + 1 = 1.5$

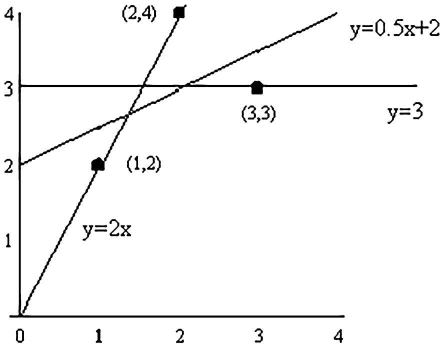

Figure 8.19: The line $y = 0.5x + 2$ yields the smallest least squares error.

8.4.2 Solution of the least squares problem

Let us now see how to find the least squares solution for the above points and in general. Suppose we have an inconsistent linear system

$$A\mathbf{x} = \mathbf{b}, \tag{8.22}$$

where A is an $m \times n$ matrix. Since for any n-vector $\mathbf{x}$, the product $A\mathbf{x}$ is never $\mathbf{b}$, the resulting error

$$\Delta = \mathbf{b} - A\mathbf{x}$$

is a nonzero m-vector for all n-vectors $\mathbf{x}$. Solving the least squares problem for (8.22) amounts to finding an n-vector $\tilde{\mathbf{x}}$ such that the length of $\Delta = \mathbf{b} - A\tilde{\mathbf{x}}$ is minimum. Then $\tilde{\mathbf{x}}$ would be our *least squares solution*.

Least squares problem: Find $\tilde{\mathbf{x}}$ such that $\|\mathbf{b} - A\tilde{\mathbf{x}}\|$ is minimum.

As $\mathbf{x}$ varies, $A\mathbf{x}$ generates $\mathrm{Col}(A)$. So $\|\mathbf{b} - A\tilde{\mathbf{x}}\|$ is minimum only if $A\tilde{\mathbf{x}}$ is the orthogonal projection $\mathbf{b}_{\mathrm{pr}}$ of $\mathbf{b}$ onto $\mathrm{Col}(A)$ by the best approximation theorem (Theorem 8.2.10). Hence

$$\|\Delta\| = \min \Leftrightarrow A\tilde{\mathbf{x}} = \mathbf{b}_{\mathrm{pr}} \Leftrightarrow \mathbf{b} - A\tilde{\mathbf{x}} = \mathbf{b}_{\mathrm{c}}.$$

We conclude that a least squares solution $\tilde{\mathbf{x}}$ for $A\mathbf{x} = \mathbf{b}$ *always exists* (because $\mathbf{b}_{\mathrm{pr}}$ exists). So we have the following theorem.

Theorem 8.4.2. *For any $m \times n$ matrix A and any m-vector $\mathbf{b}$, there is a least squares solution $\tilde{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$. In addition, if $\mathbf{b}_{\mathrm{pr}}$ is the orthogonal projection of $\mathbf{b}$ onto $\mathrm{Col}(A)$, then*

$$A\tilde{\mathbf{x}} = \mathbf{b}_{\mathrm{pr}}. \tag{8.23}$$

Because $\mathbf{b}_{\mathrm{c}} = \mathbf{b} - A\tilde{\mathbf{x}}$ is orthogonal to $\mathrm{Col}(A)$, we have that $A\mathbf{x}$ and $\mathbf{b} - A\tilde{\mathbf{x}}$ must be orthogonal for any n-vector $\mathbf{x}$. Therefore

$$\begin{aligned}
&(\mathbf{b} - A\tilde{\mathbf{x}}) \cdot A\mathbf{x} = 0 & \\
\Leftrightarrow\ & A^T(\mathbf{b} - A\tilde{\mathbf{x}}) \cdot \mathbf{x} = 0 & \text{by equation (8.1), Section 8.1} \\
\Leftrightarrow\ & A^T(\mathbf{b} - A\tilde{\mathbf{x}}) = \mathbf{0} & \text{by Theorem 8.1.9, Section 8.1} \\
\Leftrightarrow\ & A^T\mathbf{b} - A^TA\tilde{\mathbf{x}} = \mathbf{0} & \\
\Leftrightarrow\ & A^TA\tilde{\mathbf{x}} = A^T\mathbf{b}. &
\end{aligned}$$

So $\tilde{\mathbf{x}}$ is a least squares solution if and only if it satisfies the system $A^TA\tilde{\mathbf{x}} = A^T\mathbf{b}$. This system is known as the *normal equations for* $\tilde{\mathbf{x}}$. Now we know how to find $\tilde{\mathbf{x}}$. In addition, we may use $\|\Delta\| = \|\mathbf{b} - A\tilde{\mathbf{x}}\|$ to compute the *least squares error* involved.

Theorem 8.4.3 (Least squares solutions). *Let A be an $m \times n$ matrix. Then there are always least squares solutions $\tilde{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$. Furthermore, we have*

1. *$\tilde{\mathbf{x}}$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$ if and only if $\tilde{\mathbf{x}}$ is a solution of the normal equations*

$$A^TA\tilde{\mathbf{x}} = A^T\mathbf{b}. \tag{8.24}$$

The least squares error $\|\Delta\|$ *is then given by*

$$\|\Delta\| = \|\mathbf{b} - A\tilde{\mathbf{x}}\|.$$

2. *A has linearly independent columns if and only if $A^T A$ is invertible. In this case the least squares solution is unique and can be computed by*

$$\tilde{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}.$$

Proof. We only need to prove the last part of the theorem. First, we prove that A and $A^T A$ have the same null space. Indeed, if $\mathbf{v} \in \mathrm{Null}(A)$, then $A\mathbf{v} = \mathbf{0}$; so $A^T A\mathbf{v} = \mathbf{0}$, which implies $\mathbf{v} \in \mathrm{Null}(A^T A)$. Therefore $\mathrm{Null}(A) \subseteq \mathrm{Null}(A^T A)$. On the other hand,

$$\begin{aligned}
\mathbf{v} \in \mathrm{Null}(A^T A) &\Rightarrow A^T A\mathbf{v} = \mathbf{0} \\
&\Rightarrow A^T A\mathbf{v} \cdot \mathbf{v} = \mathbf{0} \cdot \mathbf{v} = 0 \\
&\Rightarrow A\mathbf{v} \cdot A\mathbf{v} = 0 \\
&\Rightarrow \|A\mathbf{v}\|^2 = 0 \\
&\Rightarrow A\mathbf{v} = \mathbf{0} \\
&\Rightarrow \mathbf{v} \in \mathrm{Null}(A).
\end{aligned}$$

Hence $\mathrm{Null}(A^T A) \subseteq \mathrm{Null}(A)$, and the two null spaces are equal. So $\mathrm{nullity}(A) = \mathrm{nullity}(A^T A)$. But by the dimension theorem

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = n = \mathrm{rank}(A^T A) + \mathrm{nullity}(A^T A).$$

Hence $\mathrm{rank}(A) = \mathrm{rank}(A^T A)$. If A has linearly independent columns, then $\mathrm{rank}(A) = n$. So $\mathrm{rank}(A^T A) = n$. Therefore the $n \times n$ matrix $A^T A$ has linearly independent columns. Hence $A^T A$ is invertible.

Conversely, if $A^T A$ is invertible, then $\mathrm{rank}(A^T A) = n$, and hence $\mathrm{rank}(A) = n$. So A has linearly independent columns. If $A^T A$ is invertible, then $A^T A\tilde{\mathbf{x}} = A^T\mathbf{b}$ implies that the unique solution is given by $\tilde{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$. □

Example 8.4.4. Solve the least squares problem and compute the least squares error for the system $A\mathbf{x} = \mathbf{b}$,

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}.$$

Use the solution to find the straight line that yields the smallest least square error for the points $(1, 2), (2, 4), (3, 3)$ discussed in the introduction of this section.

Solution. By Theorem 8.4.3 it suffices to solve the normal equations $A^T A\tilde{\mathbf{x}} = A^T\mathbf{b}$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\tilde{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

or

$$\begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}\tilde{\mathbf{x}} = \begin{bmatrix} 9 \\ 19 \end{bmatrix}.$$

The solution of this system yields the least squares solution $\tilde{\mathbf{x}} = \left[\begin{smallmatrix} 2 \\ 0.5 \end{smallmatrix}\right]$ with least squares error (Figure 8.20)

$$\|\Delta\| = \|\mathbf{b} - A\tilde{\mathbf{x}}\| = \left\| \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1/2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} \right\| = \frac{\sqrt{6}}{2}.$$

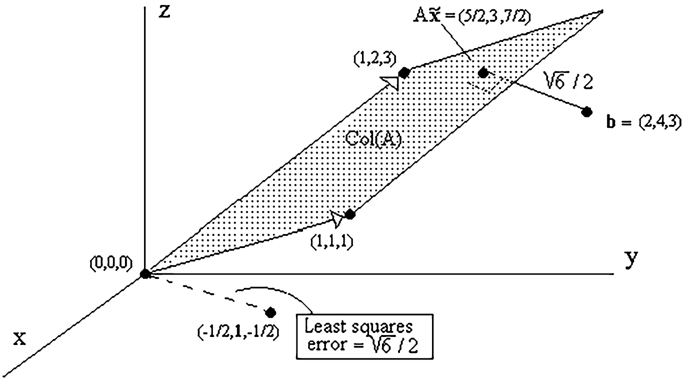

Figure 8.20: Least squares error.

Because $\tilde{\mathbf{x}} = (2, 0.5)$, the slope of the least squares line is 0.5, and its y-intercept is 2. So line equation is $y = 0.5x + 2$. This line is sketched in Figure 8.21. The total area of the squares is minimum.

Example 8.4.5. A linear algebra instructor has the following grade data. Find the least squares line for the data and use it to predict the expected percentage of Bs after the tenth semester.

Semester	**1**	**2**	**3**	**4**	**5**	**6**
Percentage of Bs	0.20	0.25	0.20	0.35	0.45	0.40

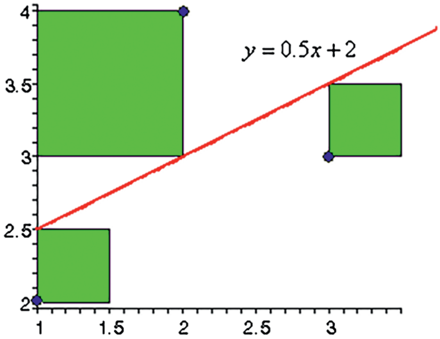

Figure 8.21: The least squares line minimizes the total area of the squares.

Solution. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.20 \\ 0.25 \\ 0.20 \\ 0.35 \\ 0.45 \\ 0.40 \end{bmatrix}.$$

Hence

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 21 \\ 21 & 91 \end{bmatrix}$$

and

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0.20 \\ 0.25 \\ 0.20 \\ 0.35 \\ 0.45 \\ 0.40 \end{bmatrix} = \begin{bmatrix} 1.85 \\ 7.35 \end{bmatrix}.$$

Therefore the normal equations are

$$\begin{bmatrix} 6 & 21 \\ 21 & 91 \end{bmatrix} \begin{bmatrix} \tilde{b} \\ \tilde{m} \end{bmatrix} = \begin{bmatrix} 1.85 \\ 7.35 \end{bmatrix}.$$

Solving then, we get $\tilde{m} = 0.05$ and $\tilde{b} = 0.13333$. Therefore the line is

$$y = 0.13333 + 0.05x.$$

If $x = 10$, then $y = 0.63333$. So roughly 63.3 % of Bs are expected after the tenth semester (Figure 8.22).

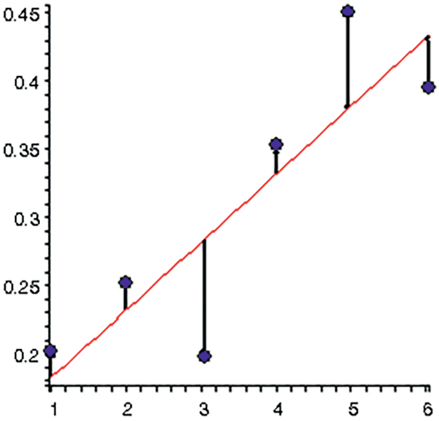

Figure 8.22: Least squares line.

If A does not have linearly independent columns, then there are several least squares solutions.

Example 8.4.6. Find all least squares solutions of the system

$$\begin{aligned} x - y &= 1, \\ x - y &= 5. \end{aligned}$$

Solution. $A = \left[\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right]$ and $\mathbf{b} = \left[\begin{smallmatrix} 1 \\ 5 \end{smallmatrix}\right]$. The normal equations

$$A^T A\tilde{\mathbf{x}} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = A^T\mathbf{b} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

have the infinite solution set $\tilde{x} = 3 + r, \tilde{y} = r, r \in \mathbf{R}$.

8.4.3 Least squares with QR factorization

The least squares solutions discussed above suffer from a frequent problem. The matrix $A^T A$ of the normal equations is usually *ill-conditioned*. This means that a small numerical error in a row reduction can cause a large error in the solution. Usually, Gauss elimination for A^TA of size $n \geq 5$ does not yield good approximate solutions. An answer to this problem is to use the QR factorization of A. The idea for this approach is that orthogonal matrices preserve lengths, so they should also preserve the length of the error vector.

Let A have linearly independent columns, and let $A = QR$ be a QR factorization (Section 8.3). Then for $\tilde{\mathbf{x}}$, a least squares solution of $A\mathbf{x} = \mathbf{b}$, we have

$$\begin{aligned} A^TA\tilde{\mathbf{x}} = A^T\mathbf{b} &\Leftrightarrow (QR)^T(QR)\tilde{\mathbf{x}} = (QR)^T\mathbf{b} \\ &\Leftrightarrow R^TQ^TQR\tilde{\mathbf{x}} = R^TQ^T\mathbf{b} \\ &\Leftrightarrow R^TR\tilde{\mathbf{x}} = R^TQ^T\mathbf{b} && \text{because } Q^TQ = I \\ &\Leftrightarrow R\tilde{\mathbf{x}} = Q^T\mathbf{b} && \text{because } R^T \text{ is invertible.} \end{aligned}$$

Note that although $R\tilde{\mathbf{x}} = Q^T\mathbf{b} \Leftrightarrow \tilde{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$, it is easier to solve $R\tilde{\mathbf{x}} = Q^T\mathbf{b}$ by back-substitution.

The above observations constitute a proof of the following theorem.

Theorem 8.4.7. *If A is an $m \times n$ matrix with linearly independent columns and if $A = QR$ is a QR factorization, then the unique least squares solution $\tilde{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ is theoretically given by*

$$\tilde{\mathbf{x}} = R^{-1}Q^T\mathbf{b},$$

which is actually computed by solving by back-substitution the system

$$R\tilde{\mathbf{x}} = Q^T\mathbf{b}.$$

Example 8.4.8. Find the least squares solution for $A\mathbf{x} = \mathbf{b}$ by using QR factorization if

$$A = \begin{bmatrix} 2 & 2 & 6 \\ 1 & 4 & -3 \\ 2 & -4 & 9 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}.$$

Solution. By applying the method of Section 8.3 we have,

$$A = QR = \begin{bmatrix} 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 9 \\ 0 & 6 & -6 \\ 0 & 0 & -3 \end{bmatrix}.$$

Since

$$Q^T\mathbf{b} = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix},$$

by Theorem 8.4.7 we can compute $\tilde{\mathbf{x}}$ by solving $R\tilde{\mathbf{x}} = Q^T\mathbf{b}$ by back-substitution. We get

$$\begin{bmatrix} 3 & 0 & 9 \\ 0 & 6 & -6 \\ 0 & 0 & -3 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \Rightarrow \tilde{\mathbf{x}} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix}.$$

Exercises 8.4

In Exercises 1–3, use the normal equations to find the least squares solution for $A\mathbf{x} = \mathbf{b}$. Then find the least squares error in each case.

1. $A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$.

2. $A = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ -1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$.

3. $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ 2 & 2 \\ -1 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$.

4. Compare the solution of the system

$$\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

with its least squares solution. Explain your answer.

In Exercises 5–7, find the least squares line for the given points. In each case, draw the points and the line.

5. $(-1, -2), (0, 1), (1, 5)$.

6. $(1, 2), (2, 3), (4, 4)$.

7. $(-1, -1), (1, 2), (3, 2), (4, 4)$.

8. Find all the least squares solutions for the system

$$\begin{aligned} x - y &= 0, \\ x - y &= 1, \\ x - y &= 2. \end{aligned}$$

9. Find all the least squares solutions for $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 0 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

10. Find all the least squares solutions for the system

$$\begin{aligned} x + y &= 1, \\ x + y &= 1.5, \\ x + y &= .5. \end{aligned}$$

11. Find all the least squares solutions for $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -4 & 0 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

12. (*Gravity acceleration*) The velocity v of a free falling body is given by the following linear relation in time t:

$$v = gt + v_0,$$

where v_0 is the initial velocity, and g is the gravity acceleration. Estimate g in m/s^2 if the following measurements are taken for the same initial velocity. Round your answer to one decimal place.

***t* (s)**	**1**	**2**	**3**	**4**	**5**
v (m/s)	12.81	22.60	32.47	42.21	52.0

13. (*Hooke's law*) Hooke's law for springs states that there is a linear relation between the spring force F and the length of the spring x:

$$F = kx + C.$$

The constant k is called *the spring constant*. Estimate the spring constant k if the following measurements are taken for the same constant C. Round your answer to one decimal place.

***x* (inch)**	**1.5**	**2**	**2.5**	**3**
F (lb)	9.1	10.9	13.1	14.9

14. (*Centroid*) Let $\mathbf{v}_1 = (x_1, y_1), \dots, \mathbf{v}_n = (x_n, y_n)$ be points in $\mathbf{R}^2$ (also viewed as vectors). Prove that the least squares line the $\mathbf{v}_i$s passes through their centroid (Figure 8.23). Recall that the centroid of $\mathbf{v}_1, \dots, \mathbf{v}_n$ is

$$\mathbf{v}_c = \frac{1}{n}(\mathbf{v}_1 + \cdots + \mathbf{v}_n).$$

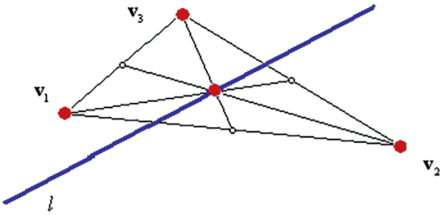

Figure 8.23: The least squares line passes through the centroid.

In Exercises 15–17, use the given QR factorization to find the least squares solution for $A\mathbf{x} = \mathbf{b}$.

15. $A = \begin{bmatrix} 0 & 3 \\ 0 & 4 \\ 5 & 10 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 3/5 \\ 0 & 4/5 \\ 1 & 0 \end{bmatrix}$, $R = \begin{bmatrix} 5 & 10 \\ 0 & 5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$.

16. $A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \\ 2 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} -2/3 & -2/3 \\ 1/3 & -2/3 \\ 2/3 & -1/3 \end{bmatrix}$, $R = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

17. $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$, $R = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}$.

Least squares when *A* has orthogonal columns

If a matrix A has nonzero orthogonal columns, then it is easy to find the least squares solution for $A\mathbf{x} = \mathbf{b}$.

18. Let A be an $m \times n$ matrix with *nonzero* orthogonal columns $\mathbf{a}_i$, and let $\tilde{\mathbf{x}}$ be the least squares solution of the system $A\mathbf{x} = \mathbf{b}$. Prove that

$$\tilde{\mathbf{x}} = \begin{bmatrix} (\mathbf{b}\cdot\mathbf{a}_1)/(\mathbf{a}_1\cdot\mathbf{a}_1) \\ (\mathbf{b}\cdot\mathbf{a}_2)/(\mathbf{a}_2\cdot\mathbf{a}_2) \\ \vdots \\ (\mathbf{b}\cdot\mathbf{a}_n)/(\mathbf{a}_n\cdot\mathbf{a}_n) \end{bmatrix}. \tag{8.25}$$

(*Hint:* A^TA has a very special form under the assumptions on A.)

19. How does (8.25) simplify when A has orthonormal columns?

20. Use Exercise 18 to find the least squares solution $\tilde{\mathbf{x}}$ for

$$A = \begin{bmatrix} -2 & -2 \\ 1 & -2 \\ 2 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}.$$

21. Use Exercise 18 to find the least squares solution $\tilde{\mathbf{x}}$ for

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix}.$$

22. Use Exercise 18 to find the least squares solution $\tilde{\mathbf{x}}$ for

$$A = \begin{bmatrix} -2/3 & -2/3 \\ 1/3 & -2/3 \\ 2/3 & -1/3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}.$$

23. Use Exercise 18 to find the least squares solution $\tilde{\mathbf{x}}$ if $abcd \neq 0$ and

$$A = \begin{bmatrix} a & b & c \\ -b & a & d \\ -c & -d & a \\ -d & c & -b \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

24. *True of false?* Explain.

(a) A least squares solution always exists for a linear system $A\mathbf{x} = \mathbf{b}$ of any size.

(b) A least squares problem for the linear system $A\mathbf{x} = \mathbf{b}$ always has a unique solution.

(c) The least squares method with QR factorization always works for any linear system $A\mathbf{x} = \mathbf{b}$.

8.5 Inner product spaces

In this section, we discuss a *useful* generalization of the dot product for vector spaces. Although the dot product in $\mathbf{R}^n$ is in the core of many applications, working directly with n-vectors may be sometimes restrictive. This is apparent with vector spaces of polynomials or functions, where it is often easier to use natural notations of these sets than vector notation.

8.5.1 Definition of inner product

We introduce a "dot product", called an *inner product*, for general vectors. To define it, we use the basic properties of the dot product outlined in Theorem 2.5.5.

Definition 8.5.1. An *inner product* on a (real) vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbf{R}$ that with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ of V associates a real number denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$. This function satisfies the following properties, or *axioms*.

For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ of V and any scalar c, we have

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$; (*Symmetry*)
2. $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$; (*Additivity*)
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$; (*Homogeneity*)
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$. Furthermore, $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$. (*Positivity*)

A real vector space with an inner product is called an *inner product space*.

To prove that a vector space is an inner product space, we must first have a function that associates a number with each pair of vectors. Then we must verify the four axioms for this function.[5]

The axioms of an inner product imply the following basic properties.

Theorem 8.5.2. *Let* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$ *be any vectors in an inner product space, and let* c *be any scalar. Then*

1. $\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle$;
2. $\langle \mathbf{u}, c\mathbf{v}\rangle = c\langle \mathbf{u}, \mathbf{v}\rangle$;
3. $\langle \mathbf{u}-\mathbf{w}, \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{w}, \mathbf{v}\rangle$;
4. $\langle \mathbf{u}, \mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{v}\rangle - \langle \mathbf{u}, \mathbf{w}\rangle$;
5. $\langle \mathbf{0}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{0}\rangle = 0$.

Proof of 1.

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle &= \langle \mathbf{v}+\mathbf{w}, \mathbf{u}\rangle && \text{by symmetry}\\ &= \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{w}, \mathbf{u}\rangle && \text{by additivity}\\ &= \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle && \text{by symmetry.}\end{aligned}$$

□

8.5.2 Examples of inner products

The inner product was designed to generalize the dot product, so the dot product in $\mathbf{R}^n$ is the first example of an inner product.

Example 8.5.3. Let $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ be any n-vectors. The dot product in $\mathbf{R}^n$

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}\cdot\mathbf{v} = \mathbf{u}^T\mathbf{v} = u_1v_1 + \cdots + u_nv_n$$

makes $\mathbf{R}^n$ an inner product space.

Solution. All axioms hold by Theorem 2.5.5.

Example 8.5.4. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be any 2-vectors. Prove that

$$\langle \mathbf{u}, \mathbf{v}\rangle = 3u_1v_1 + 4u_2v_2$$

defines an inner product in $\mathbf{R}^2$.

Solution. Symmetry: It holds, because

$$\langle \mathbf{u}, \mathbf{v}\rangle = 3u_1v_1 + 4u_2v_2 = 3v_1u_1 + 4v_2u_2 = \langle \mathbf{v}, \mathbf{u}\rangle.$$

---

[5] Often, the part $\langle \mathbf{u}, \mathbf{u}\rangle = 0 \Rightarrow \mathbf{u} = \mathbf{0}$ of the positivity axiom is the hardest to verify.

Additivity: If $\mathbf{w} = (w_1, w_2)$, then

$$\begin{aligned}\langle \mathbf{u} + \mathbf{w}, \mathbf{v}\rangle &= 3(u_1 + w_1)v_1 + 4(u_2 + w_2)v_2 \\ &= (3u_1v_1 + 4u_2v_2) + (3w_1v_1 + 4w_2v_2) \\ &= \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{w}, \mathbf{v}\rangle.\end{aligned}$$

Homogeneity: For any scalar c, we have

$$\begin{aligned}\langle c\mathbf{u}, \mathbf{v}\rangle &= 3(cu_1)v_1 + 4(cu_2)v_2 \\ &= c(3u_1v_1 + 4u_2v_2) \\ &= c\langle \mathbf{u}, \mathbf{v}\rangle.\end{aligned}$$

Positivity: We have

$$\langle \mathbf{u}, \mathbf{u}\rangle = 3u_1u_1 + 4u_2u_2 = 3u_1^2 + 4u_2^2 \geq 0.$$

This also implies that

$$\langle \mathbf{u}, \mathbf{u}\rangle = 0 \Leftrightarrow u_1 = 0 \quad \text{and} \quad u_2 = 0 \Leftrightarrow \mathbf{u} = \mathbf{0}.$$

Hence all axioms hold and $\langle \cdot, \cdot \rangle$ defines an inner product.

We have just found an inner product of $\mathbf{R}^2$ other than the dot product. So a vector space may have *several different inner products*. Example 8.5.4 is particular case of the following example.

Example 8.5.5 (Weighted dot product). Let $\mathbf{w} = (w_1, \dots, w_n)$ be a vector with strictly positive components $w_i > 0$, and let $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ be any n-vectors. Prove that

$$\langle \mathbf{u}, \mathbf{v}\rangle = w_1u_1v_1 + \cdots + w_nu_nv_n \tag{8.26}$$

defines an inner product in $\mathbf{R}^n$.

Solution. All axioms are verified as in Example 8.5.4.

The inner product of Example 8.5.5 is called the *weighted dot product* in $\mathbf{R}^n$ with *weight vector* $\mathbf{w}$ and *weights* $w_1, \dots, w_n$. It is important that all weights $w_1, \dots, w_n$ are positive. Otherwise, the positivity axiom may fail. Formula (8.26) may also be written in matrix notation as

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T W\mathbf{v} \quad \text{where } W = \begin{bmatrix} w_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & w_n \end{bmatrix}.$$

In the next two examples, we define inner products for M_{22} and P_2 that are basically identical to the dot product of $\mathbf{R}^n$. The verification of the axioms is left to the reader.

Example 8.5.6. Let A and B be 2×2 matrices with real entries,

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}.$$

The following function defines an inner product in M_{22}:

$$\langle A, B \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4.$$

Example 8.5.7. Let $p(x)$ and $q(x)$ be polynomials in P_n,

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad q(x) = b_0 + b_1 x + \cdots + b_n x^n.$$

The following function defines an inner product in P_n:

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + \cdots + a_n b_n.$$

For example, if $p(x) = 1 - x^2$ and $q(x) = -2x + x^2$, then

$$\langle p, q \rangle = 1 \cdot 0 + 0 \cdot (-2) + (-1) \cdot 1 = -1.$$

Example 8.5.8. Let $r_0, r_1, \ldots, r_n$ be $n + 1$ distinct real numbers, and let $p(x)$ and $q(x)$ be any polynomials in P_n. Prove that

$$\langle p, q \rangle = p(r_0) q(r_0) + \cdots + p(r_n) q(r_n)$$

defines an inner product in P_n.

Solution. Axioms 1–3 are easily verified. For the positivity axiom, we have

$$\langle p, p \rangle = p(r_0)^2 + \cdots + p(r_n)^2 \geq 0$$

and

$$\langle p, p \rangle = 0 \Leftrightarrow p(r_0) = 0, \quad \ldots, \quad p(r_n) = 0 \Leftrightarrow p = \mathbf{0},$$

because the polynomial p has degree at most n, so if it has more than n roots, then it has to be the zero polynomial.

Example 8.5.9. In Example 8.5.8, let $r_0 = -2$, $r_1 = 0$, $r_2 = 1$, $p(x) = 1 - x^2$, and $q(x) = -2x + x^2$. Compute $\langle p, q \rangle$.

Solution. We have

$$p(-2) = -3, \quad q(-2) = 8, \quad p(0) = 1, \quad q(0) = 0, \quad p(1) = 0, \quad q(1) = -1.$$

Therefore

$$\langle p, q\rangle = p(-2)q(-2) + p(0)q(0) + p(1)q(1) = -24.$$

Next, we introduce an important source of inner products that generalizes the weighted dot product introduced in Example 8.5.5.

Definition 8.5.10. An $n \times n$ symmetric matrix A is called *positive definite* if the number $\mathbf{x}^T A\mathbf{x}$ is strictly positive for all nonzero n-vectors $\mathbf{x}$, i. e.,

$$\mathbf{x}^T A\mathbf{x} > 0 \quad \text{for all } \mathbf{x} \neq \mathbf{0} \in \mathbf{R}^n.$$

Positive definite matrices are studied in Section 9.2. Here we only need the following property, which is discussed in Theorem 9.2.8.

! A symmetric matrix A is positive definite if and only if all its eigenvalues are strictly positive.

Example 8.5.11. Let A be any positive definite $n \times n$ matrix. Prove that for any n-vectors $\mathbf{u}$ and $\mathbf{v}$, the function

$$\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T A\mathbf{v}$$

defines an inner product in $\mathbf{R}^n$.

Solution. We need to verify the four axioms.
Symmetry: Because $A = A^T$, we have

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= \mathbf{u}^T A\mathbf{v} = \mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v} \\ &= A\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot A\mathbf{u} = \mathbf{v}^T A\mathbf{u} = \langle \mathbf{v}, \mathbf{u}\rangle.\end{aligned}$$

Additivity:

$$\begin{aligned}\langle \mathbf{u} + \mathbf{w}, \mathbf{v}\rangle &= (\mathbf{u} + \mathbf{w})^T A\mathbf{v} \\ &= \mathbf{u}^T A\mathbf{v} + \mathbf{w}^T A\mathbf{v} = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{w}, \mathbf{v}\rangle.\end{aligned}$$

Homogeneity:

$$\langle c\mathbf{u}, \mathbf{v}\rangle = (c\mathbf{u})^T A\mathbf{v} = c\mathbf{u}^T A\mathbf{v} = c\langle \mathbf{u}, \mathbf{v}\rangle.$$

Positivity: A is positive definite, so

$$\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{u}^T A \mathbf{u} > 0 \quad \text{for all } \mathbf{u} \neq \mathbf{0}.$$

This verifies the last axiom.

Example 8.5.12. Prove that the function

$$\langle \mathbf{u}, \mathbf{v} \rangle = 6u_1v_1 - 2u_2v_1 - 2u_1v_2 + 3u_2v_2$$

defines an inner product in $\mathbf{R}^2$.

Solution. $\langle \mathbf{u}, \mathbf{v} \rangle$ can be written in the form $\mathbf{u}^T A \mathbf{v}$ as follows:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

The symmetric matrix

$$A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$$

has positive eigenvalues (2 and 7). So it is positive definite. Hence $\langle \mathbf{u}, \mathbf{v} \rangle$ defines as inner product in $\mathbf{R}^2$ by Example 8.5.11.

Note that if A is not positive definite, then $\mathbf{u}^T A \mathbf{v}$ may not define an inner product. For example,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2u_1v_1 - 2u_2v_1 - 2u_1v_2 + 2u_2v_2$$

in *not* an inner product. This is seen from

$$\langle (1,1), (1,1) \rangle = 0, \quad (1,1) \neq \mathbf{0}.$$

Note that the matrix

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

has eigenvalues $0, 4$. So it is not positive definite.

Example 8.5.13 (Requires calculus). Let f and g be in $C[a, b]$, the vector space of continuous real-valued functions on $[a, b]$. Then

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx$$

defines an inner product on $C[a, b]$.

Solution. We only prove positivity and leave the verification of symmetry and homogeneity as an exercise. For any function f in $C[a,b]$, $[f(x)]^2 \geq 0$. Hence

$$\langle f, f\rangle = \int_a^b f(x)^2\,dx \geq 0$$

Let $g(x) = f(x)^2$. Then g is nonnegative and continuous, so, by a theorem of calculus,

$$\int_a^b g(x)\,dx = 0 \Leftrightarrow g = \mathbf{0}$$

($\mathbf{0}$ is the zero function, i. e., $\mathbf{0}(x) = 0$ for all $x \in [a,b]$.) Thus

$$\langle f, f\rangle = \int_a^b f(x)^2\,dx = 0 \Leftrightarrow f = \mathbf{0}.$$

This concludes the verification of the positivity axiom.

We now discuss the basic properties of inner products and a generalization of the notion of length. Proving the basic properties of the inner product is not challenging. The proofs are identical to the corresponding ones for the dot product.

8.5.3 Length and orthogonality

In an inner product space, we can define lengths, distance,s and orthogonal vectors by using formulas identical to those for the dot product.

Definition 8.5.14. Let V be an inner product space. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are called *orthogonal* if their inner product is zero,

$$\langle \mathbf{u}, \mathbf{v}\rangle = 0.$$

The *norm* (or *length*, or *magnitude*) of $\mathbf{v}$ is the nonnegative number

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}. \tag{8.27}$$

The positive square root is defined, because $\langle \mathbf{v}, \mathbf{v}\rangle \geq 0$ by the positivity axiom. Equivalently, we have

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v}\rangle. \tag{8.28}$$

We also define the *distance* between two vectors $\mathbf{u}$ and $\mathbf{v}$ as

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|. \tag{8.29}$$

Note that

$$d(\mathbf{0}, \mathbf{v}) = d(\mathbf{v}, \mathbf{0}) = \|\mathbf{v}\|.$$

A vector with norm 1 is called a *unit* vector. The set

$$\mathbf{S} = \{\mathbf{v}, \ \mathbf{v} \in V \ \text{ and } \ \|\mathbf{v}\| = 1\} \tag{8.30}$$

of all unit vectors of V is called the *unit circle* or the *unit sphere*. So **S** consists of all vectors of V of distance 1 from the origin. This is how the unit circle and sphere are defined in $\mathbf{R}^2$ and $\mathbf{R}^3$ with respect to the ordinary (dot product) norm, thus justifying the names. Note, however, that a unit circle in $\mathbf{R}^2$ may *not* have the graph of a circle in the Cartesian coordinate system.

Example 8.5.15. For the inner product $\langle \mathbf{u}, \mathbf{v} \rangle = 3u_1v_1 + 4u_2v_2$ of Example 8.5.4 and for $\mathbf{u} = \left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$, $\mathbf{v} = \left[\begin{smallmatrix} 4 \\ 3 \end{smallmatrix}\right]$, $\mathbf{w} = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$, do the following:
(a) Compute $\|\mathbf{u}\|$.
(b) Compute $d(\mathbf{e}_1, \mathbf{e}_2)$.
(c) Prove that **v** and **w** are orthogonal.
(d) Describe and sketch a graph of the unit circle S.

Solution.
(a) We have

$$\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = 3(-2)(-2) + 4 \cdot 1 \cdot 1 = 16.$$

Hence

$$\|\mathbf{u}\| = \sqrt{16} = 4.$$

(b) We have

$$\|\mathbf{e}_1 - \mathbf{e}_2\|^2 = 3 \cdot 1 \cdot 1 + 4 \cdot (-1) \cdot (-1) = 7.$$

Therefore

$$d(\mathbf{e}_1, \mathbf{e}_2) = \sqrt{7}.$$

(c) Vectors **v** and **w** are orthogonal with respect to this inner product (not with respect to the dot product!), because

$$\langle \mathbf{v}, \mathbf{w} \rangle = 3 \cdot 4 \cdot 1 + 4 \cdot 3 \cdot (-1) = 0.$$

(d) Because $\|\mathbf{p}\| = 1$ is equivalent to $\|\mathbf{p}\|^2 = 1$, we have

$$S = \{\mathbf{p}, \text{ such that } 3p_1^2 + 4p_2^2 = 1\} \subseteq \mathbf{R}^2.$$

Thus the unit sphere (circle) with respect to this inner product looks like an *ellipse* in the coordinate system equipped with the ordinary dot product, angles, and distances (Figure 8.24).

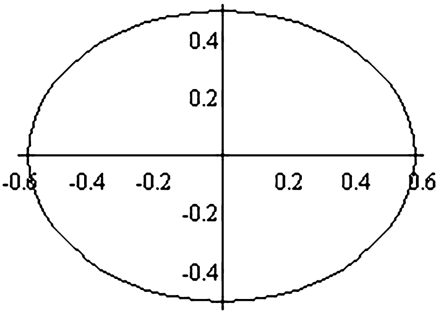

Figure 8.24: The unit circle for the inner product $3u_1v_1 + 4u_2v_2$.

8.5.4 Basic identities and inequalities

The axioms can be combined with the identity of the following theorem to generalize familiar identities of Section 2.5 and its exercises.

Theorem 8.5.16. *Let V be an inner product space. For any vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *of V, we have*

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle. \tag{8.31}$$

Proof. We have the following equalities, where we ask the reader to explain which axioms of the definition of inner product were used:

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}\rangle \\ &= \langle \mathbf{u}, \mathbf{u} + \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u} + \mathbf{v}\rangle \\ &= \langle \mathbf{u}, \mathbf{u}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle \\ &= \langle \mathbf{u}, \mathbf{u}\rangle + 2\langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle. \end{aligned}$$ □

Note that replacing $\mathbf{v}$ with $-\mathbf{v}$ in equation (8.31) yields the equation

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle. \tag{8.32}$$

Theorem 8.5.17 (Parallelogram law). *Let V be an inner product space. For any vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *of V, we have*

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\,\|\mathbf{u}\|^2 + 2\,\|\mathbf{v}\|^2 .$$

Proof. We just add equations (8.31) and (8.32). ☐

The identity of the following theorem gives the inner product in terms of the norm.

Theorem 8.5.18 (Polarization identity). *Let V be an inner product space. For any vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *of V, we have*

$$\langle \mathbf{u}, \mathbf{v}\rangle = \frac{1}{4}\,\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\,\|\mathbf{u}-\mathbf{v}\|^2 .$$

Proof. We subtract equation (8.32) from equation (8.31) and solve for $\langle \mathbf{u}, \mathbf{v}\rangle$. ☐

We also have a generalization of the Pythagorean theorem, studied in Section 2.5.

Theorem 8.5.19 (Pythagorean theorem). *Let V be an inner product space. The vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *of V are orthogonal if and only if*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 .$$

Proof. Exercise. ☐

The Cauchy–Bunyakovsky–Schwarz Inequality (CBSI)

One of the most useful consequences of the axioms is a generalization of the Cauchy–Bunyakovsky–Schwarz inequality (CBSI).

Theorem 8.5.20 (Cauchy–Bunyakovsky–Schwarz inequality).

$$|\langle \mathbf{u}, \mathbf{v}\rangle| \le \|\mathbf{u}\|\ \|\mathbf{v}\| .$$

Equality holds if and only if $\mathbf{u}$ *and* $\mathbf{v}$ *are scalar multiples of each other.*

Proof. By Theorem 8.5.2 we have

$$0 \le \langle x\,\mathbf{u}+\mathbf{v}, x\,\mathbf{u}+\mathbf{v}\rangle = x^2\,\langle \mathbf{u}, \mathbf{u}\rangle + 2x\langle\,\mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle \tag{8.33}$$

for all scalars x. This is a quadratic polynomial $p(x) = ax^2 + bx + c$ with $a = \langle \mathbf{u}, \mathbf{u}\rangle$, $b = 2\,\langle\,\mathbf{u}, \mathbf{v}\rangle$, and $c = \langle \mathbf{v}, \mathbf{v}\rangle$. Since $a \ge 0$ and $p(x) \ge 0$ for all x, the graph of $p(x)$ is a parabola in the upper half-plane that opens upward. Hence the parabola is either above the x-axis, in which case $p(x)$ has two complex roots, or is tangent to the x-axis, in which case $p(x)$ has a repeated real root. Therefore $b^2 - 4ac \le 0$. So

$$(2\,\langle \mathbf{u}, \mathbf{v}\rangle)^2 - 4\langle \mathbf{u}, \mathbf{u}\rangle\langle \mathbf{v}, \mathbf{v}\rangle \le 0 \quad \text{or} \quad 4\langle \mathbf{u}, \mathbf{v}\rangle^2 - 4\,\|\mathbf{u}\|^2\ \|\mathbf{v}\|^2 \le 0,$$

which implies the CBSI. Equality holds if and only if $b^2 - 4ac = 0$ or if and only if $p(x)$ has a double real root, say r. Hence by equation (8.33) with $x = r$ we have

$$\begin{aligned}\langle r\,\mathbf{u} + \mathbf{v}, r\,\mathbf{u} + \mathbf{v}\rangle &= 0 \\ &\Leftrightarrow \|r\,\mathbf{u} + \mathbf{v}\| = 0 \\ &\Leftrightarrow r\,\mathbf{u} + \mathbf{v} = \mathbf{0} \\ &\Leftrightarrow \mathbf{v} = -r\mathbf{u}.\end{aligned}$$

So $\mathbf{v}$ is a scalar product of $\mathbf{u}$. This proves the last claim of the theorem. ☐

Example 8.5.21. Verify the CBSI for Example 8.5.8, Section 8.5, with $r_0 = -2$, $r_1 = 0$, $r_2 = 1$, and $p(x) = 1 - x^2$, and $q(x) = -2x + x^2$.

Solution. We have

$$\begin{aligned}\langle p, q\rangle &= p(-2)q(-2) + p(0)q(0) + p(1)q(1) = -24, \\ \langle p, p\rangle &= p(-2)^2 + p(0)^2 + p(1)^2 = 10, \\ \langle q, q\rangle &= q(-2)^2 + q(0)^2 + q(1)^2 = 65.\end{aligned}$$

Hence

$$\|p\| = \sqrt{10}, \quad \|q\| = \sqrt{65},$$

and

$$\left|\langle p, q\rangle\right| = |-24| = 24 \leq \sqrt{10} \cdot \sqrt{65} \simeq 25.495$$

verifies the CBSI.

As an application of Theorem 8.5.16 and the CBSI, we have the useful *triangle inequality*.

Theorem 8.5.22 (The triangle inequality).

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|\,.$$

Proof. We have

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\,\langle \mathbf{u}, \mathbf{v}\rangle && \text{by Theorem 8.5.16} \\ &\leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| && \text{by CBSI} \\ &= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2\,.\end{aligned}$$

Therefore $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. ☐

8.5.5 The Gram–Schmidt process

The Gram–Schmidt process studied in Section 8.2 can be easily extended to inner products, thus establishing the existence of orthogonal bases for finite-dimensional inner product spaces. The formulas are the same as before, except that we replace the dot product with a general inner product.

Theorem 8.5.23 (Generalized Gram–Schmidt process). *Let V be a finite-dimensional inner product space with basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. Then V has an orthogonal basis $\mathcal{B}' = \{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, where*

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1, \\
\mathbf{u}_2 &= \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1, \\
&\vdots \\
\mathbf{u}_k &= \mathbf{v}_k - \frac{\langle \mathbf{v}_k, \mathbf{u}_1\rangle}{\langle \mathbf{u}_1, \mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_k, \mathbf{u}_2\rangle}{\langle \mathbf{u}_2, \mathbf{u}_2\rangle}\mathbf{u}_2 \cdots - \frac{\langle \mathbf{v}_k, \mathbf{u}_{k-1}\rangle}{\langle \mathbf{u}_{k-1}, \mathbf{u}_{k-1}\rangle}\mathbf{u}_{k-1}.
\end{aligned}$$

Furthermore, for $i = 1, \dots, k$, we have

$$\operatorname{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_i\} = \operatorname{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_i\}.$$

An orthonormal basis $\mathcal{B}''$ of V is obtained by normalizing $\mathcal{B}'$,

$$\mathcal{B}'' = \left\{ \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}, \dots, \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|} \right\}.$$

Example 8.5.24. Find an orthogonal basis of P_2 starting with $\{1, x, x^2\}$ and using the inner product of Example 8.5.8 with $r_0 = 0, r_1 = 1, r_2 = 2$.

Solution. Let $p_1 = 1$. We have

$$\langle 1, 1\rangle = 1^2 + 1^2 + 1^2 = 3, \quad \langle x, 1\rangle = 0 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 = 3.$$

So we let

$$p_2 = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1 = x - 1.$$

Likewise,

$$\langle x - 1, x - 1\rangle = 2, \quad \langle x^2, 1\rangle = 5, \quad \langle x^2, x - 1\rangle = 4.$$

We set

$$p_3 = x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle x^2, x - 1\rangle}{\langle x - 1, x - 1\rangle}(x - 1) = x^2 - 2x + \frac{1}{3}.$$

Therefore

$$\{p_1, p_2, p_3\} = \left\{1, x - 1, x^2 - 2x + \frac{1}{3}\right\}$$

is an orthogonal basis of P_2 with respect to the given inner product.

The best approximation theorem (Theorem 8.2.10) also generalizes to inner product spaces. We leave it to the reader to write out the details. This generalization is particularly useful when we try to approximate a function by using other functions. The kind of approximation depends on the inner product that we use.

Example 8.5.25 (Generalized best approximation). Referring to Example 8.5.24 and its solution, find a polynomial $\widetilde{p}$ in $P_1 = \operatorname{Span}\{1, x\} \subseteq P_2$ that best approximates $p(x) = 2x^2 - 1$.

Solution. By the solution of Example 8.5.24, $\{p_0, p_1\} = \{1, x - 1\}$ is an orthogonal basis of P_1. We have

$$\begin{aligned}\langle 2x^2 - 1, 1\rangle &= (-1) \cdot 1 + 1 \cdot 1 + 7 \cdot 1 = 7, \\ \langle 2x^2 - 1, x - 1\rangle &= (-1) \cdot (-1) + 1 \cdot 0 + 7 \cdot 1 = 8.\end{aligned}$$

Therefore

$$\begin{aligned}\widetilde{p} = p_{\text{pr}} &= \frac{\langle p, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p, p_1\rangle}{\langle p_1, p_1\rangle} p_1 \\ &= \frac{7}{3} + \frac{8}{2}(x - 1) = 4x - \frac{5}{3}.\end{aligned}$$

Hence $4x - 5/3$ of P_1 best approximates $2x^2 - 1$ with respect to the given inner product (Figure 8.25).

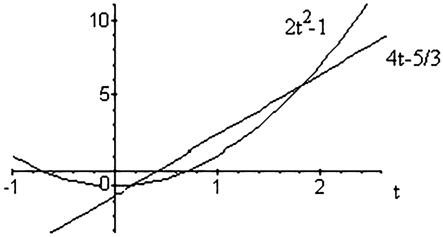

Figure 8.25: $4t - 5/3$ is the best linear polynomial approximating $2t^2 - 1$ when the distance is measured at $0, 1, 2$.

Exercises 8.5

Let $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. In Exercises 1–2, use the given inner product to

(a) compute $\langle \mathbf{u}, \mathbf{v} \rangle$, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\|\mathbf{u} + \mathbf{v}\|$,
(b) compute the distance $d(\mathbf{u}, \mathbf{v})$,
(c) verify CBSI for $\mathbf{u}$ and $\mathbf{v}$,
(d) verify the triangle inequality for $\mathbf{u}$ and $\mathbf{v}$,
(e) verify the polarization identity for $\mathbf{u}$ and $\mathbf{v}$.

1. The inner product is that of Example 8.5.4.

2. The inner product is that of Example 8.5.5 with weight vector $\mathbf{w} = (2, 5)$.

3. Referring to Exercise 1, find a vector orthogonal to $\mathbf{u}$.

4. Referring to Exercise 2, find a vector orthogonal to $\mathbf{u}$.

In Exercises 5–8, consider the inner product of Example 8.5.6 and let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

5. Find the orthogonal pairs between A, B, and C.

6. For the orthogonal pairs found in Exercise 5, verify the Pythagorean theorem.

7. Construct orthonormal pairs out of the orthogonal ones found in Exercise 5.

8. Which of A, B, and C are unit vectors?

In Exercises 9–13, determine whether for the given matrix A, the function

$$f(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v}\,, \quad \mathbf{u}, \mathbf{v} \in \mathbf{R}^2,$$

defines an inner product of $\mathbf{R}^2$ as follows: Refer to Example 8.5.11 and check to see if A is positive definite. If A is not positive definite, then find an axiom of the definition of inner product that fails.

9. $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$.

10. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

11. $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

12. $A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$.

13. $A = \begin{bmatrix} 8 & 7 \\ 1 & 2 \end{bmatrix}$.

14. *True or false?* Explain. In an inner product space:
(a) If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.
(b) If $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle$ and $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{v} = \mathbf{w}$.
(c) The sum of two unit vectors is a unit vector.

In Exercises 15–18, let $p = 1 - 2x^2$ and $q = -2x + x^2$.

(a) Compute $\langle p, q\rangle$, $\|p\|$, $\|q\|$, $\|p+q\|$.
(b) Compute the distance $d(q, p)$.
(c) Verify CBSI for p and q.
(d) Verify the triangle inequality for p and q.

15. The inner product in P_2 is that of Example 8.5.7.

16. The inner product in P_2 is that of Example 8.5.8 with $r_0 = -3, r_1 = 0, r_2 = 2$.

17. Referring to Exercise 15, find a vector orthogonal to p.

18. (*Requires calculus*) The inner product is that of Example 8.5.13 with $a = -1$ and $b = 1$.

19. Referring to Exercise 16, find a vector orthogonal to p.

20. Referring to Exercise 18, find a vector orthogonal to p.

21. Consider $\mathbf{R}^2$ equipped with the inner product of Example 8.5.12. Is the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ an orthogonal basis?

22. Consider the inner product $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T A\mathbf{v}$ in R^3, where

$$A = \begin{bmatrix} 2 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix}.$$

Let

$$\mathcal{P} = \left\{\mathbf{x} : \langle \mathbf{e}_1, \mathbf{x}\rangle = 0\right\} \subset \mathbf{R}^3.$$

(a) Write $\mathcal{P}$ as a span of linearly independent vectors.
(b) Describe $\mathcal{P}$ geometrically.

23. Let $\mathcal{B}$ be a basis of a finite-dimensional inner product space.
(a) Prove that if $\langle \mathbf{u}, \mathbf{v}\rangle = 0$ for all $\mathbf{v} \in \mathcal{B}$, then $\mathbf{u} = \mathbf{0}$.
(b) Prove that if $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{w}, \mathbf{v}\rangle$ for all $\mathbf{v} \in \mathcal{B}$, then $\mathbf{u} = \mathbf{w}$.

24. Complete the proof of Theorem 8.5.2.

25. Prove Theorem 8.5.19.

26. Suppose that $T : \mathbf{R}^n \to \mathbf{R}^n$ is an invertible linear transformation. Prove that the assignment

$$\langle \mathbf{u}, \mathbf{v}\rangle = T(\mathbf{u}) \cdot T(\mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in \mathbf{R}^n,$$

defines an inner product in $\mathbf{R}^n$.

27. Let V be an inner product space, and let T be a linear operator $T : V \to V$ such that

$$\left\|T(\mathbf{v})\right\| = \|\mathbf{v}\| \quad \text{for all } \mathbf{v} \in V.$$

Prove that T is one-to-one.

28. Let V be an inner product space, and let $\mathbf{u}$ be in V. Prove that the following transformations T and L from V to V are linear:

(a) $T(\mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle, \mathbf{v} \in V$;
(b) $L(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u} \rangle, \mathbf{v} \in V$.

29. Any finite-dimensional vector space V can be made an inner product space as follows. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V, and let $\mathbf{u}$ and $\mathbf{v}$ be any vectors in V. Then there are scalars a_i and b_i such that

$$\mathbf{u} = \sum_{i=1}^{n} a_i \mathbf{v}_i, \quad \mathbf{v} = \sum_{i=1}^{n} b_i \mathbf{v}_i.$$

Let $\langle \mathbf{u}, \mathbf{v} \rangle$ be defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} a_i b_i.$$

(a) Prove that the operation $\langle \mathbf{u}, \mathbf{v} \rangle$ makes V an inner product space.
(b) Prove that under this inner product $\mathcal{B}$ is an orthonormal basis.

Projections

We compute orthogonal projections with respect to inner products by using the same formula as before, where the dot product is replaced by the given inner product. The distance from a vector $\mathbf{u}$ to a subspace W in an inner product space is the norm $\|\mathbf{u}_c\|$ of the orthogonal component $\mathbf{u}_c$ of $\mathbf{u}$ with respect to W.

30. Consider $\mathbf{R}^2$ equipped with the inner product of Example 8.5.12. Find the orthogonal projection of $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ onto the line l through $\mathbf{0}$ and $\mathbf{e}_1$.

31. Referring to Exercise 30, find the distance from the point $(1, 1)$ to the line l.

Gram–Schmidt process

32. Consider $\mathbf{R}^2$ equipped with the inner product of Example 8.5.12. Apply the Gram–Schmidt process to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ to find an orthogonal basis for this inner product.

33. Apply the Gram–Schmidt process to find an orthogonal basis of P_2 starting with $1, x, x^2$ and using the inner product of Example 8.5.8 with $r_0 = -2, r_1 = 0, r_2 = 2$.

The remaining exercises require calculus.

34. Prove that the first four *Legendre* polynomials

$$\mathcal{L} = \left\{1, x, (3/2)x^2 - (1/2), (5/2)x^3 - (3/2)x\right\}$$

form an orthogonal basis for P_3 for the inner product of Example 8.5.13 with $a = -1$ and $b = 1$.

35. Use the Legendre polynomials and the inner product of Exercise 34 to find an orthonormal basis for P_3.

36. Consider the inner product of Example 8.5.13 with $a = 0$ and $b = \pi$. Let $f(x) = x$ and $g(x) = \sin(x)$.

(a) Compute $\langle f, g \rangle$, $\|f\|$, $\|g\|$, $\|f + g\|$.
(b) Compute the distance $d(f, g)$.
(c) Verify CBSI for f and g.
(d) Verify the triangle inequality for f and g.

8.6 Complex inner products; unitary matrices

Inner products may be defined over complex vector spaces. Complex inner product spaces are useful both in theory and applications.

8.6.1 Definition and examples

Definition 8.6.1. Let V be a complex vector space as defined in Section 4.1. Then V is called a *complex inner product space* if there is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbf{C}$ that assigns to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ of V a complex number denoted by $\langle \mathbf{u}, \mathbf{v} \rangle$ satisfying the following properties, or *axioms*.

For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and any complex scalar c, we have

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$;
2. $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$;
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$;
4. $\langle \mathbf{u}, \mathbf{u} \rangle > 0$, if $\mathbf{u} \neq \mathbf{0}$.

!

1. $\overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ in the above definition is the complex conjugate of the complex number $\langle \mathbf{v}, \mathbf{u} \rangle$.
2. Property 4 is called *positivity*, or *positive definiteness*.

The following theorem describes the basic properties of complex inner products. The reader should pay attention to Part 4 and contrast it with Part 2 of Theorem 8.5.2.

Theorem 8.6.2. *Let V be a complex inner product space. For any vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ of V and any complex scalar c, we have the following:*

1. $\langle \mathbf{0}, \mathbf{u} \rangle = 0$;
2. $\langle \mathbf{u}, \mathbf{0} \rangle = 0$;
3. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$;
4. $\langle \mathbf{u}, c\mathbf{v} \rangle = \overline{c}\langle \mathbf{u}, \mathbf{v} \rangle$.

Proof.

1. By Property 3 of the definition we have

$$\langle \mathbf{0}, \mathbf{u} \rangle = \langle 0\mathbf{0}, \mathbf{u} \rangle = 0\langle \mathbf{0}, \mathbf{u} \rangle = 0.$$

2. By Property 1 and Part 1 we have

$$\langle \mathbf{u}, \mathbf{0} \rangle = \overline{\langle \mathbf{0}, \mathbf{u} \rangle} = \overline{0} = 0.$$

3. By Properties 1 and 2 we have

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}+\mathbf{w}\rangle &= \overline{\langle \mathbf{v}+\mathbf{w}, \mathbf{u}\rangle} = \overline{\langle \mathbf{v}, \mathbf{u}\rangle + \langle \mathbf{w}, \mathbf{u}\rangle} \\ &= \overline{\langle \mathbf{v}, \mathbf{u}\rangle} + \overline{\langle \mathbf{w}, \mathbf{u}\rangle} = \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{w}\rangle .\end{aligned}$$

4. By Properties 1 and 3 we have

$$\langle \mathbf{u}, c\mathbf{v}\rangle = \overline{\langle c\mathbf{v}, \mathbf{u}\rangle} = \overline{c\langle \mathbf{v}, \mathbf{u}\rangle} = \overline{c}\,\overline{\langle \mathbf{v}, \mathbf{u}\rangle} = \overline{c}\langle \mathbf{u}, \mathbf{v}\rangle . \qquad \square$$

Our first and most basic example of a complex inner product is the complex dot product, also known as the *standard complex inner product*.

Example 8.6.3 (The complex dot product). Let $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ be in $\mathbf{C}^n$. The following function is called the *complex dot product*:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \overline{\mathbf{v}} = u_1\overline{v_1} + \cdots + u_n\overline{v_n}. \tag{8.34}$$

Prove that the complex dot product makes $\mathbf{C}^n$ a complex inner product space.

Solution. We need to verify the axioms of the definition. We verify Axioms 1 and 4 and leave Axioms 2 and 3 as exercises.

(a) Axiom 1: We have

$$\langle \mathbf{u}, \mathbf{v}\rangle = u_1\overline{v_1} + \cdots + u_n\overline{v_n} = \overline{v_1\overline{u_1} + \cdots + v_n\overline{u_n}} = \overline{\langle \mathbf{v}, \mathbf{u}\rangle}.$$

(b) Axiom 4: We have

$$\langle \mathbf{u}, \mathbf{u}\rangle = u_1\overline{u_1} + \cdots + u_n\overline{u_n} = |u_1|^2 + \cdots + |u_n|^2 \geq 0.$$

This expression is zero only if each $|u_i| = 0$ and hence only if each $u_i = 0$. So it is zero only if $\mathbf{u} = \mathbf{0}$.

Example 8.6.4. $\begin{bmatrix} 1+i \\ i \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -1+i \end{bmatrix} = (1+i)(\overline{5}) + i(\overline{-1+i}) = 6+4i.$

Property 1 in the definition of a complex inner product may be surprising in that $\overline{\langle \mathbf{v}, \mathbf{u}\rangle}$ is used instead of $\langle \mathbf{v}, \mathbf{u}\rangle$, which was used in real inner products. The reason is that $\langle \mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{v}, \mathbf{u}\rangle$ would lead to inconsistencies. For example, in $\mathbf{C}^1$, by positivity we have $\langle 1, 1\rangle > 0$ and $\langle i, i\rangle > 0$. But this contradicts the following "computation":

$$\langle i, i\rangle = i^2\langle 1, 1\rangle = -1\langle 1, 1\rangle < 0.$$

The notion of orthogonality is as in the case of real inner products.

Definition 8.6.5. We say that the vectors $\mathbf{u}$ and $\mathbf{v}$ of a complex inner product space are *orthogonal* if their inner product is zero,

$$\langle \mathbf{u}, \mathbf{v}\rangle = 0.$$

The *norm*, or *length*, or *magnitude* of a vector $\mathbf{u}$ is the positive square root

$$\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}.$$

A *unit* vector is one with norm 1. An *orthogonal* set is one with pairwise orthogonal vectors. An *orthonormal* set is an orthogonal set of unit vectors. An orthonormal set with respect to a complex inner product is also called a *unitary system*.

Note that for a complex scalar c, we have

$$\|c\mathbf{u}\| = |c| \, \|\mathbf{u}\| .$$

The useful identity (8.31) of Theorem 8.5.16 now takes the following form:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2 \operatorname{Re}\langle \mathbf{u}, \mathbf{v} \rangle.$$

Using this identity, we can prove exactly as in the real case the triangle inequality and the Cauchy–Bunyakovsky–Schwarz inequality (CBSI).

Theorem 8.6.6. *In a complex inner product space, we have*

1. $|\langle \mathbf{u}, \mathbf{v} \rangle| \le \|\mathbf{u}\| \, \|\mathbf{v}\|$; *(Cauchy–Bunyakovsky–Schwarz inequality)*
2. $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$. *(Triangle inequality)*

In the context of complex inner products spaces, we also have *orthogonal projections*, the *Gram–Schmidt process*, and the *best approximation theorem*. The statements are similar to those of the real case.

8.6.2 Unitary matrices

In Section 2.1, we defined Hermitian and skew-Hermitian matrices. These generalize the real symmetric and skew-symmetric matrices. We now add a third important class of complex matrices, the unitary matrices, which generalizes the class of orthogonal matrices.

Definition 8.6.7. A square matrix A is called *unitary* if $A^H = A^{-1}$.

Example 8.6.8. Prove that the matrix

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} i \\ -\frac{\sqrt{3}}{2} i & \frac{1}{2} \end{bmatrix}$$

is unitary.

Solution. It suffices to check that $A^H A = I$. We have

$$
\begin{aligned}
A^H A &= \overline{\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ -\frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix}}^{T} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ -\frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2}i \\ \frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ -\frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.
\end{aligned}
$$

If A is real and unitary, then $A^H A = A^T A = I$. So A is orthogonal.

Theorem 8.1.25 states that orthogonal matrices preserve dot products and norms. Likewise, unitary matrices preserve complex dot products and complex norms. We have the following theorem.

Theorem 8.6.9. *Let A be an $n \times n$ unitary matrix. Then for any complex n-vectors $\mathbf{u}$ and $\mathbf{v}$, we have with respect to the complex dot product:*

1. $A\mathbf{u} \cdot A\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$;
2. $\|A\mathbf{v}\| = \|\mathbf{v}\|$.

Proof.

1. We have

$$(A\mathbf{u}) \cdot (A\mathbf{v}) = (A\mathbf{u})^T\, \overline{(A\mathbf{v})} = \mathbf{u}^T A^T \overline{A}\,\overline{\mathbf{v}} = \mathbf{u}^T I \overline{\mathbf{v}} = \mathbf{u}^T \overline{\mathbf{v}} = \mathbf{u} \cdot \mathbf{v}.$$

2. By Part 1 we have

$$\|A\mathbf{v}\|^2 = (A\mathbf{v}) \cdot (A\mathbf{v}) = \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 .$$

Thus $\|A\mathbf{v}\| = \|\mathbf{v}\|$. ☐

Statements 1 and 2 of Theorem 8.6.9 are actually each equivalent to saying that A is unitary. The reader can verify this claim by reviewing the proof of Theorem 8.1.25.

Our next focus is information on the eigenvalues of all three classes of these special complex square matrices.

Theorem 8.6.10. *Let A be a complex square matrix.*

1. *If A is Hermitian, then its eigenvalues are real. (Thus this holds for symmetric matrices.)*
2. *If A is skew-Hermitian, then its eigenvalues are pure imaginary or* 0. *(Thus this holds for skew-symmetric matrices.)*
3. *If A is unitary, then its eigenvalues have absolute value* 1. *(Thus this holds for orthogonal matrices.)*

Proof. Let $\mathbf{v}$ be an eigenvector of A with corresponding eigenvalue λ. Then

$$A\mathbf{v} = \lambda\mathbf{v}. \tag{8.35}$$

Multiplication of (8.35) by $\overline{\mathbf{v}}^T$ yields

$$\overline{\mathbf{v}}^T A\mathbf{v} = \overline{\mathbf{v}}^T \lambda\mathbf{v} = \lambda\overline{\mathbf{v}}^T\mathbf{v}.$$

Since $\mathbf{v} \neq \mathbf{0}$, $\overline{\mathbf{v}}^T\mathbf{v} = \overline{v_1}v_1 + \cdots + \overline{v_n}v_n = |v_1|^2 + \cdots + |v_n|^2 > 0$. Hence we can divide to get

$$\lambda = \frac{\overline{\mathbf{v}}^T A\mathbf{v}}{\overline{\mathbf{v}}^T\mathbf{v}}. \tag{8.36}$$

We compute the complex conjugate of the numerator to get

$$\overline{\overline{\mathbf{v}}^T A\mathbf{v}} = \overline{\overline{\mathbf{v}}^T}\,\overline{A}\,\overline{\mathbf{v}} = \mathbf{v}^T\overline{A}\,\overline{\mathbf{v}} = \left(\mathbf{v}^T\overline{A}\,\overline{\mathbf{v}}\right)^T = \overline{\mathbf{v}}^T\overline{A}^T\mathbf{v}. \tag{8.37}$$

1. Let A be Hermitian. Then the last expression of (8.37) equals $\overline{\mathbf{v}}^T A\mathbf{v}$. Hence $\overline{\mathbf{v}}^T A\mathbf{v}$ equals its conjugate. Therefore it is real. This shows that λ in (8.36) is real.
2. Let A be skew-Hermitian. Then the last expression of (8.37) equals $-\overline{\mathbf{v}}^T A\mathbf{v}$. Hence $\overline{\mathbf{v}}^T A\mathbf{v}$ is the opposite of its conjugate. Therefore it is pure imaginary or zero. This shows that λ in (8.36) is zero or pure imaginary.
3. Let A be unitary. Then by Part 2 of Theorem 8.6.9 we have

$$|\lambda|\,\|\mathbf{v}\| = \|\lambda\mathbf{v}\| = \|A\mathbf{v}\| = \|\mathbf{v}\|.$$

Therefore $|\lambda| = 1$, because $\|\mathbf{v}\| \neq 0$. □

Just as with orthogonal matrices, unitary matrices have unit columns that are pairwise orthogonal with respect to the complex dot product.

Theorem 8.6.11. *Let A be a unitary matrix. Then the columns of A*
1. *form a unitary system with respect to the complex dot product and*
2. *are linearly independent over the set of complex numbers.*

Proof. Exercise. □

Exercises 8.6

1. Consider the complex dot product on $\mathbf{C}^2$. Let

$$\mathbf{u} = \begin{bmatrix} 1+i \\ -2i \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6i \\ 1-5i \end{bmatrix}.$$

(a) Verify the CBSI for $\mathbf{u}$ and $\mathbf{v}$.
(b) Verify the triangle inequality for $\mathbf{u}$ and $\mathbf{v}$.

2. Verify Axioms 2 and 3 for the complex dot product.

3. Let V be a complex inner product space, and let $\mathbf{u}$ be in V.

(a) Prove that

$$L(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u} \rangle, \quad \mathbf{v} \in V,$$

is linear.

(b) Is

$$T(\mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle, \quad \text{all } \mathbf{v} \in V$$

is linear? Explain.

4. Prove that for the complex dot product in $\mathbf{C}^n$,

$$\mathbf{u} \cdot A\mathbf{v} = A^H\mathbf{u} \cdot \mathbf{v} \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{C}^n.$$

5. Let V be a complex inner product space. Prove that for all $\mathbf{u}$ and $\mathbf{v}$ in V,

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\,\mathrm{Re}\langle \mathbf{u}, \mathbf{v} \rangle.$$

6. Use the identity of Exercise 5 to prove the Pythagorean theorem: If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

7. *True or false?* Explain. In an complex inner product space:

(a) If $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, then either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$.

(b) If $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle$ and $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{v} = \mathbf{w}$.

(c) The sum of two unit vectors is a unit vector.

(d) For a basis $\mathcal{B}$, if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in \mathcal{B}$, then $\mathbf{u} = \mathbf{0}$.

(e) The sum of two unitary matrices is unitary.

8. (*The Frobenius inner product*) Let $M_{nn}(\mathbf{C})$ be the set of $n \times n$ matrices with complex entries. Prove that the following defines a complex inner product on $M_{nn}(\mathbf{C})$:

$$\langle A, B \rangle = \mathrm{tr}\left(B^H A\right).$$

In Exercises 9–13, determine whether the matrix A is unitary. If it is, then find its inverse.

9. $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$

10. $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$

11. $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$

12. $A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}i & 0 \\ -\frac{\sqrt{3}}{2}i & \frac{1}{2} & 0 \\ 0 & 0 & i \end{bmatrix}$.

13. $A = \begin{bmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{bmatrix}$.

14. Prove that the *Pauli spin matrices*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

introduced in Exercises 2.1 and used in particle physics, are unitary.

15. Let

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i & 0 \\ -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & i \end{bmatrix}.$$

(a) Prove that A is unitary.
(b) Find the eigenvalues of A.
(c) Verify for A Part 3 of Theorem 8.6.10.

16. Verify Theorem 8.6.9 for the unitary matrix

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2}i \\ -\frac{\sqrt{3}}{2}i & \frac{1}{2} \end{bmatrix}.$$

17. Prove Theorem 8.6.11.

18. Prove that the product of two $n \times n$ unitary matrices is also unitary.

19. Prove that the inverse of unitary matrices is also unitary.

20. *(Requires calculus)* The formula

$$\langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}\, dx$$

defines an inner product on $C_{\mathbf{C}}[0, 1]$, the vector space of continuous complex-valued functions on $[0, 1]$. Let $f(x) = ix$ and $g(x) = e^{ix}$. (Note that by Euler's formula $e^{ix} = \cos x + i \sin x$. Conclude that $\overline{e^{ix}} = e^{-ix}$.)
(a) Compute $\langle f, g \rangle$, $\|f\|$, $\|g\|$, $\|f + g\|$.
(b) Verify CBSI for f and g.
(c) Verify the triangle inequality for f and g.

8.7 Polynomial and continuous least squares

In Section 8.4, we studied least squares line fitting for planar data points. However, not all sets of data can be satisfactorily approximated by straight lines. Often, we have to use quadratic or cubic polynomials or other functions. The question then arises: Which function is suitable in a given situation and how to do we compute it? This is the subject of *trend analysis*.

8.7.1 Polynomial least squares

Let us explore the polynomial fitting of data.

We seek a polynomial q of degree at most $n-1$ that best approximates a set of points $(a_1, b_1), \ldots, (a_m, b_m)$. Let

$$q(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1} x^{n-1}.$$

Evaluation of q at the x-coordinates of the points may not quite yield the corresponding y-coordinates. Suppose that there are errors $\delta_1, \ldots, \delta_m$ in the y-coordinates. We have

$$\begin{aligned} b_1 &= \alpha_0 + \alpha_1 a_1 + \cdots + \alpha_{n-1} a_1^{n-1} + \delta_1, \\ b_2 &= \alpha_0 + \alpha_1 a_2 + \cdots + \alpha_{n-1} a_2^{n-1} + \delta_2, \\ &\vdots \\ b_m &= \alpha_0 + \alpha_1 a_m + \cdots + \alpha_{n-1} a_m^{n-1} + \delta_m. \end{aligned}$$

In matrix notation,

$$\mathbf{b} = A\boldsymbol{\alpha} + \Delta,$$

where

$$\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \quad \boldsymbol{\alpha} = \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{bmatrix}, \quad \Delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_m \end{bmatrix},$$

and A is the $m \times n$ coefficient matrix

$$A = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_m & a_m^2 & \cdots & a_m^{n-1} \end{bmatrix}. \tag{8.38}$$

The goal is to find a vector $\boldsymbol{\alpha}$ that minimizes the length of the error vector $\|\Delta\| = \|\mathbf{b} - A\boldsymbol{\alpha}\|$. As we saw in Section 8.4, this amounts to solving for $\boldsymbol{\alpha}$ the normal equations

$$A^T A\boldsymbol{\alpha} = A^T\mathbf{b}.$$

We know that least squares solutions always exist, but what about uniqueness? After all, we want *one* polynomial that best fits the data. The following theorem gives an answer.

Theorem 8.7.1. *Suppose that the data points* $(a_1, b_1), \ldots, (a_m, b_m)$ *have all different x-coordinates. For each positive integer n such that* $n \leq m$, *there is a unique polynomial*

$$q(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1}$$

that minimizes $\|\Delta\|$.

Proof. Let n be a positive integer such that $n \leq m$. By Part 2 of Theorem 8.4.3 it suffices to prove that the matrix A given by equation (8.38) has linearly independent columns if all a_i are distinct. We prove this claim for $m = 4$ and $n = 3$ and leave the general case as an exercise. We have

$$\begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \\ 1 & a_4 & a_4^2 \end{bmatrix} \sim \begin{bmatrix} 1 & a_1 & a_1^2 \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 \\ 0 & a_3 - a_1 & a_3^2 - a_1^2 \\ 0 & a_4 - a_1 & a_4^2 - a_1^2 \end{bmatrix} \sim \begin{bmatrix} 1 & a_1 & a_1^2 \\ 0 & a_2 - a_1 & a_2^2 - a_1^2 \\ 0 & 0 & (a_1 - a_3)(a_2 - a_3) \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence all the columns of A are pivot columns, because the a_i are distinct. Thus A has linearly independent columns, as claimed. □

The unique polynomial of Theorem 8.7.1 is called the *least squares polynomial* of degree $n - 1$ for these points. Of particular interest is when A is square, in which case $m = n$.

Theorem 8.7.2. *Suppose that the n data points* $(a_1, b_1), \ldots, (a_n, b_n)$ *have all distinct x-coordinates. Then* $\Delta = \mathbf{0}$, *and the unique least squares polynomial*

$$q(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_{n-1}x^{n-1}$$

passes through all the points. So in this case,

$$q(a_i) = b_i, \quad i = 1, \ldots, n.$$

Proof. Exercise. □

The unique polynomial of Theorem 8.7.2 is called the *interpolating polynomial* for the points. Note that if A above is square, then its transpose

$$A^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}$$

is the Vandermonde matrix studied in the miniproject Section 6.7.

Example 8.7.3. For the following grades data, find
(a) the least squares line,
(b) the least squares quadratic.

Semester	**1**	**2**	**3**	**4**	**5**	**6**
Percentage of Cs	0.20	0.25	0.25	0.35	0.35	0.30

Solution.
(a) To compute the least squares line, let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.20 \\ 0.25 \\ 0.25 \\ 0.35 \\ 0.35 \\ 0.30 \end{bmatrix}.$$

Then

$$A^T A\boldsymbol{\alpha} = \begin{bmatrix} 6 & 21 \\ 21 & 91 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = A^T\mathbf{b} = \begin{bmatrix} 1.7 \\ 6.4 \end{bmatrix}$$

with solution $\boldsymbol{\alpha} = (0.19333, 0.0257)$. So the least squares line is

$$y = 0.19333 + 0.0257x.$$

(b) To compute the least squares quadratic, let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}.$$

Then the normal equations are

$$A^T A\boldsymbol{a} = \begin{bmatrix} 6 & 21 & 91 \\ 21 & 91 & 441 \\ 91 & 441 & 2275 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = A^T \mathbf{b} = \begin{bmatrix} 1.7 \\ 6.4 \\ 28.6 \end{bmatrix}$$

with unique solution $\boldsymbol{a} = (0.11, 0.0882, -0.0089)$. Hence the least squares quadratic is

$$y = 0.11 + 0.0882x - 0.0089x^2.$$

Figure 8.26 shows the two least squares curves and the points. If the quadratic is a better approximation (it seems to be), then the number of Cs is expected to decrease overall.

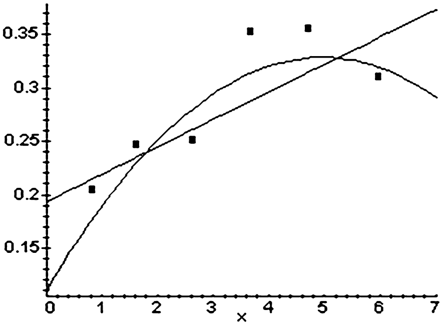

Figure 8.26: Least squares line and quadratic for the same data.

Least squares methods are not restricted to planar data points. We can fit space points by using *surfaces* instead of curves. The simplest surfaces are planes. Other surfaces are defined by equations $f(x, y, z) = 0$, where f is a polynomial in three variables x, y, and z.

Example 8.7.4. Find a least squares plane for the following space data points:

$$(1, 0.5, 0.75), \quad (2, 1.2, 2.5), \quad (3, 2.5, -0.5), \quad (4, 2, 0.5)$$

Solution. Since all data points have different xy-coordinates, we may assume that the plane has equation of the form $z = ax + by + c$. When we plug-in a point (x_i, y_i, z_i) into this equation, there is an error δ_i, so that $z_i = ax_i + by_i + c + \delta_i$. If we use the given points, then we get an overdetermined linear system $\mathbf{b} = A\boldsymbol{a} + \Delta$. Explicitly, we have

$$\begin{bmatrix} 0.75 \\ 2.5 \\ -0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 1 \\ 2 & 1.2 & 1 \\ 3 & 2.5 & 1 \\ 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix}.$$

We find a vector $\boldsymbol{a} = (a, b, c)$ that minimizes the length of the error vector $\|\Delta\| = \|\mathbf{b} - A\boldsymbol{a}\|$. This amounts to solving for $\boldsymbol{a}$ the normal equations $A^T A\boldsymbol{a} = A^T\mathbf{b}$. We have

$$[A^T A : A^T\mathbf{b}] = \begin{bmatrix} 30 & 18.4 & 10 & 6.25 \\ 18.4 & 11.94 & 6.2 & 3.125 \\ 10 & 6.2 & 4 & 3.25 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0.36343 \\ 0 & 1 & 0 & -1.2731 \\ 0 & 0 & 1 & 1.8773 \end{bmatrix}.$$

So the least squares plane shown in Figure 8.27 is

$$z = 0.36343x - 1.2731y + 1.8773.$$

The least squares error is $\|\Delta\| = \|\mathbf{b} - A\boldsymbol{a}\| = 1.7083$. The least squares error is the sum of the volumes of the cubes shown.

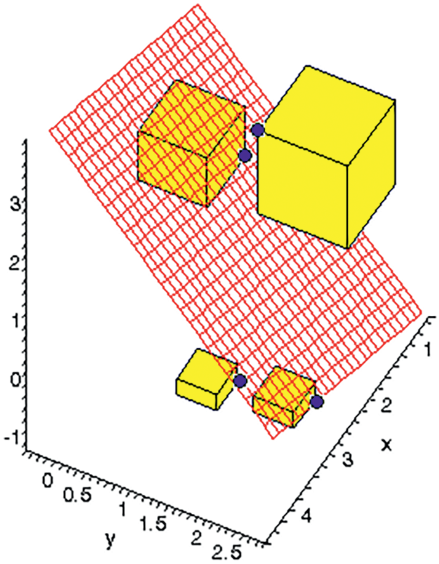

Figure 8.27: Least squares plane for space points. The least squares error is the sum of the volumes of the cubes.

8.7.2 Continuous least squares (requires calculus)

We already know how to compute least squares fitting for a finite sets of points (the discrete case). We address now the question of finding a fitting curve for an infinite continuous set of data points.

Suppose, we want the straight line $y = b + mx$ that best approximates the function $f(x) = x^2$ on the interval $[0, 1]$.

Unless we pick finitely many points, we can no longer use the dot product. We can however use the inner product for continuous functions of Example 8.5.13, Section 8.5.

We need to find b and m that “minimize”the error vector

$$\Delta = x^2 - (b + mx) \tag{8.39}$$

by using the integral inner product. In other words, we must minimize

$$\|\Delta\|^2 = \int_0^1 (x^2 - (b + mx))^2 dx.$$

Equation (8.39) can be written in vector notation as

$$\Delta = x^2 - A\mathbf{x}, \tag{8.40}$$

where

$$A = \begin{bmatrix} 1 & x \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} b \\ m \end{bmatrix}.$$

We know from Section 8.4 that if $\tilde{\mathbf{x}} = \left[\begin{smallmatrix} \tilde{b} \\ \tilde{m} \end{smallmatrix}\right]$ is a least squares solution for the system $A\mathbf{x} = x^2$, then $A\tilde{\mathbf{x}}$ must be the unique projection of x^2 onto $\mathrm{Col}(A)$, and we have a formula for the projection. This is valid, provided that we use an orthogonal basis of $\mathrm{Col}(A)$. It is easy to see that $\{1, x\}$ yields the orthogonal basis $\{1, x-1/2\}$ by the Gram–Schmidt process. So we have

$$\begin{aligned} A\tilde{\mathbf{x}} = x^2_{\mathrm{pr}} &= \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle x^2, x - 1/2\rangle}{\langle x - 1/2, x - 1/2\rangle}(x - 1/2) \\ &= -1/6 + x. \end{aligned}$$

Now

$$\int_0^1 x^2 \cdot 1 dx = 1/3, \quad \int_0^1 x^2 \cdot (x - 1/2) dx = 1/12,$$

$$\int_0^1 1 \cdot 1 dx = 1, \quad \int_0^1 (x - 1/2) \cdot (x - 1/2) dx = 1/12.$$

Therefore

$$A\tilde{\mathbf{x}} = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} \tilde{b} \\ \tilde{m} \end{bmatrix} = -1/6 + x.$$

Hence $\widetilde{b} = -1/6$ and $\widetilde{m} = 1$. So the least squares line that best approximates x^2 on $[0, 1]$ is (Figure 8.28)

$$y = -1/6 + x.$$

Note that the answer depends on the interval we choose.

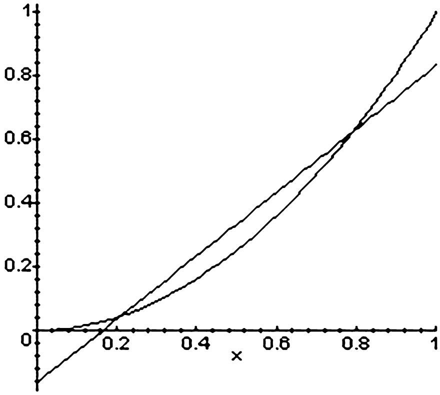

Figure 8.28: The least squares line for x^2 on $[0, 1]$.

Actually, we do not need to orthogonalize. We can use the corresponding "normal equations"

$$A^T A \widetilde{\mathbf{x}} = A^T x^2.$$

The "matrix multiplication" this time uses the current inner product, not the dot product. So by $A^T A$ we mean

$$A^T A = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 & x \end{bmatrix} = \begin{bmatrix} \langle 1, 1 \rangle & \langle 1, x \rangle \\ \langle x, 1 \rangle & \langle x, x \rangle \end{bmatrix},$$

and $A^T x^2$ is

$$A^T x^2 = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} x^2 \end{bmatrix} = \begin{bmatrix} \langle 1, x^2 \rangle \\ \langle x, x^2 \rangle \end{bmatrix}.$$

We compute the corresponding integrals of the entries to get the system

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \widetilde{b} \\ \widetilde{m} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix}$$

with the same solution $\widetilde{b} = -1/6$, $\widetilde{m} = 1$, as before. The justification of these steps is left to the reader.

Exercises 8.7

In Exercises 1–3, find the least squares quadratic $q(x)$ passing through the given points. Then evaluate $q(6)$.

1. $(1,-3), (2,0), (3,4), (4,13), (5,20)$.

2. $(1,-1), (2,0), (3,1), (4,4), (5,8)$.

3. $(-2,1), (-1,2), (0,4), (1,6), (2,9)$.

In Exercises 4–5, find the least squares cubic $q(x)$ passing through the points. Then evaluate $q(3)$.

4. $(-2,-8), (-1,-2), (0,0), (1,8), (2,12)$.

5. $(-2,-2), (-1,0), (0,1), (1,-3), (2,7)$.

6. Prove Theorem 8.7.1.

7. Prove Theorem 8.7.2.

In Exercises 8–11, find the least squares line that approximates $f(x)$.

8. $f(x) = x^2$ on $[0,2]$.

9. $f(x) = x^2$ on $[1,2]$.

10. $f(x) = x^3$ on $[0,1]$.

11. $f(x) = x^3$ on $[0,2]$.

12. Find the least squares plane and error for the points $(2,-1.1,0), (-1.1,3,-1), (5,6.1,-10), (3.1,3,-5)$.

13. Use the normal equation approach with $A = \begin{bmatrix} 1 & x & x^2 \end{bmatrix}$ to find the least squares quadratic that approximates $f(x) = x^3$ on $[0,2]$.

14. The displacement $x(t)$ of a spring-mass system of mass m moving without friction is

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

with constants A and B and $\omega = \sqrt{k/m}$, where k is the spring constant (Figure 8.29).

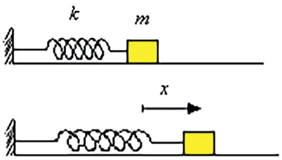

Figure 8.29: Spring-mass system.

In a certain experiment, we have $\omega = 4$ and the following measurements.

t (s)	1	2	3
x (inch)	−1.96	−0.44	2.53

(a) Use least squares fitting to estimate A and B.
(b) Estimate the mass location at time $t = 5$.

15. (*Falling body*) The displacement $x = x(t)$ of a free falling body is given by a quadratic relation in time t as follows:

$$x = \frac{1}{2}gt^2 + v_0 t + x_0,$$

where v_0 is the initial velocity, x_0 is the initial displacement, and g is the gravity acceleration. Suppose the following measurements are taken for the same initial velocity and initial displacement.

t (s)	1	2	3	4
x (m)	6.8	18.7	40.	71.5

Estimate (round your answer to one decimal place)
(a) the gravity acceleration g,
(b) the initial velocity v_0,
(c) the initial displacement x_0.

8.8 Special topic: The NFL rating of quarterbacks

In this section, we discuss a few of the many applications of the dot products. Not only the applications of orthogonality, least squares, etc., are very interesting, but they can also be fun. Let us begin with how NFL rates quarterbacks.

We present the essence of an interesting paper by Roger W. Johnson, published in *The College Mathematics Journal*, Vol., No. 5, November 1993. In this paper, Johnson finds the formula for the rating of a quarterback, given a table of All-Time Leading Passers through the 1989–1990 season.[6] Although we follow Johnson's exposition almost to the letter, we use slightly more recent data, partly to see if the old formula is still valid.

Tables 8.1 and 8.2 were taken from *The Sports illustrated, 1995 Sports Almanac*, a paper by Peter King. Table 8.1 shows the 1993 NFL Individual Leading Passers for the American Football Conference.

Table 8.2 displays the same information for the National Football Conference.

Given the attempts (Att), completions (Comp), yards, touch downs (TD), interceptions (Int), and ratings, we want to find a formula for the rating. As Johnson points out, it is known that this rating depends on the percentages of the completions, touchdowns, and interceptions and also on the average gain per pass attempt (computed in Tables 8.3 and 8.4). However, there seems to be no published formula of the ratings. Let us assume

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[6] The author is grateful to Professor W. David Joyner for bringing Johnson's paper to his attention.

Table 8.1: 1993 American Football Conference.

Player	Att	Comp	Yards	TD	Int	Rating
Elway	551	348	4030	25	10	92.8
Montana	298	181	2144	13	7	87.4
Testaverde	230	130	1797	14	9	85.7
Esiason	473	288	3421	16	11	84.5
Mitchell	233	133	1773	12	8	84.2
Hostetler	419	236	3242	14	10	82.5
Kelly	470	288	3382	18	18	79.9
O'Donnell	486	270	3208	14	7	79.5
George	407	234	2526	8	6	76.3
DeBerg	227	136	1707	7	10	75.3

Table 8.2: 1993 National Football Conference.

Player	Att	Comp	Yards	TD	Int	Rating
Young	462	314	4023	29	16	101.5
Aikman	392	271	3100	15	6	99.0
Simms	400	247	3038	15	9	88.3
Brister	309	181	1905	14	5	84.9
Hebert	430	263	2978	24	17	84.0
Buerlein	418	258	3164	18	17	82.5
McMahon	331	200	1967	9	8	76.2
Favre	522	318	3303	19	24	72.2
Harbaugh	325	200	2002	7	11	72.1
Wilson	388	221	2457	12	15	70.1

that the rating depends *linearly* on these four quantities and a constant, say,

$$\text{Rating} = x_1 + x_2\,(\%\ \text{Comp}) + x_3\,(\%\ \text{TD}) + x_4\,(\%\ \text{Int}) + x_5\,(\text{Yds/Att}). \tag{8.41}$$

We want to compute the unknown coefficients $\mathbf{x} = (x_1, \dots, x_5)$ using Tables 8.3 and 8.4. This yields a system of 20 equations (10 per table) and 5 unknowns. Let A be the coefficient matrix of this system. Then

$$A\mathbf{x} = \mathbf{b}, \tag{8.42}$$

where $\mathbf{b}$ is the vector of all the ratings. Note that A has a first column of 1s; the remaining of its columns are the percentages of the tables.

It seems reasonable to use the data from only 5 players to get a square system in $\mathbf{x}$. However, if we compare the solutions using the first five players of the American conference and the last five players of the National conference, then the two answers differ by $(.82, -.04, 0, -.2, .28)$ (rounded to 2 decimal places). Clearly, system (8.42) is inconsistent.

Table 8.3: American Football Conference.

Player	% Comp	% TD's	% Int's	Yds/Att
Elway	63.1579	4.5372	1.8149	7.3140
Montana	60.7383	4.3624	2.3490	7.1946
Testaverde	56.5217	6.0870	3.9130	7.8130
Esiason	60.8879	3.3827	2.3256	7.2326
Mitchell	57.0815	5.1502	3.4335	7.6094
Hostetler	56.3246	3.3413	2.3866	7.7375
Kelly	61.2766	3.8298	3.8298	7.1957
O'Donnell	55.5556	2.8807	1.4403	6.6008
George	57.4939	1.9656	1.4742	6.2064
DeBerg	59.9119	3.0837	4.4053	7.5198

Table 8.4: National Football Conference.

Player	% Comp	% TD's	% Int's	Yds/Att
Young	67.9654	6.2771	3.4632	8.7078
Aikman	69.1327	3.8265	1.5306	7.9082
Simms	61.7500	3.7500	2.2500	7.5950
Brister	58.5761	4.5307	1.6181	6.1650
Hebert	61.1628	5.5814	3.9535	6.9256
Buerlein	61.7225	4.3062	4.0670	7.5694
McMahon	60.4230	2.7190	2.4169	5.9426
Favre	60.9195	3.6398	4.5977	6.3276
Harbaugh	61.5385	2.1538	3.3846	6.1600
Wilson	56.9588	3.0928	3.8660	6.3325

So we have to find an optimal solution by using least squares. The normal equations are

$$A^T A\mathbf{x} = A^T\mathbf{b}.$$

Using MATLAB (bank notation format), we get the system

$$\begin{bmatrix} 20.00 & 1209.10 & 78.50 & 58.52 & 142.06 \\ 1209.10 & 73333.88 & 4766.20 & 3536.22 & 8609.83 \\ 78.50 & 4766.20 & 335.31 & 236.62 & 568.22 \\ 58.52 & 3536.22 & 236.62 & 192.91 & 417.78 \\ 142.06 & 8609.83 & 568.22 & 417.78 & 1019.51 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1658.90 \\ 100652.76 \\ 6634.23 \\ 4794.16 \\ 11871.57 \end{bmatrix}$$

with solution

$$\mathbf{x} = (2.0589, 0.8321, 3.3178, -4.1666, 4.1884).$$

Therefore the formula for the ratings is

$$\begin{aligned}\text{Rating} = 2.0589 + 0.8321\,(\%\ \text{Comp}) + 3.3178\,(\%\ \text{TD})\\ - 4.1666\,(\%\ \text{Int}) + 4.1884\,(\text{Yds}/\text{Att}).\end{aligned}$$

In fact, if we compute the product $A\mathbf{x}$, then we get all the ratings to the displayed accuracy.

These coefficients with the 4 decimal places are not very handy. There is a rational approximation found in Johnson's paper:

$$\text{Rating} = \frac{1}{24}(50 + 20\,(\%\ \text{Comp}) + 80\,(\%\ \text{TD}) - 100\,(\%\ \text{Int}) + 100\,(\text{Yds}/\text{Att})),$$

which yields the same accuracy. It seems reasonable to assume that this is the correct formula for the ratings.

! We used data from both conferences to get a more accurate least squares approximation.

8.9 Miniprojects

8.9.1 The Pauli spin and Dirac matrices

In this project, we explore the basic properties of certain matrices with complex entries that play an important role in nuclear physics and quantum mechanics. In Exercises 2.1, we introduced the *Pauli spin matrices*

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which were used by Pauli in the computation of the electron spin.

In 1927, P. A. M. Dirac,[7] working in quantum mechanics, generalized Pauli's spin matrices to the following, known as *Dirac matrices*:

$$\alpha_x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_y = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},$$

---

[7] **Paul Adrien Maurice Dirac**, born in Bristol, England, in 1902. In 1926, he obtained a doctorate from Cambridge University. He then studied under Bohr in Copenhagen and under Born in Göttingen. In 1932, he became the Lucasian Professor of mathematics at Cambridge (the chair once held by Newton). He won the Nobel Prize in Physics in 1933. He is known for his work on quantum mechanics, elementary particles, and the theory of antimatter.

$$\alpha_z = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The first three Dirac matrices are block matrices in the Pauli spin matrices:

$$\alpha_x = \begin{bmatrix} \mathbf{0} & \sigma_x \\ \sigma_x & \mathbf{0} \end{bmatrix} \quad \alpha_y = \begin{bmatrix} \mathbf{0} & \sigma_y \\ \sigma_y & \mathbf{0} \end{bmatrix} \quad \alpha_z = \begin{bmatrix} \mathbf{0} & \sigma_z \\ \sigma_z & \mathbf{0} \end{bmatrix}.$$

A square matrix A is *involutory* if $A^{-1} = A$ or, equivalently, if $A^2 = I$. For example, $-I$ is involutory.

Problem A. Prove that the Pauli spin matrices and the Dirac matrices are

1. Hermitian,
2. unitary,
3. involutory.

Problem B. Prove that the Pauli spin matrices satisfy the relations

$$\sigma_x \sigma_y = i\,\sigma_z, \quad \sigma_y \sigma_z = i\,\sigma_x, \quad \sigma_z \sigma_x = i\,\sigma_y$$

and that they *anticommute*, i. e.,

$$\sigma_x \sigma_y = -\sigma_y \sigma_x, \quad \sigma_y \sigma_z = -\sigma_z \sigma_y, \quad \sigma_x \sigma_z = -\sigma_z \sigma_x.$$

Problem C. Prove that the Dirac matrices satisfy the relations

$$\alpha_x \beta = -\beta\,\alpha_x \quad \alpha_y\beta = -\beta\,\alpha_y \quad \alpha_z \beta = -\beta\,\alpha_z$$

and that they *anticommute*, i. e.,

$$\alpha_x \alpha_y = -\alpha_y \alpha_x, \quad \alpha_y \alpha_z = -\alpha_z \alpha_y, \quad \alpha_x \alpha_z = -\alpha_z \alpha_x.$$

8.9.2 Rigid motions in $\mathbf{R}^n$

A transformation $f : \mathbf{R}^n \to \mathbf{R}^n$ is a *rigid motion* if

$$\|f(\mathbf{u}) - f(\mathbf{v})\| = \|\mathbf{u} - \mathbf{v}\|$$

for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$. A rigid motion is not necessarily linear, but it preserves distances in $\mathbf{R}^n$.

Problem A.

1. Prove that a translation $f : \mathbf{R}^n \to \mathbf{R}^n$ is a rigid motion. Recall that a translation by a fixed vector $\mathbf{b}$ is a transformation of the form
$$f(\mathbf{v}) = \mathbf{v} + \mathbf{b}, \quad \mathbf{v} \in \mathbf{R}^n.$$
2. Prove that an orthogonal transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ is a rigid motion. Recall that an orthogonal transformation is transformation of the form
$$T(\mathbf{v}) = A\mathbf{v}, \quad \mathbf{v} \in \mathbf{R}^n,$$
for an orthogonal matrix A.

Problem B.

1. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a rigid motion. Let T be the transformation $T : \mathbf{R}^n \to \mathbf{R}^n$ defined by
$$T(\mathbf{v}) = f(\mathbf{v}) - f(\mathbf{0}), \quad \mathbf{v} \in \mathbf{R}^n.$$
Prove that T is an orthogonal transformation as follows:
   (a) Prove that $\|T(\mathbf{v})\|^2 = \|\mathbf{v}\|^2$ and conclude that $\|T(\mathbf{v})\| = \|\mathbf{v}\|$.
   (b) Use Part (a) to prove that
$$\left\|T(\mathbf{u}) - T(\mathbf{v})\right\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\left\langle T(\mathbf{u}), T(\mathbf{v})\right\rangle.$$
Compare this relation with
$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle$$
to conclude that $\langle T(\mathbf{u}), T(\mathbf{v})\rangle = \langle \mathbf{u}, \mathbf{v}\rangle$.
   (c) Use Part (b) to prove by expansion the following relation:
$$\left\|T(c_1\mathbf{u} + c_2\mathbf{v}) - c_1T(\mathbf{u}) - c_2T(\mathbf{v})\right\|^2 = 0.$$
Conclude that T is *linear*.
   (d) Now use Part 3 of Theorem 8.1.25, Section 8.1, to prove that T is an orthogonal transformation.
2. Use Part 1 to prove the following important theorem.

Theorem 8.9.1. *Every rigid motion $f : \mathbf{R}^n \to \mathbf{R}^n$ is a composition of a translation T followed by an orthogonal transformation Q. So*
$$f = Q \circ T.$$
Furthermore, this decomposition is unique. So if $f = Q' \circ T'$ with orthogonal transformation Q' and translation T', then $Q = Q'$ and $T = T'$.

8.9.3 Volume of the parallelepiped and the Gram determinant

In this project, you are to prove a formula for the volume V of the parallelepiped with adjacent sides the 3-vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in V in terms of the dot product (Figure 8.30).

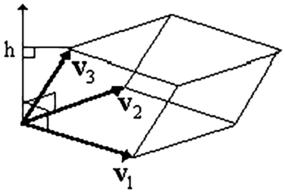

Figure 8.30: A parallelepiped with adjacent sides the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Recall from Exercise 24, Section 8.2, that the *Gram determinant* of a set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ of n-vectors ($k \le n$) is the determinant $\det(A)$ of the matrix A with (i,j) entry $\mathbf{v}_i \cdot \mathbf{v}_j$.

Problem. Prove the following relation for the volume V of the parallelepiped with adjacent sides the 3-vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$:

$$V^2 = \begin{vmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \mathbf{v}_1 \cdot \mathbf{v}_3 \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \mathbf{v}_2 \cdot \mathbf{v}_3 \\ \mathbf{v}_3 \cdot \mathbf{v}_1 & \mathbf{v}_3 \cdot \mathbf{v}_2 & \mathbf{v}_3 \cdot \mathbf{v}_3 \end{vmatrix}.$$

8.10 Technology-aided problems and answers

In Exercises 1–3, let

$$A = \begin{bmatrix} 1 & 4 & 69 \\ 2 & -3 & 28 \\ -3 & 2 & -37 \\ 4 & 2 & -59 \end{bmatrix}.$$

1. Prove that A has orthogonal columns by (a) using the dot product; (b) using Theorem 8.1.10, Section 8.1.

2. Let A_1 be the matrix obtained by adding $\mathbf{e}_4 \in \mathbf{R}^4$ as a last column to A. Apply the Gram–Schmidt process to the columns of A_1 to orthonormalize them. Form a matrix A_2 with these columns and verify relation (8.4).

3. Verify Bessel's inequality (8.1.18) with the set of orthonormal vectors consisting of the first 3 columns of A_2 of Exercise 2 and with $\mathbf{u} = (1, 2, 3, 4)$ in $\mathbf{R}^4$.

4. Find a matrix whose columns are orthonormal and span the column space of

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

5. Write a short program that computes the orthogonal projection of a vector **u** onto the span of a set S of orthogonal vectors.

6. Test your program of Exercise 5 with $\mathbf{u} = (1, -1, 2)$ and $S = \{(-1, 4, 1), (5, 1, 1)\}$.

7. Modify your program of Exercise 5 so that it computes orthogonal projection of a vector onto the span of any finite set of linearly independent but not necessarily orthogonal vectors.

8. Test your program of Exercise 7 with $\mathbf{u} = (1, 1, 1)$ and $S = \{(1, -1, 2), (-1, 4, 1)\}$.

9. Compute the QR factorization of

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

and verify your answer.

10. Find the least squares line through the ten points of x^2 with x-coordinates $0, 1, \ldots, 9$. On the same graph, plot this line and x^2 over $[0, 9]$.

11. Find and plot the least squares quadratic through the points $(-2, 2), (-1, 0), (2, 4), (3, 7), (4, 9)$.

12. Consider the inner product of Example 8.5.13, Section 8.5, on $[0, 1]$. Use your program to compute the values $\langle x^2, \sin(x)\rangle$, $\|x^2\|$, $\|\sin(x)\|$. Verify the CBSI inequality.

13. Verify the CBSI inequality using the inner product of Example 8.5.8, Section 8.5, with $r_0 = -10$, $r_1 = 3$, $r_2 = 15$ for $\mathbf{u} = x^2 + x - 1$ and $\mathbf{v} = -x^2 + 2x$.

14. Write and test a function with arguments **u**, **v**, and **w** that computes the weighted dot product $\langle \mathbf{u}, \mathbf{v}\rangle$ with weight vector **w** (Example 8.5.5, Sec. 8.5).

15. Use Exercise 14 to find the weighted dot product $\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{w}}$ of $\mathbf{u} = (-7, 3, 1)$ and $\mathbf{v} = (1, -9, 6)$ with weight vector $\mathbf{w} = (1, 2, 3)$. Also, compute $\langle \mathbf{u}, \mathbf{u}\rangle_{\mathbf{w}}$ to check if the square norm is positive.

16. Write a program that finds the Householder matrix of any unit vector **u** of $\mathbf{R}^n$. Use your program to find the reflection of $\mathbf{u} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ about the line through the origin that is perpendicular to **u**.

8.10.1 Selected solutions with Mathematica

```
A = {{1,4,69},{2,-3,28},{-3,2,-37},{4,2,-59}}
(* Exercise 1. *)
AT=Transpose[A];              (* We can access the columns easier by transposition.*)
Dot[AT[[1]],AT[[2]]]                      (* Etc.. Repeat with the other two pairs.*)
AT . A        (* (A^T)A is diagonal with diag. entries norms^2 of the columns of A.*)
Norm[AT[[1]]]^2                  (* Checking the squares of norms of vectors. *)
(* Exercise 2. *)
e4 = {{0},{0},{0},{1}}                     (* e_4  *)
A1 = Join[A, e4, 2]
A2 = Orthogonalize[Transpose[A1]]        (* Gram-Schmidt on the columns of A.*)
Transpose[A2] . A2  (* (1.4) is true, the command yields an orthonormal set. *)
(* Exercise 3. *)
```

```
u = {1,2,3,4}
Norm[u]^2-((u.A2[[1]])^2+(u.A2[[2]])^2 + (u.A2[[3]])^2) // N
(* The rows of A2 were the orthonormalized set. We checked the first
three rows against u and after numerical evaluation of the difference
we got a positive number. *)
(* Exercise 4. *)
B = {{1,2,3},{4,5,6},{7,8,9}}
RowReduce[Transpose[B]] (* Reduce B^T. So B has dependent columns *)
(* We orthogonalize the first two columns that are lin. independent
and span col(B). *)
Orthogonalize[{Transpose[B][[1]],Transpose[B][[2]]}]
B1 = Transpose[%]  (* The transpose has the required columns. *)
(* Exercise 5. *)
proj[u_,lis_] :=
Sum[Dot[u,lis[[i]]]/Dot[lis[[i]],lis[[i]]]*lis[[i]],{i,1,Length[lis]}]
(* Exercise 6. *)
pr=proj[{1,-1,2},{{-1,4,1},{5,1,1}}]   (* The projection vector. *)
orthu = {1,-1,2} - pr  (* Checking that the complement is orthogonal *)
orthu . {-1,4,1}        (* to the subspace spanned by S. *)
orthu . {5,1,1}
(* Exercise 7. *)
ProjAny[u_,lis_] :=
Module[{lis1=Orthogonalize[lis],i,n=Length[lis]},
Sum[Dot[u,lis1[[i]]]/Dot[lis1[[i]],lis1[[i]]]*lis1[[i]],{i,1,n}]
//Simplify]
(* Modified the program: we orthonormalized the vectors of S.  *)
(* Exercise 8. *)
ProjAny[{1,1,1},{{1,-1,2},{-1,4,1}}]
(* Can verify the answer as before to check. *)
(* Exercise 9. *)
CC = {{1,2,3,4},{2,2,3,4},{3,3,3,4},{4,4,4,4}}(* C is already used . *)
QRDecomposition[N[CC]]  (* Need to numerically evaluate CC first.    *)
Q = %[[1]]; R= %[[2]];  (* Warning: Q is such that  (Q^T)R is CC.    *)
Transpose[Q] . R - CC   (* The difference is approx. the zero matrix,*)
Transpose[Q] . Q        (* and Q is orthogonal, since (Q^T)Q=I.      *)
(* Exercise 10. *)
(* The least squares line can be computed in one step. *)
Fit[{{0,0},{1,1},{2,4},{3,9},{4,16},{5,25},
{6,36},{7,49},{8,64},{9,81}},{1,s},s]
(* or by using the discussed procedure. *)
A = {{1,0},{1,1},{1,2},{1,3},{1,4},{1,5},{1,6},{1,7},{1,8},{1,9}}
AT = Transpose[A]
b = {{0},{1},{4},{9},{16},{25},{36},{49},{64},{81}}
LinearSolve[AT.A,AT.b]    (* Solve the normal equations to get 9x-12.*)
LeastSquares[A, b]   (* or better in one step. We get line 9x-12 *)
Plot[{x^2,9x-12},{x,0,9}] (* The least squares line and x^2 plotted. *)
(* Exercise 11. *)
(* Least squares quadratic in one step. *)
quad =Fit[{{-2, 2}, {-1, 0}, {2, 4}, {3, 7}, {4, 9}}, {1, s, s^2}, s]
```

```
(* or by using the discussed procedure. *)
A = {{1,-2,4}, {1,-1,1}, {1,2,4}, {1,3,9}, {1,4,16}}
b = {{2},{0},{4},{7},{9}}
LinearSolve[Transpose[A].A,Transpose[A].b]
(* Or by using a command *)
LeastSquares[A, b]
N[%]   (* Numerically it matches the answer from the "Fit" command. *)
Plot[quad, {s, -2.5, 4.5}]   (* Plotting the least squares quadratic*)
(* Exercise 12. *)
a=Integrate[x^2 Sin[x],{x,0,1}]       (* <x^2, sin(x)>.                *)
b=Sqrt[Integrate[x^2,{x,0,1}]]        (* Norm(x^2).                    *)
c=Sqrt[Integrate[Sin[x],{x,0,1}]]     (* Norm(sin(x)).                 *)
N[b*c- a]           (* Norm(x^2)*Norm(sin(x))-<x^2, sin(x)> is >0. OK. *)
(* Exercise 13. *)
(* Define the inner product function with input p, q, and r.*)
PolyInnerProd[p_, q_, r_] :=
Sum[(p /. x -> r[[i]])*(q /. x -> r[[i]]), {i, 1, Length[r]}]
p = x^2 + x - 1; q = -x^2 + 2 x; r = {-10, 3, 15};        (* Data. *)
pq = PolyInnerProd[p, q, r]            (* The inner product <p,q>. *)
np = Sqrt[PolyInnerProd[p, p, r]]                (* The norm of p. *)
nq = Sqrt[PolyInnerProd[q, q, r]]                (* The norm of p. *)
N[np * nq - pq]                    (* Positive difference, CBSI OK. *)
(* Exercises 14 and 15. *)
WeightedDotProd[u_,v_,w_] := Sum[u[[i]]*v[[i]]*w[[i]], {i,1,Length[u]}]
WeightedDotProd[{-7, 3, 1}, {1, -9, 6}, {1, 2, 3}]
WeightedDotProd[{-7, 3, 1}, {-7, 3, 1}, {1, 2, 3}]
(* Exercise 16. *)
householder[u_]:=IdentityMatrix[Length[u]]-2 KroneckerProduct[u, u]
u = {1/Sqrt[2], -1/Sqrt[2]}
householder[u]                        (* The Householder matrix. *)
householder[u] . u     (* This is reflection about the line y=x. *)
```

8.10.2 Selected solutions with MATLAB

```
% Exercise 1.
A = [1 4 69; 2 -3 28; -3 2 -37; 4 2 -59]
% Exercise 1
dot(A(:,1), A(:,2))              % Etc.. Repeat with the other two pairs.
A.'*A              % (A^T)A is diagonal with diag. entries norms^2 of
norm(A(:,1))^2,norm(A(:,2))^2,norm(A(:,3))^2       % the columns of A.
% Exercise 2.
e4 = [0;0;0;1]                 % e4.
A1 = [A e4]                    % A1.
A2=orth(A1)                    % Gram-Schmidt on the columns of A.
A2.'*A2            % The matrix is already orthogonal! GS normalizes too.
% Exercise 3.
u=[1;2;3;4]
```

```
norm(u)^2-(dot(u,A2(:,1))^2+dot(u,A2(:,2))^2+dot(u,A2(:,3))^2)
% The difference is positive, inequality verified.
% Exercise 4.
B = [1 2 3; 4 5 6; 7 8 9]
B1 = orth(B)       % GS on the columns of B to form a new matrix B1
[B1 B]             % whose columns span Col(B) since, the first 2
rref(ans)          % columns of [B1,B] are pivot columns.
% Exercise 5.
function [A] = proj(u,lis)          % In an m-file called proj.m type
[m,n]=size(lis);            % the code on the left for
s = zeros(1,n);             % the orthogonal projection.
for i=1:m
s =  s + dot(u,lis(i,:))/dot(lis(i,:),lis(i,:))*lis(i,:);
end
A = s;
end
% Exercise 6.
pr=proj([1 -1 2],[-1 4 1; 5 1 1])   % The projection vector.
orthu = [1 -1 2]-pr  % Checking that the complement is orthogonal
dot(orthu , [-1 4 1]) % to the subspace spanned by S.
dot(orthu, [5 1 1])
% Exercise 7.
function [A] = proj_any(u,lis) % In an m-file called proj_any.m
[m,n]=size(lis)            % type the code on the left for
s = zeros(1,n);             % the orthogonal projection.
lis1 = orth(lis’)’
for i=1:m
s =  s + dot(u,lis1(i,:))/dot(lis1(i,:),lis1(i,:))*lis1(i,:);
end
A = s;
end
% Exercise 8.
proj_any([1,1,1],[1 -1 2; -1 4 1])
% Exercise 9.
C = [1 2 3 4; 2 2 3 4; 3 3 3 4; 4 4 4 4]
[Q,R] = qr(C)            % QR factorization.
C - Q*R                  % The difference is approx. the zero matrix,
Q.’ * Q                  % and Q is orthogonal, since (Q^T)Q=I.
% Exercise 10.
A = [1 0; 1 1; 1 2; 1 3; 1 4; 1 5; 1 6; 1 7; 1 8; 1 9]
b = [0;1;4;9;16;25;36;49;64;81] % No need for normal equations lscov does it
lscov(A,b,diag(ones(10,1)))     % in one step. Look it up. See also nnls.
x = 0:.1:9;                  % Plotting the least squares line
y1 = 9*x-12;                 % 9x-12 and x^2 in one graph
y2 = x.^2;                   % on [0,9].
plot(x,y1,x,y2);
% Exercise 11.
A = [1,-2,4; 1,-1,1; 1,2,4; 1,3,9; 1,4,16]
b = [2;0;4;7;9]
```

```
ls=lscov(A,b,diag(ones(5,1)))
x = 0:.1:5;
y = ls(1) + ls(2)*x + ls(3)*x.^2;
plot(x,y)
% Exercise 12.
xsin = @(x) x.^2.*sin(x);
a=integral(xsin,0,1)
xx = @(x) x.^2;
b=sqrt(integral(xx,0,1))
ssin= @(x) sin(x);
c=sqrt(integral(ssin,0,1))
b*c-a
% Exercise 13.
function [A] = PolyInnerProd(p,q,r)  % In an m-file type
p1=polyval(p,r);
q1=polyval(q,r);
A=dot(p1,q1)
end                              % end of file
p=[1 1 -1];
q=[-1 2 0];
r=[-10 3 15];
pq=PolyInnerProd(p,q,r)
np = sqrt(PolyInnerProd(p, p, r))
nq = sqrt(PolyInnerProd(q, q, r))
np*nq - pq
% Exercises 14 and 15.
function [A] = WeightedDotProd(u,v,w)  % In an m-file type
A=sum((u.*v).*w)
end                                % end of file
WeightedDotProd([-7, 3, 1], [1, -9, 6], [1, 2, 3])
WeightedDotProd([-7, 3, 1], [-7, 3, 1], [1, 2, 3])
% Exercise 16.
function [A] = householder(u) % In an m-file type
A=eye(length(u)) - 2*u*u'
end                          % end of file
u = [1/sqrt(2);-1/sqrt(2)]
householder(u)          % The Householder matrix.
householder(u)*u  % This is reflection about the line y=x.
=
```

8.10.3 Selected solutions with Maple

```
with(LinearAlgebra);
A := Matrix([[1,4,69],[2,-3,28],[-3,2,-37],[4,2,-59]]);
# Exercise 1.
DotProduct(Column(A,1),Column(A,2));    # Etc.
Transpose(A).A;  #(A^T)A is diagonal with diag. entries norms^2
```

```
# the columns of A.
VectorNorm(Column(A, 1),2)^2; VectorNorm(Column(A, 2),2)^2; # Etc.
# Exercise 2.
A1 := <A|Vector([0,0,0,1])>;   # Add e4 and Gram-Schmidt in the
L1:=GramSchmidt([seq(Column(A1,i),i=1..4)]);  # columns of A1.
# They are not unit. Need to normalize to unit vectors.
L2:= [seq(Normalize(L1[i],Euclidean),i=1..4)];
A2:=<L2[1]|L2[2]|L2[3]|L2[4]>;
Transpose(A2) . A2;
# Exercise 3.
u:=Vector([1,2,3,4]);
Norm(u, 2)^2 - DotProduct(u, Column(A2, 1))^2
- DotProduct(u, Column(A2, 2))^2
- DotProduct(u, Column(A2, 3))^2; # Got positive, inequality verified.
# Exercise 4.
B := Matrix([[1,2,3],[4,5,6],[7,8,9]]);
ReducedRowEchelonForm(Transpose(B));   # Columns 1 and 2 span col(B).
# We orthogonalize the first two columns that are lin.
# independent and span col(B).
L1:=GramSchmidt([Column(B, 1), Column(B, 2)]);
# They are not unit. Need to normalize to unit vectors.
L2:= [Normalize(L1[1],Euclidean),Normalize(L1[2],Euclidean)];
B1:=<L2[1]|L2[2]>;  # The required matrix.
# Exercise 5.
proj := proc(u,lis) local i, s;     # proj for projection.
s := [seq(0,i=1..nops(u))];  # 0 list converted to vector in the loop.
for i from 1 to nops(lis) do
s := s + DotProduct(u,lis[i])/
DotProduct(lis[i],lis[i])*lis[i] od:
s; end:
# Exercise 6.
pr:=proj([1,-1,2],[[-1,4,1],[5,1,1]]); # The projection vector.
orthu := [1, -1, 2] - pr;  # Checking that the complement is
DotProduct(orthu, [-1,4,1]); # orthogonal to the subspace
DotProduct(orthu, [5,1,1]);  # spanned by S.
# Exercise 7.
ProjAny := proc(u,lis) local i, s, lis1;   # Proj_Any for projection.
s := [seq(0,i=1..nops(u))];  # 0 list converted to vector in the loop.
lis1 := [seq(convert(lis[i], Vector), i = 1 .. nops(lis))];
lis1 := GramSchmidt(lis1);
lis1 := [seq(convert(lis1[i], list), i = 1 .. nops(lis1))];
for i from 1 to nops(lis1) do
s := s + DotProduct(u,lis1[i])/
DotProduct(lis1[i],lis1[i])*lis1[i] od:
s; end:
# Exercise 8.
ProjAny([1, 1, 1], [[1, -1, 2], [-1, 4, 1]]);
# Can verify the answer as before to check.
# Exercise 9.
```

```
C := Matrix([[1,2,3,4],[2,2,3,4],[3,3,3,4],[4,4,4,4]]);
Q, R := QRDecomposition(C, fullspan);  # QR factorization.
C - Q.R;         # The difference is the zero matrix,
Transpose(Q).Q; # and Q is orthogonal, since (Q^T)Q=I.
# Exercise 10.
A := Matrix(10,2,[1,0,1,1,1,2,1,3,1,4,1,5,1,6,1,7,1,8,1,9]);
b := Vector([0,1,4,9,16,25,36,49,64,81]); # No need for normal
LeastSquares(A, b); # We get line 9x-12 then the least
plot({x^2,9*x-12},x=0..9); # squares line and x^2 plotted.
# Exercise 11.
restart;
with(CurveFitting);
LeastSquares([[-2, 2], [-1, 0], [2, 4], [3, 7], [4, 9]],
v, curve = a*v^2 + b*v + c);
quad:=evalf(%);
plot(quad, v=-2.5..4.5);
# Exercise 12.
a :=int(x^2*sin(x),x=0..1);                          # <x^2, sin(x)>.
b :=sqrt(int(x^2,x=0..1));                           # Norm(x^2).
c :=sqrt(int(sin(x),x=0..1));                        # Norm(sin(x)).
evalf(b*c - a);    # Norm(x^2)*Norm(sin(x))-<x^2, sin(x)> is >0. OK.
# Exercise 13.
PolyInnerProd  := proc(p,q,r) local i;
sum(subs(x=r[i],p)*subs(x=r[i],q), i=1..nops(r));
end:
p := x^2 + x - 1; q := -x^2 + 2 x; r := [-10, 3, 15];   # Data
pq := PolyInnerProd(p, q, r); # The inner product < p, q >.
np := sqrt(PolyInnerProd(p, p, r)); # The norm of p.
nq := sqrt(PolyInnerProd(q, q, r)); # The norm of q.
evalf(np*nq - pq);  # Positive difference, CBSI OK.
# Exercises 14 and 15.
WeightedDotProd := proc(u,v,w) local i;
sum(u[i]*v[i]*w[i], i=1..nops(u)); end:
WeightedDotProd([-7, 3, 1], [1, -9, 6], [1, 2, 3]);
WeightedDotProd([-7, 3, 1], [-7, 3, 1], [1, 2, 3]);
# Exercise 16.
with(LinearAlgebra);
householder := proc(u)
IdentityMatrix(nops(u))-2*Vector(u).Transpose(Vector(u)); end:
u := [1/sqrt(2), -1/sqrt(2)]; # The Householder matrix.
householder(u).Vector(u); # This is reflection about the line y=x.
```

9 Quadratic forms, SVD, wavelets

> Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.
>
> Joseph Fourier, French mathematician (Figure 9.1).

Figure 9.1: Jean-Baptiste Joseph Fourier.
Portrait by Julien-Léopold Boilly Public Domain, https://commons.wikimedia.org/w/index.php?curid=114366437.
Jean-Baptiste Joseph Fourier (1768–1830) was born in Auxerre and died in Paris, France. He studied mathematics at École Normale in Paris under Lagrange and Laplace. He taught at the Collège de France and at the École Polytechnique. He was involved in the French Revolution, and he followed Napoleon to Egypt. He did important mathematical work on the theory of heat. He wrote a memoir *On the Propagation of Heat in Solid Bodies*. He was elected to the Académie des Sciences.

Introduction

In this chapter, we examine three topics related to orthogonality. The first one, in Section 9.1, is about *orthogonalization of symmetric matrices*. Every symmetric matrix A has a special factorization as a product $A = QDQ^T$ with orthogonal Q and diagonal D. In general, factorizations that involve orthogonal matrices are very useful in practice. The reason is that orthogonal matrices preserve the lengths of vectors. Therefore they *preserve the lengths of error vectors*.

The second topic, discussed in Section 9.2, is about *quadratic forms and conic sections*. Quadratic forms are expressions of the form $\mathbf{x}^T A\mathbf{x}$, where A is a symmetric $n \times n$ matrix, and $\mathbf{x}$ is an n-variable vector. In simple terms, quadratic forms are homogeneous polynomials in several variables.[1] Conic sections are examples of quadratic forms. Using orthogonalization of symmetric matrices, we may change the coordinates in such a way that the quadratic form is transformed into $\mathbf{y}^T D\mathbf{y}$ where D is diagonal. The effect is that the corresponding homogeneous polynomial has no cross terms, i. e., it is of the form

$$a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2.$$

This is a great and frequently used simplification.

---

[1] A homogeneous polynomial is one whose terms have the same total degree in its variables.

https://doi.org/10.1515/9783111331850-009

Our third topic, discussed in Section 9.3, is a most useful factorization of any matrix, the *singular value decomposition* (SVD). The SVD of an $m \times n$ matrix A is a factorization of the form

$$A = U\Sigma V^T,$$

where U and V are orthogonal, and Σ is an $m \times n$ matrix with a diagonal upper left block of positive entries of decreasing magnitude and the remaining entries 0. The positive entries of Σ are the square roots of the eigenvalues of the symmetric matrix $A^T A$. The SVD yields one of the most reliable estimations of the *rank* of a matrix. It is also used in *image compression* discussed in Section 9.4.

The chapter ends with an application of orthogonality to *Fourier series* in Section 9.5.

Historical note

The singular value decomposition is first found in 1873 in Eugenio Beltrami's paper *Sulle funzioni bilineari*, Giorn. di Mat. 11 (1873) pp. 98–106.[2] It is also found in 1874 in Camille Jordan's paper *Mémoire sur les formes bilineáires*, J. Math. Pures Appl. Ser. 2, 19 (1874), pp. 35–54 (Figure 9.2).[3]

Figure 9.2: **Eugenio Beltrami** and **Camille Jordan**. (Wikimedia, public domain.)
http://www-groups.dcs.st-and.ac.uk/~history/PictDisplay/Jordan.html
http://www.math.uni-hamburg.de/home/grothkopf/fotos/math-ges/.

---

[2] *Eugenio Beltrami* (1835–1900) was born in Cremona, Lombardy, Austrian Empire (now Italy), and died in Rome, Italy. He taught mathematics at the Universities of Bologna, Pisa, Rome, and Pavia. He is famous for his work in non-Euclidean geometry and in the differential geometry on curves and surfaces.

[3] *Marie Ennemond Camille Jordan* (1838–1922) was born in La Croix-Rousse, Lyon, France, and died in Paris, France. He became professor of analysis at École Polytechnique and at the Collège de France. He is noted for his work in topology, finite groups, linear and multilinear algebra, the theory of numbers, differential equations, and mechanics. Note that Camille Jordan is *not* the person after whom Gauss–Jordan elimination is named. Gauss–Jordan bares Wilhelm Jordan's name, who applied this elimination method to surveying.

The first numerical algorithm that computes the SVD was published in 1965. It is due to G. H. Colub and W. Kahan in their paper *Calculating the singular values and pseudoinverse*, SIAM J. Numerical Analysis, Ser. B, 2 (1965) pp. 205–224.

9.1 Orthogonalization of symmetric matrices

In this section, we study some interesting properties of symmetric matrices.

From Section 7.2 we know that an $n \times n$ matrix A is diagonalizable if it is similar to a diagonal matrix. In this case, there are an invertible matrix P and a diagonal matrix D such that

$$P^{-1}AP = D.$$

Recall that the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues.

We are interested in diagonalizable matrices for which P can be *orthogonal*.

Definition 9.1.1. We say that square matrix A is *orthogonally diagonalizable* if it can be diagonalized by an invertible orthogonal matrix Q and a diagonal matrix D. Hence $Q^{-1}AQ = D$, or, equivalently,

$$Q^T AQ = D,$$

because $Q^{-1} = Q^T$. In general, we say that two $n \times n$ matrices A and B are *orthogonally similar* if there is an orthogonal matrix Q such that

$$Q^T AQ = B.$$

So an orthogonally diagonalizable matrix is orthogonally similar to a diagonal matrix.

Theorem 9.1.2. *An orthogonally diagonalizable matrix is symmetric.*

Proof. If A is orthogonally diagonalizable, then $Q^{-1}AQ = D$ for some orthogonal matrix Q. So

$$A = QDQ^{-1} = QDQ^T,$$

which implies that

$$A^T = (QDQ^T)^T = (Q^T)^T D^T Q^T = QDQ^T = A.$$

Therefore A is symmetric. □

It is surprising that the converse of Theorem 9.1.2 is true, that is, *any symmetric matrix is orthogonally diagonalizable*. This is the claim of Theorem 9.1.8, which is not easy to prove.

Theorem 9.1.3. *A real symmetric matrix has only real eigenvalues.*

Proof. Let A be a symmetric matrix, and let $\mathbf{v}$ be an eigenvector with corresponding eigenvalue λ. Then $A\mathbf{v} = \lambda\mathbf{v}$. Also, $\overline{A} = A$ because A has real entries. Hence

$$\begin{aligned}
\overline{A\mathbf{v}} = \overline{\lambda\mathbf{v}} &\Rightarrow \overline{A}\,\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}} \\
&\Rightarrow A\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}} \\
&\Rightarrow (A\overline{\mathbf{v}})^T = (\overline{\lambda}\,\overline{\mathbf{v}})^T \\
&\Rightarrow \overline{\mathbf{v}}^T A^T = \overline{\lambda}\,\overline{\mathbf{v}}^T \\
&\Rightarrow \overline{\mathbf{v}}^T A = \overline{\lambda}\,\overline{\mathbf{v}}^T \\
&\Rightarrow (\overline{\mathbf{v}}^T A)\mathbf{v} = (\overline{\lambda}\,\overline{\mathbf{v}}^T)\mathbf{v} \\
&\Rightarrow \overline{\mathbf{v}}^T (A\mathbf{v}) = \overline{\lambda}(\overline{\mathbf{v}}^T \mathbf{v}) \\
&\Rightarrow \overline{\mathbf{v}}^T (\lambda\mathbf{v}) = \overline{\lambda}(\overline{\mathbf{v}}^T \mathbf{v}) \\
&\Rightarrow (\lambda - \overline{\lambda})(\overline{\mathbf{v}}^T \mathbf{v}) = 0.
\end{aligned}$$

Since $\mathbf{v} \neq \mathbf{0}$, we have $\overline{\mathbf{v}}^T\mathbf{v} \neq 0$, so that $\lambda - \overline{\lambda} = 0$, or $\lambda = \overline{\lambda}$. Therefore the eigenvalue λ is real. □

! Theorem 9.1.3 is a particular case of Theorem 8.6.10.

Theorem 9.1.4. *Any two eigenvectors of a symmetric matrix A that correspond to different eigenvalues are orthogonal.*

Proof. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two eigenvectors of A with corresponding eigenvalues λ_1 and λ_2 such that $\lambda_1 \neq \lambda_2$. Because $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$, $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$, and $A^T = A$, we have

$$\begin{aligned}
\lambda_1\mathbf{v}_1 \cdot \mathbf{v}_2 &= (A\mathbf{v}_1) \cdot \mathbf{v}_2 \\
&= \mathbf{v}_1 \cdot (A^T\mathbf{v}_2) \\
&= \mathbf{v}_1 \cdot A\mathbf{v}_2 \\
&= \mathbf{v}_1 \cdot \lambda_2\mathbf{v}_2 \\
&= \lambda_2\mathbf{v}_1 \cdot \mathbf{v}_2.
\end{aligned}$$

Thus

$$(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = 0.$$

Since $\lambda_1 - \lambda_2 \neq 0$, it follows that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, as stated. □

Example 9.1.5. Verify Theorems 9.1.3 and 9.1.4 for

$$A = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 2 & 4 \\ 1 & 4 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}.$$

Solution.

1. The eigenvalues of A are -3, 0, 6 with the corresponding eigenvectors

   $$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

   We have $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$. So the eigenvalues are all real, and the eigenvectors are all orthogonal, as claimed by Theorems 9.1.3 and 9.1.4.
2. The matrix B has the characteristic polynomial $(6-\lambda)(\lambda+3)^2$ and hence real eigenvalues -3 and 6. Furthermore,

   $$E_{-3} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}, \quad E_6 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

   For the eigenvectors, $(-1,1,0) \cdot (1,1,1) = 0$ and $(-1,0,1) \cdot (1,1,1) = 0$. Note, however, that $(-1,1,0) \cdot (-1,0,1) = 1 \neq 0$.

Although eigenvectors that correspond to *different* eigenvalues are orthogonal, eigenvectors that correspond to the *same* eigenvalue do *not have to be orthogonal.* (See matrix B of Example 9.1.5.)

Example 9.1.6. Orthogonally diagonalize matrices A and B of Example 9.1.5.

Solution.

1. In Example 9.1.5, we found basic eigenvectors of A that were already orthogonal. If we normalize them, then they will remain orthogonal. Hence

   $$Q = \begin{bmatrix} -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ -1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$$

   is orthogonal with columns eigenvectors of A. So Q orthogonally diagonalizes A. It is easy to check that

   $$Q^T A Q = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

2. For B, the eigenvectors $(-1,1,0)$ and $(-1,0,1)$ were not orthogonal, but we may apply Gram–Schmidt to orthogonalize them. We easily get $(-1,1,0)$ and $(-1/2,-1/2,1)$. It is important to note that the Gram–Schmidt process did not alter the span of the original vectors, so the new vector $(-1/2,-1/2,1)$ is still in E_{-3} So it is an eigenvector of A corresponding to -3. Hence it must be orthogonal to $(1,1,1)$. (It is.) Now $(-1,1,0)$, $(-1/2,-1/2,1)$, and $(1,1,1)$ are orthogonal, so we must normalize them to get an orthogonal matrix Q that orthogonally diagonalizes B:

   $$Q = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}.$$

   Again, it is easy to check that

   $$Q^T BQ = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

In Example 9.1.6, we saw how to orthogonally diagonalize a symmetric matrix: If necessary, we apply the Gram–Schmidt process to orthogonalize eigenvectors that correspond to the same eigenvalue. Then normalize the orthogonal eigenvectors and use them as columns for Q. It remains to prove that all symmetric matrices are diagonalizable. To prove this, we use a classical theorem due to Schur (1909). Its proof is in the end of the section.

Theorem 9.1.7 (Schur's decomposition). *Any real square matrix A with only real eigenvalues is orthogonally similar to an upper triangular matrix T. So there exist an orthogonal matrix Q and an upper triangular matrix T such that $Q^T AQ = T$ or, equivalently,*

$$A = QTQ^T.$$

We have arrived at a major result on symmetric matrices, the spectral theorem.

Theorem 9.1.8 (The spectral theorem). *A square matrix is real symmetric if and only if it is orthogonally diagonalizable.*

Proof. We already know that if A is orthogonally diagonalizable, then it is symmetric by Theorem 9.1.2. Conversely, let A be symmetric. Then by Theorem 9.1.3, A has real eigenvalues. Hence by Schur's decomposition theorem there are an orthogonal matrix Q and an upper triangular matrix D such that

$$Q^T AQ = D.$$

Because $A^T = A$, we have

$$D^T = \left(Q^T A Q\right)^T = Q^T A^T Q = Q^T A Q = D.$$

We see that D is symmetric and upper triangular, so it is diagonal. Therefore A is orthogonally diagonalizable. ☐

Now that we know that symmetric matrices are orthogonally diagonalizable, we can describe the procedure applied in Example 9.1.6 to find Q and D.

Algorithm (Diagonalization of a symmetric matrix).

Input: $n \times n$ symmetric matrix A.

1. Compute all the eigenvalues of A. Let $\lambda_1, \dots, \lambda_k$ be all the *distinct* ones. (They are all real, and some are possibly multiple.)
2. Find a basis $\mathcal{B}_i$ of eigenvectors for each eigenspace E_{λ_i}, $i = 1, \dots, k$. (The union $\mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ is a basis of eigenvectors of A, because A is symmetric, hence diagonalizable.)
3. Apply, if necessary, the Gram–Schmidt process to each $\mathcal{B}_i$ to get orthogonal sets $\mathcal{B}_i'$. (So each $\mathcal{B}_i'$ is automatically linearly independent. Since $\mathrm{Span}(\mathcal{B}_i) = \mathrm{Span}(\mathcal{B}_i')$, each $\mathcal{B}_i'$ forms an orthogonal basis for E_{λ_i}.)
4. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be the vectors of $\mathcal{B}_1', \dots, \mathcal{B}_k'$. They form an orthogonal basis of eigenvectors of A (since the vectors from the same $\mathcal{B}_i'$ are orthogonal and the vectors from different $\mathcal{B}_i'$ are also orthogonal as they correspond to distinct eigenvalues).
5. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the normalizations of the $\mathbf{u}_i$s. These form an orthonormal basis of eigenvectors of A.
6. Let $Q = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$. Q is orthogonal.
7. Let D be the diagonal matrix with diagonal entries the corresponding eigenvalues in the same order and with repeated entries for multiple eigenvalues.
8. Q and D orthogonally diagonalize A.

Output: Q orthogonal and D diagonal such that $Q^T A Q = D$.

9.1.1 Proof of Schur's decomposition theorem and example

Proof. Let $\mathbf{v}_1$ be a unit eigenvector of A, and let λ_1 be the corresponding (real) eigenvalue. By the Gram–Schmidt process there is an orthogonal matrix $Q_1 = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$ with first column $\mathbf{v}_1$. Then

$$Q_1^T A Q_1 = \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} [A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_n] = \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} [\lambda_1 \mathbf{v}_1 \ \cdots \ A\mathbf{v}_n] = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline \mathbf{0} & A_1 \end{array}\right],$$

because by orthogonality $\lambda_1 \mathbf{v}_1^T \mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_1 = \lambda_1$ (since $\mathbf{v}_1$ is unit) and $\lambda_1 \mathbf{v}_i^T \mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_i = \lambda_1 0 = 0$ for $i \neq 1$. Now if λ_2 is an eigenvalue of A_1, then it is also an eigenvalue of A; hence λ_2 is real. We apply the same procedure to the $(n-1) \times (n-1)$ matrix A_1 to find an orthogonal matrix Q_2 such that

$$Q_2^T A_1 Q_2 = \begin{bmatrix} \lambda_2 & \vdots & * \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & A_2 \end{bmatrix}.$$

The matrix $Q_2' = \begin{bmatrix} 1 & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & Q_2 \end{bmatrix}$ has the property

$$\begin{aligned} Q_2'^T (Q_1^T A Q_1) Q_2' &= \begin{bmatrix} 1 & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & Q_2^T \end{bmatrix} \begin{bmatrix} \lambda_1 & \vdots & * \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & A_1 \end{bmatrix} \begin{bmatrix} 1 & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & Q_2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & \vdots & * \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & Q_2^T A_1 Q_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & * & \vdots & * \\ 0 & \lambda_2 & \vdots & * \\ \cdots & \cdots & \vdots & \cdots \\ \mathbf{0} & \mathbf{0} & \vdots & A_2 \end{bmatrix}. \end{aligned}$$

Continuing the same way, after $n-1$ steps, we obtain the orthogonal matrix

$$Q = Q_1 \begin{bmatrix} 1 & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & Q_2 \end{bmatrix} \begin{bmatrix} I_2 & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & Q_3 \end{bmatrix} \cdots \begin{bmatrix} I_{n-2} & \vdots & \mathbf{0} \\ \cdots & \vdots & \cdots \\ \mathbf{0} & \vdots & Q_{n-1} \end{bmatrix}$$

such that $Q^T A Q$ is upper triangular,

$$Q^T A Q = \begin{bmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ 0 & 0 & \ddots & * \\ \vdots & & & \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

This completes the proof of Schur's decomposition theorem. □

Example 9.1.9. Find the Schur decomposition of $A = \left[\begin{smallmatrix} 9 & 8 \\ 1 & 2 \end{smallmatrix}\right]$.

Solution. Both eigenvalues 1 and 10 of A are real, and 1 has the unit eigenvector $\mathbf{v}_1 = \left[\begin{smallmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{smallmatrix}\right]$. By the proof of the theorem we need an orthogonal matrix $Q_1 = [\mathbf{v}_1 \ \mathbf{v}_2]$. We may take $Q_1 = \left[\begin{smallmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{smallmatrix}\right]$. Then $T = Q_1^T A Q_1 = \left[\begin{smallmatrix} 1 & -7 \\ 0 & 10 \end{smallmatrix}\right]$. So we have

$$\begin{bmatrix} 9 & 8 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & -7 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

Exercises 9.1

In Exercises 1–10, orthogonally diagonalize the matrix A.

1. $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$.

2. $A = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$.

3. $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$.

4. $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$.

5. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix}$.

6. $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$.

7. $A = \begin{bmatrix} 5 & -4 & 0 \\ -4 & 3 & 4 \\ 0 & 4 & 1 \end{bmatrix}$.

8. $A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$.

9. $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{bmatrix}$.

10. $A = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$.

11. Prove that if a matrix A is both symmetric and orthogonal, then $A^2 = I$. Conclude that $A = A^{-1}$.

12. Without computation, find the inverse of $A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$.

13. Let A and B be $n \times n$ and orthogonally diagonalizable with real entries, and let c be any real scalar. Use the spectral theorem to prove that the following are also orthogonally diagonalizable:

(a) $A + B$;
(b) $A - B$;
(c) cA;
(d) A^2.

14. Prove that if A is real symmetric, then A^2 has nonnegative eigenvalues.

15. Prove that the geometric and algebraic multiplicities of each eigenvalue of a symmetric matrix are equal.

16. Modify the proof of Theorem 9.1.3 to prove that a skew-symmetric matrix has eigenvalues that are either zero or pure imaginary.

17. If A is skew-symmetric, then prove that $A + I$ is invertible. (*Hint:* Use Exercise 16.)

In Exercises 18–21, find the Schur decomposition of A.

18. $A = \begin{bmatrix} 9 & 8 \\ 2 & 3 \end{bmatrix}$.

19. $A = \begin{bmatrix} 6 & 5 \\ 1 & 2 \end{bmatrix}$.

20. $A = \begin{bmatrix} 10 & 9 \\ 2 & 3 \end{bmatrix}$.

21. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 1 & 2 \end{bmatrix}$.

Spectral decomposition

22. Let $A = QDQ^T$ be an orthogonal diagonalization of A. If $Q = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$ and D has diagonal entries $\lambda_1, \dots, \lambda_n$, then prove that

$$A = \lambda_1 \mathbf{v}_1 \mathbf{v}_1^T + \cdots + \lambda_n \mathbf{v}_n \mathbf{v}_n^T.$$

This is called the *spectral decomposition* of A.

In Exercises 23–24, find the spectral decomposition of A (as defined in Exercise 22).

23. $A = \begin{bmatrix} 12 & 3 \\ 3 & 4 \end{bmatrix}$.

24. $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$.

25. $A = \begin{bmatrix} 13 & 4 \\ 4 & 7 \end{bmatrix}$.

9.2 Quadratic Forms and conic sections

In this section, we use the diagonalization of symmetric matrices, discussed in Section 9.1, to study quadratic polynomials in several variables such as

$$ax^2 + by^2 + cxy + dx + ey + f$$

or

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + kz + l.$$

Actually, we are only interested in the quadratic (or *principal*) terms of the polynomials

$$ax^2 + by^2 + cxy \tag{9.1}$$

and

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz. \tag{9.2}$$

Expressions of the form (9.1) and (9.2) are called *quadratic forms*. Quadratic forms can be written as matrix products $\mathbf{x}^T A\mathbf{x}$. For example,

$$3x^2 + 7y^2 - 2xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

or even

$$3x^2 + 7y^2 - 2xy = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

We prefer the first decomposition over the second, because in the first one the matrix is symmetric. In fact, when we write a quadratic form as $\mathbf{x}^T A\mathbf{x}$, we may always assume that A is symmetric; A is called the *associated* matrix of the quadratic form. For example, we have

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & \frac{1}{2}d & \frac{1}{2}e \\ \frac{1}{2}d & b & \frac{1}{2}f \\ \frac{1}{2}e & \frac{1}{2}f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Symmetric matrices are preferred, because they can be *orthogonally diagonalized.*

Definition 9.2.1. A *quadratic form* in n variables is a function q of the form

$$q : \mathbf{R}^n \to \mathbf{R}, \quad q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x},$$

where A is an $n \times n$ *symmetric* matrix. We say that A is the matrix *associated* with q.

Example 9.2.2. Prove that the dot product in $\mathbf{R}^n$ defines a quadratic form q defined by

$$q(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2 .$$

What is the associated matrix?

Solution. Because

$$q(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T\mathbf{x} = \mathbf{x}^T I\mathbf{x},$$

we see that q is a quadratic form with associated matrix I.

Quadratic forms in two variables, $q(x,y) = ax^2 + by^2 + cxy$, are related to *conic sections*. In fact, if $c = 0$, then the equation $q(x,y) = ax^2 + by^2 = 1$ defines in general an *ellipse* or a *hyperbola* in *standard position*.[4] This means that the principal axes of these conics are the x- and y-axes (Figure 9.3). In this case the matrix of the quadratic form is *diagonal*.

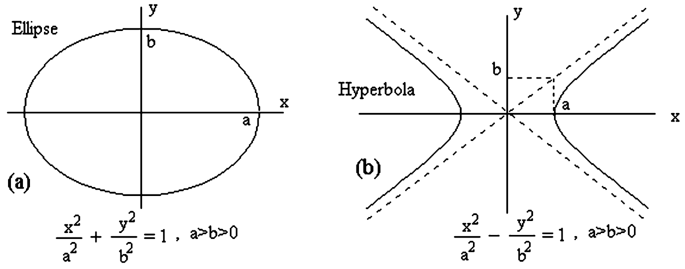
Ellipse
y
b
x
a
(a)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a>b>0
y
b
Hyperbola
a
x
(b)
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, a>b>0

Figure 9.3: Ellipse and hyperbola in standard position.

For the general case $q(x,y) = ax^2 + by^2 + cxy$, $c \neq 0$, we change the variables so that with respect to the new variables, say, x', y', there is no cross term, that is, $q(x',y') = a'x'^2 + b'y'^2$. Then $q(x',y') = 1$ can be identified as a conic section.

9.2.1 Diagonalization of quadratic forms

Let $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be any quadratic form in n-variables. Since A is symmetric, it can be orthogonally diagonalized, say, by orthogonal Q and diagonal D,

$$Q^{-1}AQ = D \quad \text{or} \quad Q^T AQ = D.$$

---

[4] Parabolas, such as $y = 2x^2$ or $y^2 = 3x$, are also conic sections, but because they contain nonquadratic terms, we do not include them in our study.

Using the change of variables

$$\mathbf{x} = Q\mathbf{y} \quad \text{or} \quad \mathbf{y} = Q^{-1}\mathbf{x} = Q^T\mathbf{x}, \tag{9.3}$$

we get

$$\begin{aligned} q(\mathbf{x}) &= \mathbf{x}^T A\mathbf{x} \\ &= (Q\mathbf{y})^T AQ\mathbf{y} \\ &= \mathbf{y}^T Q^T AQ\mathbf{y} \\ &= \mathbf{y}^T D\mathbf{y}. \end{aligned}$$

So q has the same values as a quadratic form in the new variables $\mathbf{y}$ with matrix D. In the new variables, there are no cross terms, because D is diagonal. This process is called *diagonalization* of q. Since the diagonal entries of D are the eigenvalues of A, we have proved the following theorem.

Theorem 9.2.3 (Principal axes theorem). *Let A be an $n \times n$ symmetric matrix orthogonally diagonalized by Q and D. Then the change of variables $\mathbf{x} = Q\mathbf{y}$ transforms the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ into the form $\mathbf{y}^T D\mathbf{y}$ that has no cross terms. In fact, if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A and if $\mathbf{y} = (y_1, \dots, y_n)$, then*

$$q(\mathbf{x}) = q(\mathbf{y}) = \mathbf{y}^T D\mathbf{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2.$$

> We are interested in the *orthogonal* diagonalization of quadratic forms, because the change of variables is then done by an *orthogonal linear transformation* $\mathbf{y} = Q\mathbf{x}$ (meaning Q is orthogonal). So lengths and angles are preserved. Therefore the shapes of the curves, surfaces, and solids are also preserved in the new coordinates.

9.2.2 Applications of quadratic forms to geometry

The principal axes theorem (Theorem 9.2.3) can be used for $n = 2, 3$ to identify conic sections and quadric surfaces with cross terms.

Conic sections: Ellipses–hyperbolas

Example 9.2.4. Use diagonalization to identify the conic sections $q_1(x, y) = 1$ and $q_2(x, y) = 1$, where
(a) $q_1(x, y) = 2x^2 + 2y^2 - 2xy$;
(b) $q_2(x, y) = -x^2 - y^2 + 6xy$.

Solution. Let $A_1 = \left[\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right]$ and $A_2 = \left[\begin{smallmatrix} -1 & 3 \\ 3 & -1 \end{smallmatrix}\right]$ be the corresponding matrices. They are diagonalized by Q_1, D_1 and Q_2, D_2, where

$$Q_1 = Q_2 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}.$$

Hence

$$q_1(x',y') = \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = x'^2 + 3y'^2$$

and

$$q_2(x',y') = \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 2x'^2 - 4y'^2.$$

We have the ellipse $x'^2 + 3y'^2 = 1$ and the hyperbola $2x'^2 - 4y'^2 = 1$. To sketch them, we find the vectors that map to $(1,0)$ and $(0,1)$ in x' and y'. We have

$$Q_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = Q_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},$$
$$Q_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = Q_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

So $(1,0)$ and $(0,1)$ in the new systems are $(1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$ in the old. Hence the ellipse and parabola are rotated 45° degrees from the standard position. (Figure 9.4).

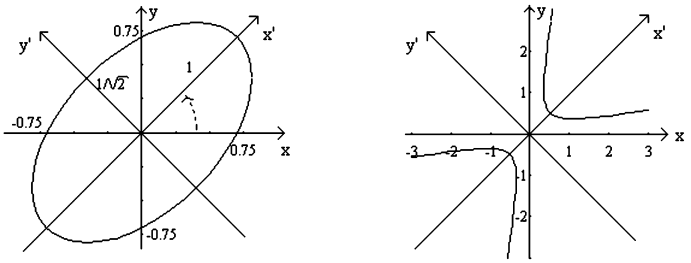

Figure 9.4: Conics in nonstandard position.

Example 9.2.5. Use diagonalization to identify the conic sections $q_1(x,y) = 1$ and $q_2(x,y) = 1$, where
(a) $q_1(x,y) = 2x^2 + 2y^2 + 2xy$;
(b) $q_2(x,y) = 2x^2 + 2y^2 - 4xy$.

Solution. It is easy to see that q_1 and q_2 are respectively diagonalized by

$$Q_1 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

and

$$Q_2 = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}.$$

Hence

$$q_1(x',y') = \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = x'^2 + 3y'^2$$

and

$$q_2(x',y') = \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 4y'^2.$$

So $q_1(x',y') = x'^2 + 3y'^2 = 1$ is an ellipse, just as in Example 9.2.4. This time, however, $Q_1(1,0) = (-1/\sqrt{2}, 1/\sqrt{2})$ and $Q_1(0,1) = (1/\sqrt{2}, 1/\sqrt{2})$, so the positive x'-axis is the half-line at an angle of 135°, and the positive y'-axis is the half-line at an angle of 45°. In fact, since

$$Q_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix},$$

the transformation defined by Q_1 is a rotation by $-45°$ (the second matrix) followed by a reflection about $y = x$ (the first matrix) (Figure 9.5a).

As of $q_2(x',y') = 4y'^2 = 1$, we have $y' = \pm 1/2$, so this time, we do not get an ellipse or hyperbola but two straight lines in the $x'y'$-system (Figure 9.5b). This was an example of a *degenerate* quadratic form.

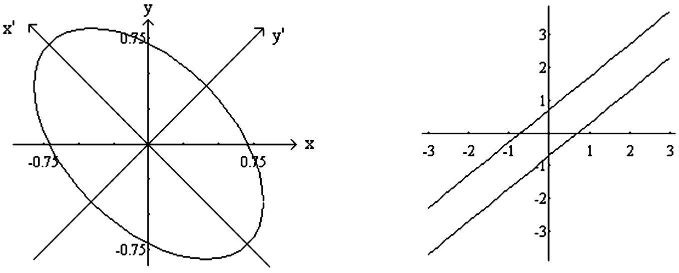

Figure 9.5: (a) Rotation followed by reflection. (b) Degenerate form: Two parallel lines.

Quadric surfaces: Ellipsoids

We way apply these methods to identify quadric surfaces. For example, an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

for $a, b, c > 0$ is an ellipsoid in standard position. The cross-sections of such a surface with coordinates planes are ellipses. Figure 9.6a shows the ellipsoid $3x^2 + 6y^2 + 9z^2 = 1$.

Example 9.2.6. Identify the quadric surface $5x^2 + 6y^2 + 7z^2 + 4xy + 4yz = 1$.

Solution. The matrix of $q(x, y, z) = 5x^2 + 6y^2 + 7z^2 + 4xy + 4yz$ is

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$$

with eigenvalues 2, 6, and 9 and the corresponding eigenvectors, $(2, -2, 1)$, $(2, 1, -2)$, and $(1, 2, 2)$. These are already orthogonal, so we normalize them to get

$$Q = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix}.$$

Using the change of variables $\mathbf{y} = Q^T\mathbf{x}$, we get

$$q(x, y, z) = \begin{bmatrix} x' & y' & z' \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = 3x'^2 + 6y'^2 + 9z'^2.$$

Therefore $q(x, y, z) = 1$ takes the form $3x'^2 + 6y'^2 + 9z'^2 = 1$ in the new system. The graph is an ellipsoid. Figure 9.6b shows this ellipsoid as somewhat turned and titled, compared with the one with the same equation in standard position.

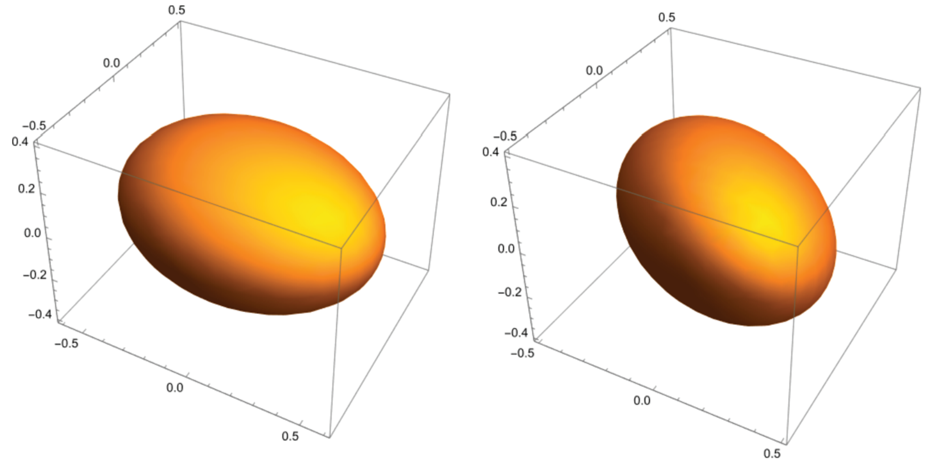

Figure 9.6: Ellipsoids in: (a) standard and (b) nonstandard positions.

9.2.3 Positive and negative definite quadratic forms

Let $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form with symmetric A.

Definition 9.2.7. 1. If $q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, then q and A are called *positive definite*.
2. If $q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$, then q and A are called *negative definite*.
3. If $q(\mathbf{x})$ takes on both positive and negative values, then q and A are called *indefinite*.

In addition to these basic types of forms, we have *positive* and *negative semidefinite* quadratic forms and symmetric matrices according to whether $q(\mathbf{x}) \geq 0$ or $q(\mathbf{x}) \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$.

Theorem 9.2.3 can be easily used to identify the type of a quadratic form by looking at the signs of the eigenvalues of its matrix.

Theorem 9.2.8. *Let $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form with symmetric matrix A. Then $q(\mathbf{x})$ and A are*
1. *positive definite if and only if all the eigenvalues of A are positive,*
2. *negative definite if and only if all the eigenvalues of A are negative,*
3. *indefinite if and only if A has positive and negative eigenvalues.*

Proof. Exercise. ☐

The appearance of the signs in the formula of a quadratic form can be deceiving. For example, the form $q(\mathbf{x}) = x^2 + y^2 + 10xy$ is not positive definite. In fact, $q(1, -1) = -8 < 0$.

Example 9.2.9. By Theorem 9.2.8 we can check the eigenvalues to verify that A is positive semidefinite, B is negative definite, and C is indefinite, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -10 & a \\ 0 & -100 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}.$$

Example 9.2.10 (Relativity). Prove that the following quadratic form q, used in the theory of relativity to define distance in space-time, is indefinite:

$$q(\mathbf{x}) = \begin{bmatrix} x & y & z & t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = x^2 + y^2 + z^2 - t^2.$$

Solution. By Theorem 9.2.8, q is indefinite, since its matrix has both positive and negative eigenvalues.

Theorem 9.2.11. *A positive definite matrix is invertible.*

Proof. If A is positive definite, then its eigenvalues are all positive by Theorem 9.2.8. So 0 is not an eigenvalue. Hence $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Therefore A is invertible. ☐

Further properties of positive definite matrices are explored in the exercises.

Exercises 9.2

In Exercises 1–4, evaluate the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for the given A and $\mathbf{x}$.

1. $A = \begin{bmatrix} -2 & 2 \\ 2 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

2. $A = \begin{bmatrix} 4 & 7 \\ 7 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

3. $A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 0 & 1 \\ 2 & 1 & 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

4. $A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 0 & 1 \\ 2 & 1 & 5 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

In Exercises 5–12, find the symmetric matrix A of the quadratic form.

5. $q(x,y) = 3x^2 - 6xy + 3y^2$.

6. $q(x,y) = -x^2 + 10xy - y^2$.

7. $q(x,y) = -4x^2 + 2xy - 4y^2$.

8. $q(x,y) = 6x^2 - 2xy + 6y^2$.

9. $q(x,y,z) = 2x^2 + 2xz + 2z^2$.

10. $q(x,y,z) = x^2 + 2y^2 + 8yz + 2z^2$.

11. $q(x,y,z) = 5x^2 - 8xy + 3y^2 + 12yz + z^2$.

12. $q(x,y,z,w) = x^2 + 2xy + y^2 + 2z^2 + 4zw + 2w^2$.

In Exercises 13–17, orthogonally diagonalize the quadratic form. Use a change of variables to rewrite the form without cross terms.

13. $q(x,y) = 3x^2 - 2xy + 3y^2$.

14. $q(x,y) = -x^2 + 4xy - y^2$.

15. $q(x,y) = -4x^2 + 4xy - 4y^2$.

16. $q(x,y,z) = x^2 + 2y^2 - 4yz + 2z^2$.

17. $q(x,y,z) = 2x^2 - 2xz + 2z^2$.

18. Identify the conic section $q(x,y) = 3x^2 - 2xy + 3y^2 = 1$.

19. Identify the conic section $q(x,y) = 5x^2 - 8xy + 5y^2 = 1$.

20. Identify the quadric surface $q(x,y,z) = 6x^2 + 8xy + 4xz + 10y^2 + 12yz + 11z^2 = 1$.

21. Prove Theorem 9.2.8.

22. Prove that the quadratic form $q(x,y) = ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$.

23. Prove that the sum of two positive definite matrices is positive definite.

24. Prove that the inverse of a positive definite matrix is positive definite.

25. Prove that if A is positive definite, then A^k is positive definite.

26. Is the product of two positive definite matrices a positive definite matrix? If the answer is "yes", then prove the statement. Otherwise, give an example.

27. Prove that A is positive definite if and only if there exists an invertible matrix P such that $A = P^T P$. (*Hint:* Orthogonally diagonalize A and use Theorem 9.2.8.)

28. Prove that a positive definite matrix A has a *square root*, i. e., there is a positive definite matrix R such that $A = R^2$. (*Hint:* Orthogonally diagonalize A and use Theorem 9.2.8.)

29. Prove that the matrix R defined in Exercise 28 is unique.

30. Prove that if A is symmetric, then A^2 is positive semidefinite.

31. Prove that if A is skew-symmetric, then A^2 is negative semidefinite.

The familiar completion of the square

$$A^2 + bA = \left(A^2 + 2(b/2)A + b^2/4\right) - b^2/4 = (A + b/2)^2 - b^2/4$$

can be used to convert a two-variable quadratic form into one without cross terms.

32. Let $q(x,y) = ax^2 + bxy + cy^2$. If $a \neq 0$, then complete the square to write q in the form $aX^2 + By^2$ for some constant B and a new variable X depending on x and y.

33. Apply the formula from Exercise 32 to write $q(x,y) = 3x^2 - 2xy + 3y^2$ without cross terms.

34. Referring to Exercise 32, if $a = 0$ and $c \neq 0$, then can you still complete the square and write q without cross terms?

35. Referring to Exercise 32, if $a = 0$ and $c = 0$, then can you write q without cross terms?

9.3 The singular value decomposition (SVD)

We have seen that factorizations of matrices into factors with special properties are useful. A factorization is of particular interest if some of the factors are orthogonal matrices. The reason is that orthogonal transformations preserve norms and angles. In particular, they preserve the lengths of the error vectors, which are inevitable in numerical calculations.

We study an important factorization that applies to *any* matrix A. It is called the *singular value decomposition (SVD)*.[5] Among the many applications of SVD, the most reliable is estimation of the rank of a matrix.

We seek to factor an $m \times n$ matrix A as

$$A = U\Sigma V^T,$$

where U is $m \times m$ and V is $n \times n$, and they are both orthogonal. The matrix Σ is an $m \times n$ matrix with a diagonal upper left block of positive entries of decreasing magnitude and the remaining entries 0. So

$$\Sigma = \begin{bmatrix} D & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{0} \end{bmatrix}, \quad \text{where } D = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}, \tag{9.4}$$

and

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0, \qquad r \leq m, n.$$

Here are some examples for Σ with $r = 2$:

$$\begin{bmatrix} 6 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the corresponding Ds

$$\begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix}, \quad \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}, \quad \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}.$$

9.3.1 Singular values; finding V, Σ, and U

First, we define V, then find the σ_is along the diagonal of D to form Σ. Consider the $n \times n$ symmetric matrix A^TA. By the spectral theorem, A^TA is orthogonally diagonalizable and has real eigenvalues, say, $\lambda_1, \dots, \lambda_n$. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be the corresponding eigenvectors. These form an orthonormal basis of $\mathbf{R}^n$. V is defined by

$$V = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n].$$

All the eigenvalues λ_i are nonnegative, because

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[5] This method is already mentioned in E. Beltrami's "Sulle Funzioni Bilineari", *Giornale di Mathematische 11*, (1873) pp. 98–106.

$$\begin{aligned}0 &\le \|A\mathbf{v}_i\|^2\\ &= (A\mathbf{v}_i)^T A\mathbf{v}_i\\ &= \mathbf{v}_i^T A^T A\mathbf{v}_i\\ &= \mathbf{v}_i^T \lambda_i \mathbf{v}_i\\ &= \lambda_i \|\mathbf{v}_i\|^2\\ &= \lambda_i.\end{aligned}$$

By renumbering if necessary we order the λs from the largest to smallest and define σ_i by

$$\sigma_1 = \sqrt{\lambda_1} \ge \cdots \ge \sigma_n = \sqrt{\lambda_n} \ge 0.$$

Hence we have

$$\sigma_i = \|A\mathbf{v}_i\|, \quad i = 1, \dots, n. \tag{9.5}$$

The numbers $\sigma_1, \dots, \sigma_n$ are called the *singular values* of A, and they carry important information about A. Let r be the integer such that

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0 \quad \text{and} \quad \sigma_{r+1} = \cdots = \sigma_n = 0.$$

So $\sigma_1, \dots, \sigma_r$ are the nonzero singular values of A ordered by magnitude. These are the diagonal entries of D in Σ.

Example 9.3.1. Compute V and Σ for the matrices

$$\text{(a)}\quad A = \begin{bmatrix} 2 & 4\\ 1 & -4\\ -2 & 2\end{bmatrix}, \quad \text{(b)}\quad A = \begin{bmatrix} -2 & 1 & 2\\ 6 & 6 & 3\end{bmatrix}, \quad \text{(c)}\quad A = \begin{bmatrix} 0 & 6 & 6\\ -6 & -3 & 0\\ 6 & 0 & -3\end{bmatrix}.$$

Solution. We need A^TA, its eigenvalues, and the corresponding basic eigenvectors. We have

	A^TA	Eigenvalues	Eigenvectors
(a)	$\begin{bmatrix} 9 & 0\\ 0 & 36\end{bmatrix}$	$36, 9$	$\begin{bmatrix} 0\\ 1\end{bmatrix}, \begin{bmatrix} 1\\ 0\end{bmatrix}$
(b)	$\begin{bmatrix} 40 & 34 & 14\\ 34 & 37 & 20\\ 14 & 20 & 13\end{bmatrix}$	$81, 9, 0$	$\begin{bmatrix} 2\\ 2\\ 1\end{bmatrix}, \begin{bmatrix} -2\\ 1\\ 2\end{bmatrix}, \begin{bmatrix} 1\\ -2\\ 2\end{bmatrix}$
(c)	$\begin{bmatrix} 72 & 18 & -18\\ 18 & 45 & 36\\ -18 & 36 & 45\end{bmatrix}$	$81, 81, 0$	$\begin{bmatrix} -2\\ 0\\ 1\end{bmatrix}, \begin{bmatrix} 2\\ 1\\ 0\end{bmatrix}, \begin{bmatrix} 1\\ -2\\ 2\end{bmatrix}$

(a) The singular values of A are $\sigma_1 = \sqrt{36} = 6$ and $\sigma_2 = \sqrt{9} = 3$. The eigenvectors are already orthonormal, so

$$V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 6 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}.$$

(b) The singular values of A are $\sigma_1 = 9$, $\sigma_2 = 3$, and $\sigma_3 = 0$. The eigenvectors are orthogonal and need normalization. So

$$V = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}.$$

(c) The singular values of A are $\sigma_1 = 9$, $\sigma_2 = 9$, and $\sigma_3 = 0$. We now need to orthonormalize the eigenvectors. Orthogonalizing the first two that belong to E_{81}, we get $(-2, 0, 1)$ and $(2/5, 1, 4/5)$ by the one-step Gram–Schmidt process. So normalizing the orthogonal set $\{(-2, 0, 1), (2/5, 1, 4/5), (1, -2, 2)\}$ yields

$$V = \begin{bmatrix} -2/\sqrt{5} & 2/(3\sqrt{5}) & 1/3 \\ 0 & 1/(3\sqrt{5}) & -2/3 \\ 1/\sqrt{5} & 4/(3\sqrt{5}) & 2/3 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The computation of V involves choices, so V is not unique. In (c) of Example 9.3.1, instead of the eigenvectors $\mathbf{v}_1 = (-2, 0, 1)$ and $\mathbf{v}_1 = (2, 1, 0)$, we could have used the linear combinations $\mathbf{v}_1 + 2\mathbf{v}_2 = (2, 2, 1)$ and $2\mathbf{v}_1 + \mathbf{v}_2 = (-2, 1, 2)$. These are orthogonal, so after normalization, we get a different V,

$$V = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}.$$

We now come to the definition of U. It is done in two steps.

Step 1. We form

$$\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i \quad \text{for } i = 1, \ldots, r. \tag{9.6}$$

These vectors are *orthonormal*, since

$$\begin{aligned} \mathbf{u}_i \cdot \mathbf{u}_j &= \frac{1}{\sigma_i \sigma_j}(A\mathbf{v}_i \cdot A\mathbf{v}_j) = \frac{1}{\sigma_i \sigma_j}(A^T A\mathbf{v}_i) \cdot \mathbf{v}_j \\ &= \frac{\lambda_i}{\sigma_i \sigma_j} \mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \end{aligned} \tag{9.7}$$

because the $\mathbf{v}_i$s are orthonormal, and for $i = j$, we have $\lambda_i/\sigma_i^2 = 1$ by the definition of the singular values.

Step 2. We extend the set $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ to an orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of $\mathbf{R}^m$. This is necessary only if $r < m$. We define

$$U = [\mathbf{u}_1 \; \mathbf{u}_2 \; \cdots \; \mathbf{u}_m] \,.$$

Uniqueness of Σ: Note that if a *square* matrix A can be factored as $A = Q_1 S Q_2$ with orthogonal Q_1, Q_2 and diagonal S, then $A^T A = Q_2^T S^2 Q_2$. So $A^T A$ and S^2 have the same eigenvalues. Thus the eigenvalues of S are the singular values of A. If we choose an S where these are in descending order, then we see that Σ in SVD must be *unique*.

Example 9.3.2. Find an SVD for the matrix A of Example 9.3.1(b).

Solution. We have $\sigma_1 = 9$, $\sigma_2 = 3$, $\mathbf{v}_1 = (2/3, 2/3, 1/3)$, and $\mathbf{v}_2 = (-2/3, 1/3, 2/3)$. Hence

$$\mathbf{u}_1 = \frac{1}{9}\begin{bmatrix} -2 & 1 & 2 \\ 6 & 6 & 3 \end{bmatrix}\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$\mathbf{u}_2 = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ 6 & 6 & 3 \end{bmatrix}\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since $m = r = 2$, the set $\{\mathbf{u}_1, \mathbf{u}_2\}$ needs no extension. So

$$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An SVD of $A = U\Sigma V^T$ is then

$$\begin{bmatrix} -2 & 1 & 2 \\ 6 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}^T .$$

One way to extend an orthonormal set $S = \{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ to an orthonormal basis $\mathcal{B} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is outlined by the following steps.

1. Form $S' = \{\mathbf{u}_1, \dots, \mathbf{u}_r, \mathbf{e}_1, \dots, \mathbf{e}_m\}$ and find the pivot columns of the matrix with columns these vectors.
2. Form the subset S'' of S' that consists of the pivot columns. S'' is a basis of $\mathbf{R}^m$.
3. Apply the Gram–Schmidt to S'' and normalize the resulting vectors to get $\mathcal{B}$.

Example 9.3.3 (Extension to orthonormal basis). Find an SVD for A of Example 9.3.1(a).

Solution. To find U, we have

$$\mathbf{u}_1 = \frac{1}{6}\begin{bmatrix} 2 & 4 \\ 1 & -4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix},$$

$$\mathbf{u}_2 = \frac{1}{3}\begin{bmatrix} 2 & 4 \\ 1 & -4 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}.$$

Now we need to extend $\{\mathbf{u}_1, \mathbf{u}_2\}$ to an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbf{R}^3$. Since

$$\begin{bmatrix} 2/3 & 2/3 & 1 & 0 & 0 \\ -2/3 & 1/3 & 0 & 1 & 0 \\ 1/3 & -2/3 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 2 \end{bmatrix},$$

the first three columns are pivot columns, so $\{\mathbf{u}_1, \mathbf{u}_2, (1, 0, 0)\}$ forms a basis of $\mathbf{R}^3$. Gram–Schmidt and normalization now yield $\mathbf{u}_3 = (1/2, 2/3, 2/3)$. So

$$U = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}.$$

An SVD of $A = U\Sigma V^T$ is then

$$\begin{bmatrix} 2 & 4 \\ 1 & -4 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}\begin{bmatrix} 6 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T.$$

We leave it to the reader to verify the following SVD for A of Example 9.3.1(c) and to find another one based on the first V:

$$\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}^T.$$

We have almost proved the following basic theorem.

Theorem 9.3.4. *Let A be an $m \times n$ matrix, and let $\sigma_1, \dots, \sigma_r$ be all its nonzero singular values. Then there are orthogonal matrices U ($m \times m$) and V ($n \times n$) and an $m \times n$ matrix Σ of the form* (9.4) *such that*

$$A = U\Sigma V^T.$$

Proof. The matrices U, V, and Σ (of the indicated sizes) have been already explicitly defined; U and V are orthogonal by construction. It only remains to prove that $A = U\Sigma V^T$, or $AV = U\Sigma$, because $V^T = V^{-1}$. By (9.6) we have

$$\sigma_i \mathbf{u}_i = A\mathbf{v}_i \quad \text{for } i = 1, \ldots, r,$$

and by (9.5) we have $\|A\mathbf{v}_i\| = \sigma_i = 0$ for $i = r+1, \ldots, n$. So

$$A\mathbf{v}_i = \mathbf{0} \quad \text{for } i = r+1, \ldots, n. \tag{9.8}$$

Therefore

$$\begin{aligned} AV &= [A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_n] \\ &= [A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_r \ \mathbf{0} \ \cdots \ \mathbf{0}] \\ &= [\sigma_1\mathbf{u}_1 \ \cdots \ \sigma_r\mathbf{u}_r \ \mathbf{0} \ \cdots \ \mathbf{0}] \\ &= [\mathbf{u}_1 \ \cdots \ \mathbf{u}_m] \begin{bmatrix} \sigma_1 & & 0 & \vdots & \\ & \ddots & & \vdots & \mathbf{0} \\ 0 & & \sigma_r & \vdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & \mathbf{0} & \vdots & \mathbf{0} \end{bmatrix} \\ &= U\Sigma. \end{aligned}$$

□

The matrices U, Σ, and V and r (the number of nonzero singular values) provide important information on A.

Theorem 9.3.5. *Let V, Σ, U be singular value decomposition matrices for an $m \times n$ matrix A. Let $\sigma_1, \ldots, \sigma_r$ be all the nonzero singular values of A. Then*

1. *The rank of A is r;*
2. *$\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ is an orthonormal basis for* $\mathrm{Col}(A)$*;*
3. *$\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m\}$ is an orthonormal basis for* $\mathrm{Null}(A^T)$*;*
4. *$\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for* $\mathrm{Row}(A)$*;*
5. *$\{\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for* $\mathrm{Null}(A)$*.*

Proof.

Parts 1 and 2. Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$. Then $\mathcal{B}$ is orthonormal by (9.7) and thus linearly independent; it is a subset of $\mathrm{Col}(A)$ by (9.6). Because $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbf{R}^n$, the set $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_n\}$ spans $\mathrm{Col}(A)$. Therefore $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ spans $\mathrm{Col}(A)$ by (9.8). So the dimension of $\mathrm{Col}(A)$ is $\leq r$, and thus it is exactly r, since $\mathcal{B}$ is a linearly independent subset with r elements. So $\mathcal{B}$ is an orthonormal basis of $\mathrm{Col}(A)$, and $\mathrm{rank}(A) = r$.

Part 3. By Part 2, $\{\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m\}$ is an orthonormal basis for the orthogonal complement of $\mathrm{Col}(A)$. The claim now follows from $(\mathrm{Col}(A))^{\perp} = \mathrm{Null}(A^T)$ of Theorem 8.2.6.

Part 5. $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is an orthonormal subset of Null(A) by (9.8). By the rank theorem (Theorem 4.6.19) the nullity of A is $n - \text{rank}(A) = n - r$. Hence the dimension of Null(A) is $n - r$, so $\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$ is an orthonormal basis.

Part 4. By Part 5, $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is an orthonormal basis for the orthogonal complement of Null(A). Since $(\text{Null}(A))^{\perp} = (\text{Row}(A)^{\perp})^{\perp} = \text{Row}(A)$ by Theorem 8.2.6, the claim follows. □

Note on the numerical computation of rank

One of the most important applications of the SVD is in the computation of the rank of a matrix by using Theorem 9.3.5. Numerical reduction of large matrices often yields the wrong rank due to the accumulation of round-off errors. Entries that should have been zero, could be replaced by small numbers. This is propagated, repeated, and magnified during row reduction. So Gauss elimination can be unreliable in the computation of the rank. On the other hand, when we factor a matrix by using SVD, we can see that most of the round-off errors occur in the computation of Σ. So, for instance, we may choose to discard very small values of σ_i by replacing them by 0 and then count the remaining σ_is to estimate the rank.

9.3.2 Pseudoinverse

Let A be an $m \times n$ matrix. We may use the SVD of A to define an $n \times m$ matrix A^+ such that in the special case where A is invertible (so $m = n$) $A^+ = A^{-1}$. The matrix A^+ has several interesting properties and gives an optimal solution to the least squares problem studied in Section 8.4.

Definition 9.3.6. Let $A = U\Sigma V^T$ be an SVD for an $m \times n$ matrix A. The *pseudoinverse* (or *Moore–Penrose inverse*) of A is the $n \times m$ matrix

$$A^+ = V\Sigma^+ U^T \tag{9.9}$$

with the $n \times m$ matrix

$$\Sigma^+ = \left[\begin{array}{ccc} D^{-1} & \vdots & \mathbf{0} \\ \cdots & \cdots & \cdots \\ \mathbf{0} & \vdots & \mathbf{0} \end{array}\right],$$

where D is, as before, the $r \times r$ diagonal matrix with diagonal entries the positive singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ of A.

Example 9.3.7. Compute the pseudoinverse of $A = \left[\begin{smallmatrix} 2 & 0 & 0 \\ 0 & 0 & -6 \end{smallmatrix}\right]$.

Solution. An SVD for A is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^T .$$

Hence $\Sigma^+ = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$, and

$$A^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{6} \end{bmatrix} .$$

Note that *if* A is $n \times n$ invertible, then $n = r$, and $\Sigma = D$. So Σ is $n \times n$ and invertible. Moreover, $\Sigma\Sigma^+ = I_n$. Therefore

$$AA^+ = AV\Sigma^+U^T = U\Sigma V^T V\Sigma^+ U^T = U\Sigma\Sigma^+U^T = UU^T = I.$$

This holds only if A is invertible!

Hence, in this case, the pseudoinverse is the same as the inverse, $A^+ = A^{-1}$.

Roger Penrose proved that A^+ is the *unique* matrix B that satisfies the *Moore–Penrose conditions*:

1. $ABA = A$;
2. $BAB = B$;
3. $(AB)^T = AB$;
4. $(BA)^T = BA$.

It is instructive to verify these conditions for the pair (A, A^+) of Example 9.3.7. The verification for any pair (A, A^+) is discussed in the exercises. Although we do not prove it, we *use* the uniqueness part of Penrose's statement. So if we can prove that the pair (A, B) satisfies the conditions, then B is the unique pseudoinverse of A. So $B = A^+$.

The pseudoinverse A^+ is used in the solution of the least squares problem, as we see next.

9.3.3 SVD and least squares

Recall from Section 8.4 that a least squares solution for the possibly inconsistent system $A\mathbf{x} = \mathbf{b}$ is a vector $\tilde{\mathbf{x}}$ that minimizes the length of the error vector $\Delta = \mathbf{b} - A\tilde{\mathbf{x}}$,

$$\|\Delta\| = \|\mathbf{b} - A\tilde{\mathbf{x}}\| = \min .$$

The vector $\tilde{\mathbf{x}}$ in not necessarily unique. If A is $m \times n$ with rank $r < n$, then its nullity is ≥ 1. In this case, any vector of the form $\tilde{\mathbf{x}} + \mathbf{z}$ with $\mathbf{z} \neq \mathbf{0}$ in $\mathrm{Null}(A)$ will also be a least squares solution, because

$$\mathbf{b} - A(\tilde{\mathbf{x}} + \mathbf{z}) = \mathbf{b} - A\tilde{\mathbf{x}} - A\mathbf{z} = \mathbf{b} - A\tilde{\mathbf{x}}.$$

If, however, we demand that $\tilde{\mathbf{x}}$ has also the minimum length, then such a solution *is* unique and can be computed by using the Moore–Penrose inverse of A.

Theorem 9.3.8. *The least squares problem $A\mathbf{x} = \mathbf{b}$ has a unique least squares solution $\tilde{\mathbf{x}}$ of minimal length given by*

$$\tilde{\mathbf{x}} = A^{+}\mathbf{b}.$$

Proof. Let $\mathbf{x}$ be an n-vector, and let $\mathbf{y} = (y_1, \ldots, y_n)$ be $V^T\mathbf{x}$. The matrix U^T is orthogonal, because U is. Hence $\|U^T\mathbf{z}\| = \|\mathbf{z}\|$ for any m-vector $\mathbf{z}$. We have

$$\|\mathbf{b} - A\mathbf{x}\| = \left\|\mathbf{b} - U\Sigma V^T\mathbf{x}\right\| = \left\|U^T\mathbf{b} - \Sigma V^T\mathbf{x}\right\| = S_1 + S_2,$$

where

$$\begin{aligned} S_1 &= \left(\mathbf{u}_1^T\mathbf{b} - \sigma_1 y_1\right)^2 + \cdots + \left(\mathbf{u}_r^T\mathbf{b} - \sigma_r y_r\right)^2 \\ S_2 &= \left(\mathbf{u}_{r+1}^T\mathbf{b}\right)^2 + \cdots + \left(\mathbf{u}_m^T\mathbf{b}\right)^2, \end{aligned}$$

because Σ has only r nonzero entries located at the upper left $r \times r$ block.

The sum S_2 is fixed, so $\|\mathbf{b} - A\mathbf{x}\|$ is minimized if S_1 is minimum. In fact, if we could choose $\mathbf{x} = (x_1, \ldots, x_n)$ such that

$$\mathbf{u}_i^T\mathbf{b} = \sigma_i y_i, \quad i = 1, \ldots, r,$$

then S_1 would be 0. So all we need is a vector $\mathbf{x}$ of the form

$$\mathbf{x} = \left(V\frac{\mathbf{u}_1^T\mathbf{b}}{\sigma_1}, \ldots, V\frac{\mathbf{u}_r^T\mathbf{b}}{\sigma_r}, *, \ldots, *\right).$$

Any such $\mathbf{x}$ would a least squares solution, because it minimizes $\|\mathbf{b} - A\mathbf{x}\|$. To get such $\mathbf{x}$ of least magnitude, we have to set the last $n - r$ coordinates equal to 0. So

$$\tilde{\mathbf{x}} = \left(V\frac{\mathbf{u}_1^T\mathbf{b}}{\sigma_1}, \ldots, V\frac{\mathbf{u}_r^T\mathbf{b}}{\sigma_r}, 0, \ldots, 0\right)$$

is the only least squares solution of minimal length. Moreover, we may rewrite $\tilde{\mathbf{x}}$ as

$$\tilde{\mathbf{x}} = V\Sigma^{+}U^T\mathbf{b} \Rightarrow \tilde{\mathbf{x}} = A^{+}\mathbf{b}.$$ ☐

Example 9.3.9. Find the minimum length least squares solution of

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution. If A is the coefficient matrix, then by Example 9.3.7 we have

$$\tilde{\mathbf{x}} = A^+ \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{6} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{3} \end{bmatrix}.$$

9.3.4 The polar decomposition of a square matrix

An interesting and useful consequence of the SVD for a *square* matrix A is the *polar decomposition* of A.

Theorem 9.3.10 (Polar decomposition). *Any square matrix A can be factored as*

$$A = PQ, \tag{9.10}$$

where P is positive semidefinite, and Q is orthogonal. Furthermore,

1. *the matrix P is unique, and*
2. *if A is invertible, then Q is also unique.*

Proof. If A is $n \times n$, then so are U, Σ, and V in an SVD of A. Therefore,

$$A = U\Sigma V^T = U\Sigma(U^T U)V^T = (U\Sigma U^T)UV^T,$$

and we let $P = U\Sigma U^T$ and $Q = UV^T$. The rest of the proof is left as an exercise. □

The polar decomposition is analogous to writing a complex number z in polar form $z = re^{i\theta}$, where $r \geq 0$ is the magnitude of z, and θ is its argument, with $|e^{i\theta}| = 1$. In the polar decomposition, P and Q play the roles of r and $e^{i\theta}$, respectively.

Example 9.3.11. Find the polar decomposition of $A = \left[\begin{smallmatrix} -2 & 0 \\ 0 & -5 \end{smallmatrix}\right]$.

Solution. From the SVD of A we have

$$\begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T.$$

We set

$$P = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Clearly, $A = PQ$ with positive definite P and orthogonal Q.

Example 9.3.12. We can verify the following polar decomposition:

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Exercises 9.3

In Exercises 1–3, find the singular values of the matrix.

1. $\begin{bmatrix} 0 & 0 \\ 0 & -2 \\ 3 & 0 \end{bmatrix}$.

2. $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

In Exercises 4–11, find an SVD for the matrix.

4. $\begin{bmatrix} -2 & 0 \\ 0 & 0 \\ 0 & 5 \end{bmatrix}$.

5. $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

6. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

7. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

8. $\begin{bmatrix} -5 & 0 & -5 \\ 0 & 4 & 0 \\ -5 & 0 & -5 \end{bmatrix}$.

9. $\begin{bmatrix} 2 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 8 \end{bmatrix}$.

10. $\begin{bmatrix} 1 & 6 & -4 \\ -2 & 6 & 2 \\ 2 & 3 & 4 \end{bmatrix}$.

11. $\begin{bmatrix} 2 & 6 & -4 \\ -4 & 6 & 2 \\ 4 & 3 & 4 \end{bmatrix}$.

In Exercises 12–13, find an SVD by working with the transpose of the matrix.

12. $\begin{bmatrix} 2 & -4 & 4 \\ 6 & 6 & 3 \\ -4 & 2 & 4 \end{bmatrix}$.

13. $\begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 6 & -6 & 3 \end{bmatrix}$.

14. Prove that a symmetric matrix of rank r can be written as a sum of r symmetric matrices of rank 1. (*Hint:* Use SVD.)

15. Complete the proof of Theorem 9.3.10.

In Exercises 16–19, compute the pseudoinverses and verify the Moore–Penrose properties.

16. $\begin{bmatrix} -2 & 0 \\ 0 & 0 \\ 0 & 5 \end{bmatrix}^+$.

17. $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}^+$.

18. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}^+$.

19. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}^+$.

In Exercises 20–21, compute and compare A^+ and A^{-1}.

20. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

21. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

22. Let A be any matrix. Prove that the pair (A, A^+) satisfies the Moore–Penrose conditions. (*Hint:* Verify the conditions for (Σ, Σ^+) first.)

In Exercises 23–24, prove the identities by verifying the Moore–Penrose conditions.

23. $\begin{bmatrix} -3 & 0 \\ 0 & 0 \\ 0 & 4 \end{bmatrix}^+ = \begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$.

24. $\begin{bmatrix} -2 & 6 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}^{+} = \begin{bmatrix} -\frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{2}{27} & \frac{2}{27} & \frac{1}{27} \end{bmatrix}$.

25. Prove that $A^{++} = A$. (*Hint:* Verify the Moore–Penrose conditions for (A^{+}, A).)

26. Prove that $(A^{T})^{+} = (A^{+})^{T}$. Conclude that if A is symmetric, then so is A^{+}. (*Hint:* Verify the Moore–Penrose conditions for $(A^{T}, (A^{+})^{T})$.)

In Exercises 27–29, solve the least squares problem for $A\mathbf{x} = \mathbf{b}$ by using A^{+}.

27. $A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \\ 0 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

28. $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

29. $A = \begin{bmatrix} -2 & 6 \\ 1 & 6 \\ 2 & 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

In Exercises 30–33, compute the polar decomposition of A.

30. $A = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$.

31. $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$.

32. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

33. $A = \begin{bmatrix} 1 & 6 & -4 \\ -2 & 6 & 2 \\ 2 & 3 & 4 \end{bmatrix}$.

34. Let the square matrix A have polar decomposition $A = PQ$. Find an SVD for A.

9.4 Special topic: SVD and image compression

There are many and interesting applications of the SVD. The key idea upon which these applications are based is that when working with large matrices, the numerical errors in finding the SVD occur in the computation of Σ. So we may impose conditions to restrict or improve Σ as needed. Recall that in the estimation of the rank of a large matrix, we discard any unusually small singular values and use the remaining to estimate the rank.

These ideas find great use in image processing, especially in image compression and image watermarking. In image compression, we may reduce the large size of an image in a simple way as follows. The image file is a very large matrix representing the color values or the grayscale values of each pixel in the image. We compute the SVD and reduce its Σ by discarding small singular values to get a new Σ_1. Then U and V are resized by discarding all entries that correspond to the deleted singular values. The new U_1 and V_1 are used to create a new reduced image matrix A_1 by letting $A_1 = U_1 \Sigma V_1^T$. The amount of compression is a choice based on which the desired features of the picture need to be retained.

In the following example, the author used MATLAB to covert a colored photograph he had taken to first covert it to grayscale and then display it along with the rank of the grayscale matrix (Figure 9.7).

```
A = imread('Maine.jpg');
A = rgb2gray(A);
imshow(A)
title(['Original (',sprintf('Rank %d)',rank(double(A)))])
```

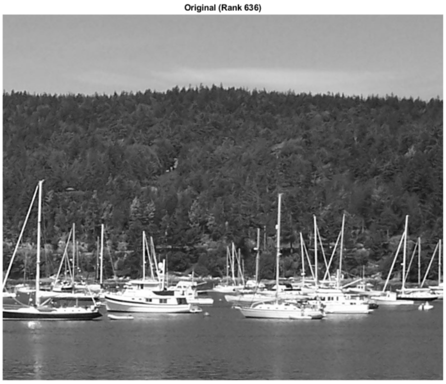

Figure 9.7: Maine. Original picture by the author converted to grayscale. The image matrix has rank 636.

The original grayscale image matrix has rank *636*. The next two images show SVD compressions of ranks *376* and *63* (Figure 9.8).

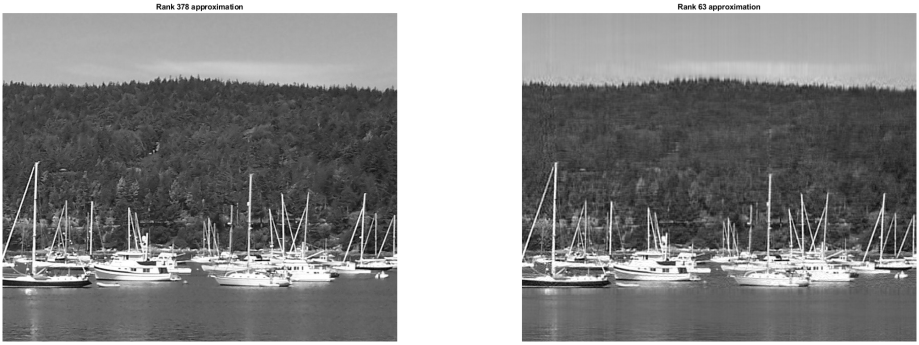

Figure 9.8: Compressed images down to ranks 376 and 63.

The MATLAB code for the compression is:

```
[U1,S1,V1] = svdsketch(double(A),1e-2);
A1 = uint8(U1*S1*V1');
imshow(uint8(A1))
title(sprintf('Rank %d approximation',size(S1,1)))

[U2,S2,V2] = svdsketch(double(A),1e-1);
A2 = uint8(U2*S2*V2');
imshow(A2)
title(sprintf('Rank %d approximation',size(S2,1)))
```

9.5 Fourier series and polynomials

This section requires basic knowledge of integration.

In many applications, we need to analyze a function (such as one representing a sound wave) in terms of its periodicity. Most functions, however, are not periodic, so we try to approximate them using periodic functions like $\sin x$ and $\cos x$. This idea goes back to *Euler*. However, it flourished with the work of *Fourier*.

Let $\mathcal{B}$ be the set of trigonometric functions defined on $[-\pi, \pi]$,

$$\mathcal{B} = \{1, \cos x, \cos 2x, \dots, \cos nx,\ \sin x, \sin 2x, \dots, \sin nx\}. \tag{9.11}$$

Definition 9.5.1. A *trigonometric polynomial* is a linear combination of elements of $\mathcal{B}$,

$$p(x) = a_0 + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx.$$

If not both a_n and b_n are zero, then we say that $p(x)$ has *order n*.

It is a basic fact that any function f in $C[-\pi, \pi]$ can be approximated by a trigonometric polynomial. “Approximated” means that f and some p are close with respect to the norm defined by the integral inner product of Example 8.5.13.

Let $T_n[-\pi, \pi]$ be the subspace of $C[-\pi, \pi]$ that consists of all trigonometric polynomials of order at most n. Then $T_n[-\pi, \pi] = \mathrm{Span}(\mathcal{B})$. We have the following basic theorem.

Theorem 9.5.2. *The set $\mathcal{B}$ defined by* (9.11) *is an orthogonal basis of $T_n[-\pi, \pi]$.*

Proof. It is clear that $\mathcal{B}$ spans $T_n[-\pi, \pi]$. We leave as an exercise the fact that $\mathcal{B}$ is linearly independent. To prove that $\mathcal{B}$ is orthogonal, we need to prove that any two distinct functions are orthogonal. So we need to verify the following relations:

$$\begin{aligned}
\langle 1, \cos nx \rangle &= 0, & n &= 1, 2, \dots, \\
\langle 1, \sin nx \rangle &= 0, & n &= 1, 2, \dots, \\
\langle \cos mx, \cos nx \rangle &= 0, & m &\neq n, \\
\langle \cos mx, \sin nx \rangle &= 0, & m, n &= 1, 2, \dots, \\
\langle \sin mx, \sin nx \rangle &= 0, & m &\neq n.
\end{aligned}$$

To prove the third identity, we have

$$\begin{aligned}
\langle \cos mx, \cos nx \rangle &= \int_{-\pi}^{\pi} \cos mx \cos nx \, dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} (\cos(m+n)x + \cos(m-n)x) \, dx \\
&= \frac{1}{2} \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n} \right]_{-\pi}^{\pi} = 0.
\end{aligned}$$

In the second step, we used a trigonometric identity. In the last step, we used the fact that $\sin(k\pi) = 0$ for integer k. The remaining identities are proved similarly by using appropriate trigonometric identities. ☐

It is easy to compute the norms of the functions of $\mathcal{B}$. For example, by using the half-angle formula we have

$$\begin{aligned}
\|\cos kx\|^2 &= \langle \cos kx, \cos kx \rangle \\
&= \int_{-\pi}^{\pi} \cos^2 kx \, dx \\
&= \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos 2kx) \, dx \\
&= \frac{1}{2} \left[x + \frac{\sin 2kx}{2k} \right]_{-\pi}^{\pi} = \pi.
\end{aligned}$$

Similarly, we can compute for $\|1\|^2$ and $\|\sin kx\|^2$. We get

$$\|1\| = \sqrt{2\pi}, \quad \|\cos kx\| = \sqrt{\pi}, \quad \|\sin kx\| = \sqrt{\pi}. \tag{9.12}$$

Now to approximate f, we need the orthogonal projection f_{pr} of f onto $T_n[-\pi, \pi]$ with respect to the orthogonal basis $\mathcal{B}$. Let

$$f_{\mathrm{pr}}(x) = a_0 + a_1 \cos x + \cdots + a_n \cos nx + b_1 \sin x + \cdots + b_n \sin nx. \qquad (9.13)$$

Then the Fourier coefficients are given (just as in the case of the dot product) by

$$a_0 = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle}, \quad a_k = \frac{\langle f, \cos kx\rangle}{\langle \cos kx, \cos kx\rangle}, \quad b_k = \frac{\langle f, \sin kx\rangle}{\langle \sin kx, \sin kx\rangle}.$$

Hence by (9.12) we get for $k \geq 1$,

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \quad k \geq 1, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx, \quad k \geq 1. \end{aligned} \qquad (9.14)$$

These formulas are due to Euler. Fourier used them in his work on the *heat equation* in physics.

The trigonometric polynomial that approximates f given by (9.13) and (9.14) is called the nth-order *Fourier polynomial* (or *Fourier approximation*) of f on the interval $[-\pi, \pi]$.

Example 9.5.3. Find the nth-order Fourier polynomial of $f(x) = x$ on $[-\pi, \pi]$.

Solution. We have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = \frac{1}{2\pi}\left.\frac{x^2}{2}\right|_{-\pi}^{\pi} dx = 0.$$

For $k \geq 1$, by integration by parts we get

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\ \cos kx dx = \frac{1}{\pi}\left[\frac{\cos kx}{k^2} + \frac{x \sin kx}{k}\right]_{-\pi}^{\pi} = 0, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\ \sin kx dx = \frac{1}{\pi}\left[\frac{\sin kx}{k^2} - \frac{x \cos kx}{k}\right]_{-\pi}^{\pi} = \frac{2(-1)^{k+1}}{k}, \end{aligned}$$

because $\cos k\pi = (-1)^k$ for any integer k. Therefore the Fourier approximation p_n of f is

$$p_n(x) = 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \cdots + \frac{2(-1)^{n+1}}{n}\sin nx.$$

Figure 9.9 shows $f(x) = x$ sketched with $p_2(x)$ and $p_3(x)$ on $[-\pi, \pi]$.

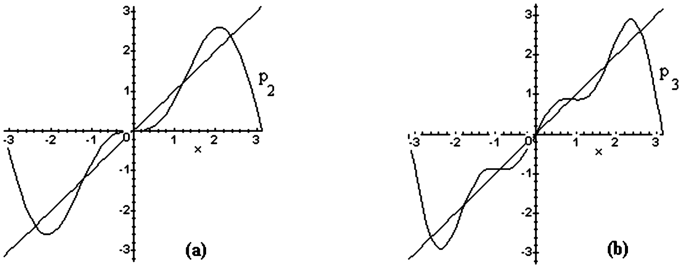

Figure 9.9: The Fourier approximations of orders 2 and 3 for x on $[-\pi, \pi]$.

As n grows, the polynomials p_n get closer to f. Taking the limit as $n \to \infty$ yields an infinite series. We write

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The right-hand side is called the *Fourier series* of f on $[-\pi, \pi]$. Fourier studied conditions on the function f under which the Fourier series converges.

Exercises 9.5

In Exercises 1–3, find the Fourier coefficients a_0, a_n, and b_n of f.

1. $f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0, \\ 1 & \text{if } 0 < x < \pi. \end{cases}$

2. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0, \\ 1 & \text{if } 0 < x < \pi. \end{cases}$

3. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0, \\ 1 & \text{if } 0 < x < \pi/2, \\ 0 & \text{if } \pi/2 < x < \pi. \end{cases}$

4. Let $f(x) = -x$ on $[-\pi, \pi]$.
(a) Find the nth-order Fourier polynomial of $f(x)$.
(b) Plot in one graph $f(x)$ and its second-order Fourier polynomial.

In Exercises 5–11, use the integral inner product on the given interval to
(a) prove that the set is orthogonal and
(b) find the norm of each function.

5. $\{\sin(x), \dots, \sin(nx)\}$ on $[0, \pi]$.

6. $\{1, \cos(x), \dots, \cos(nx)\}$ on $[0, 2\pi]$.

7. $\{\sin(\pi x), \dots, \sin(n\pi x)\}$ on $[-1, 1]$.

8. $\{1, \cos(\pi x), \dots, \cos(n\pi x)\}$ on $[-1, 1]$.

9. $\{1, \cos(\pi x), \dots, \cos(n\pi x)\}$ on $[0, 2]$.

10. $\{1, \cos(\frac{\pi x}{L}), \dots, \cos(\frac{n\pi x}{L})\}$ on $[0, L]$.

11. $\{\sin(\frac{\pi x}{L}), \dots, \sin(\frac{n\pi x}{L})\}$ on $[0, L]$.

12. (*Fourier cosine polynomials*) Consider the orthogonal set of Exercise 10. Let f be a continuous function on $[0, L]$. Then as in the case of Fourier polynomials, f can be approximated by an orthogonal projection f_{pr} of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{k} a_n \cos\left(\frac{n\pi x}{L}\right).$$

Prove that the coefficients a_n are computed by

$$a_0 = \frac{2}{L}\int_0^L f(x)\, dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx.$$

13. (*Fourier sine polynomials*) Consider the orthogonal set of Exercise 11. Let f be a continuous function on $[0, L]$. Then as in the case of Fourier polynomials, f can be approximated by an orthogonal projection f_{pr} of the form

$$f(x) = \sum_{n=1}^{k} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Prove that the coefficients b_n are computed by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$$

9.6 Application to wavelets

This section requires basic knowledge of integration.

One of the latest and most important applications of inner products is in the theory of wavelets.[6] This theory has become quite significant. It targets many of the problems that Fourier polynomials were designed to solve. These are usually problems involving waves, frequencies, amplitudes, etc. In many cases the results from using wavelets are more favorable compared with those using Fourier analysis.

Some of the main contributors of wavelet theory are *Alfred Haar* (1885–1933), *Jean Morlet* (1942–2007), *Ingrid Daubechies* (born 1954), *Yves Meyer* (born 1939), *Stéphane Mallat* (born 1961), *Ronald Coifman* (born 1938), *Terence Tao* (born 1975), as well as several other researchers.

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[6] The author is in debt to professor P. R. Turner for allowing him to read and use his notes on this topic.

We illustrate some of the highlights of the theory. Additional information is supplied in Section 9.7.1 on the miniprojects.

We define the *mother wavelet* ψ (Figure 9.10) by

$$\psi(x) = \psi_{0,0}(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/2, \\ -1 & \text{if } 1/2 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

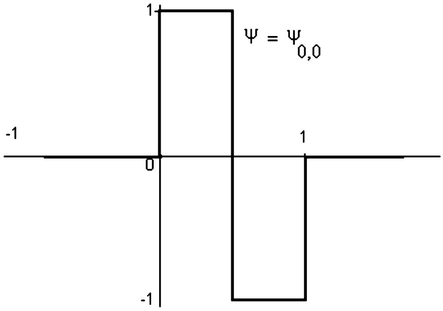

Figure 9.10: The mother wavelet.

For any pair of integers m and n, we define the *Haar* (or *basic*) *wavelets* $\psi_{m,n}$ in terms of the mother wavelet by

$$\psi_{m,n}(x) = 2^{-m/2}\psi(2^{-m}x - n).$$

As we will see in Project 1, Section 9.7.1, this is equivalent to the full definition

$$\psi_{m,n}(x) = \begin{cases} 2^{-m/2} & \text{if } 2^m n \le x \le 2^m(n+1/2), \\ -2^{-m/2} & \text{if } 2^m(n+1/2) < x \le 2^m(n+1), \\ 0 & \text{otherwise.} \end{cases}$$

Figure 9.11 shows the basic wavelets $\psi_{-2,-3}$, $\psi_{0,1}$, $\psi_{1,2}$, and $\psi_{2,2}$.

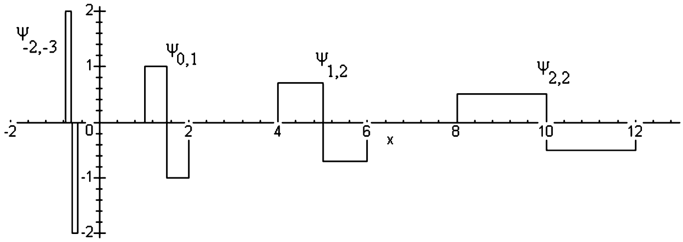

Figure 9.11: Some basic wavelets.

The interval $I_{m,n} = [2^m n, 2^m(n+1)]$, which is the only set over which $\psi_{m,n}$ is nonzero, is called the *support* of the wavelet. In general, the support of a function f is the set of points x such that $f(x) \neq 0$. For example, the support of $\psi_{-2,-3}$ is $[-3/4, -1/2]$, wheeas that of $\psi_{2,2}$ is $[8, 12]$.

For functions f and g, we consider the usual inner product, except that we integrate over the *entire real line,*

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx. \tag{9.15}$$

The indefinite integral is not always defined. However, it can be proved that it is defined for functions with finite norm

$$\|f\| = \left(\int_{-\infty}^{\infty} f(x)^2\,dx\right)^{1/2} < \infty. \tag{9.16}$$

The set of functions that satisfy this condition, denoted by L_2, is a vector space under the usual addition and scalar multiplication of functions. It is also an inner product space with (9.15) as the defining inner product. The functions of L_2 are called *L_2-functions*. The basic wavelets are L_2-functions.

The first interesting fact is that all the basic wavelets are units, i. e.,

$$\|\psi_{m,n}\| = 1$$

for all integers m and n. This is seen from the following calculation:

$$\begin{aligned}
\|\psi_{m,n}\|^2 &= \int_{-\infty}^{\infty} \psi_{m,n}(x)^2\,dx \\
&= \int_{-\infty}^{2^m n} \psi_{m,n}(x)^2\,dx + \int_{2^m n}^{2^m(n+1/2)} \psi_{m,n}(x)^2\,dx \\
&\quad + \int_{2^m(n+1/2)}^{2^m(n+1)} \psi_{m,n}(x)^2\,dx + \int_{2^m(n+1)}^{\infty} \psi_{m,n}(x)^2\,dx \\
&= 0 + \int_{2^m n}^{2^m(n+1/2)} 2^{-m}\,dx + \int_{2^m(n+1/2)}^{2^m(n+1)} 2^{-m}\,dx + 0 \\
&= 2^{-m}\,x\Big|_{2^m n}^{2^m(n+1/2)} + 2^{-m}\,x\Big|_{2^m(n+1/2)}^{2^m(n+1)} \\
&= \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

Also, any two basic wavelets are *orthogonal*. So, for $(m_1, n_1) \neq (m_2, n_2)$, we have

$$\langle \psi_{m_1,n_1}, \psi_{m_2,n_2} \rangle = \int_{-\infty}^{\infty} \psi_{m_1,n_1}(x)\, \psi_{m_2,n_2}(x)\, dx = 0. \tag{9.17}$$

The proof of this basic fact is discussed in Project 1, Section 9.7.1. We have the following theorem.

Theorem 9.6.1. *All the basic wavelets $\psi_{m,n}$ form an orthonormal set.*

Just as functions can be approximated by trigonometric polynomials, they can also be approximated by linear combinations of basic wavelets. It turns out that this is the case for all functions of L_2. If f is any L_2-function and V is the span of finitely many Haar wavelets, then the projection f_{pr} of f onto V is a linear combination

$$f_{\mathrm{pr}}(x) = \sum_{m,n} c_{m,n} \psi_{m,n}(x),$$

where m and n take on values from two finite sets. The coefficients $c_{m,n}$ are computed as usual by

$$c_{m,n} = \frac{\langle f, \psi_{m,n} \rangle}{\langle \psi_{m,n}, \psi_{m,n} \rangle} = \int_{-\infty}^{\infty} f(x) \psi_{m,n}(x)\, dx,$$

because $\langle \psi_{m,n}, \psi_{m,n} \rangle = \|\psi_{m,n}\|^2 = 1$. This time the integral is not indefinite, because the support of $\psi_{m,n}$ is a finite interval. We have

$$\begin{aligned} c_{m,n} &= \int_{2^m n}^{2^m(n+1)} f(x) \psi_{m,n}(x)\, dx \\ &= \int_{2^m n}^{2^m(n+1/2)} f(x) 2^{-m/2}\, dx + \int_{2^m(n+1/2)}^{2^m(n+1)} f(x)(-2^{-m/2})\, dx. \end{aligned}$$

So we may write

$$\begin{aligned} c_{m,n} &= A_{m,n} - B_{m,n}, \\ A_{m,n} &= 2^{-m/2} \int_{2^m n}^{2^m(n+1/2)} f(x)\, dx, \\ B_{m,n} &= 2^{-m/2} \int_{2^m(n+1/2)}^{2^m(n+1)} f(x)\, dx. \end{aligned} \tag{9.18}$$

Formulas (9.18) yield the coefficients c_{mn} of f_{pr} as linear combinations of $\psi_{m,n}$. These are analogous to formulas (9.14) that give the coefficients in the trigonometric polynomial approximation of f. To properly approximate a function f, we need to take the coefficients of *all* Haar wavelets $\psi_{m,n}$ into account, infinitely many of which may be nonzero. So just as with the Fourier series, we write f as an infinite series in terms of $\psi_{m,n}$:

$$f(x)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_{m,n}\psi_{m,n}(x).$$

Example 9.6.2. Let

$$f(x)=\begin{cases}1, & 0\le x\le 1,\\ 0 & \text{otherwise}\end{cases}$$

(Figure 9.12), and let V_k be the span of Haar wavelets

$$V_k=\mathrm{Span}\{\psi_{1,0},\psi_{2,0},\dots,\psi_{k,0}\}.$$

Approximate f by computing f_{pr} with respect to V_k.

Solution. Let $c_{1,0},\dots,c_{k,0}$ be scalars such that for all x,

$$f_{\mathrm{pr}}(x)=c_{1,0}\psi_{1,0}(x)+\cdots+c_{k,0}\psi_{k,0}(x).$$

In Project 1, Section 9.7.1, we will see that for $m=1,\dots,k$,

$$c_{m,0}=2^{-m/2}.$$

Therefore

$$f_{\mathrm{pr}}(x)=2^{-1/2}\psi_{1,0}(x)+2^{-2/2}\psi_{2,0}(x)+\cdots+2^{-k/2}\psi_{k,0}(x).$$

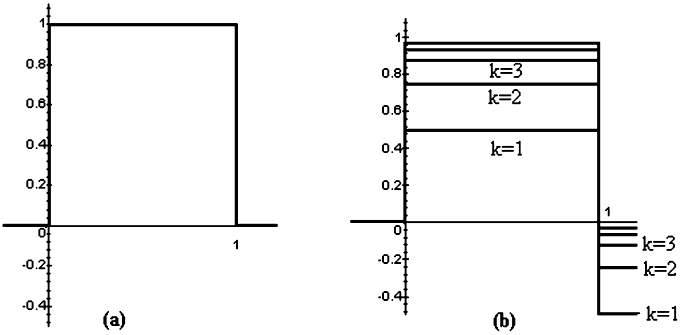

Figure 9.12: Approximation by Haar wavelets.

Figure 9.12b shows the graphs of f_{pr} for $k = 1, \ldots, 5$. It is clear that as k grows, f_{pr} approaches f very fast.

Exercises 9.6

1. Let $f(x) = \begin{cases} -1 & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$

Write f_{pr} as

$$f_{\mathrm{pr}}(x) = \sum_{m=1}^{k} c_{m,0}\psi_{m,0}(x)$$

and prove that

$$c_{m,0} = -2^{-m/2}.$$

Sketch the graphs of f and f_{pr} for (a) $k = 2$ and (b) $k = 3$.

2. Let $f(x) = \begin{cases} 1 & \text{if } -1 \le x \le 0, \\ 0 & \text{otherwise.} \end{cases}$

Write f_{pr} as

$$f_{\mathrm{pr}}(x) = \sum_{m=1}^{k} c_{m,-1}\psi_{m,-1}(x)$$

and prove that

$$c_{m,-1} = -2^{-m/2}.$$

Sketch the graphs of f and f_{pr} for (a) $k = 2$ and (b) $k = 3$.

9.7 Miniprojects

This section requires basic knowledge of integration.

9.7.1 Wavelets

In this project, you are guided to prove some claims made in wavelet theory of Section 9.6.

Problem A. Prove that the definition of the basic wavelets in terms of the mother wavelet

$$\psi_{m,n}(x) = 2^{-m/2}\psi(2^{-m}x - n)$$

is equivalent to the full definition

$$\psi_{m,n}(x) = \begin{cases} 2^{-m/2} & \text{if } 2^m n \le x \le 2^m(n+1/2), \\ -2^{-m/2} & \text{if } 2^m(n+1/2) < x \le 2^m(n+1), \\ 0 & \text{otherwise.} \end{cases}$$

Problem B. Use the steps below to prove that the basic wavelets are *orthogonal*, i. e., for $(m_1, n_1) \neq (m_2, n_2)$,

$$\langle \psi_{m_1,n_1}, \psi_{m_2,n_2} \rangle = \int_{-\infty}^{\infty} \psi_{m_1,n_1}(x)\, \psi_{m_2,n_2}(x)\, dx = 0. \tag{9.19}$$

Recall that $I_{m,n} = [2^m n, 2^m(n+1)]$ is the support of $\psi_{m,n}$.

1. If $m_1 = m_2$ and $n_1 \neq n_2$, then prove that the intersection $I_{m_1,n_1} \cap I_{m_2,n_2}$ contains at most one point.
2. If $m_1 > m_2$, then prove that either $I_{m_1,n_1} \cap I_{m_2,n_2}$ contains at most one point or I_{m_2,n_2} is contained in I_{m_1,n_1}.
3. If $(m_1, n_1) \neq (m_2, n_2)$ and if I_{m_2,n_2} is contained in I_{m_1,n_1}, then prove that I_{m_2,n_2} is contained either in $[2^{m_1}n_1, 2^{m_1}(n_1 + 1/2)]$ or in $[2^{m_1}(n_1 + 1/2), 2^{m_1}(n_1 + 1)]$.
4. Use Parts 1, 2, and 3 to prove (9.19) for $(m_1, n_1) \neq (m_2, n_2)$.

Problem C. Let

$$f(x) = \begin{cases} 1 & 0 \le x \le 1m \\ 0 & \text{otherwise,} \end{cases}$$

Let $V_k = \operatorname{Span}\{\psi_{1,0}, \psi_{2,0}, \dots, \psi_{k,0}\}$, and let V be any span of Haar wavelets containing V_k. Using the following steps, prove that the projection f_{pr} with respect to V is given by

$$f_{\text{pr}}(x) = 2^{-1/2}\psi_{1,0}(x) + 2^{-2/2}\psi_{2,0}(x) + \cdots + 2^{-k/2}\psi_{k,0}(x).$$

First, let

$$c_{m,n} = \int_{-\infty}^{\infty} f(x)\psi_{m,n}(x)\, dx = \int_0^1 \psi_{m,n}(x)\, dx.$$

1. If $m \ge 0$ and $n \neq 0$, then prove that $c_{m,n} = 0$.
2. If $m = 0$ and $n = 0$, then prove that $c_{0,0} = 0$.
3. If $m \le 0$ and $n \neq 0$, then prove that the intersection $I_{m,n} \cap [0,1]$ is either at most one point or $I_{m,n}$. Conclude that $c_{m,n} = 0$.
4. If $m > 0$ and $n = 0$, then prove that

$$c_{m,0} = 2^{-m/2}.$$

The reason we are interested in f above is that if we can prove that f can be approximated by wavelets, then so can all piecewise constant functions. These functions are *dense* in L_2, i. e., they can approximate any L_2-function. So the basic wavelets would approximate any L_2-function. The relation between f and the wavelets $\psi_{k,0}$ is a strong one. It can be proved that for all x,

$$f(x) = \sum_{m=1}^{\infty} c_{m,0}\psi_{m,0}(x).$$

9.7.2 An image compression project generated by *ChatGPT*

The following project was generated by *ChatGPT* of *OpenAI*. We include it here to give the reader a glimpse of the future, where AI will be too important to be ignored. The project itself requires both programming knowledge and access to certain libraries. This project outlines the main steps of how to apply SVD to image compression. Note that in Section 9.4 the author used a special command of MATLAB to perform the image compression. Here the steps involve a more elaborate strategy.

Project Title: Image Compression and Reconstruction by Using SVD

Objective: To implement image compression and reconstruction using singular value decomposition (SVD) and assess the trade-off between image quality and compression ratio.

Tools/Libraries: Python, NumPy (for matrix operations), PIL (Python Imaging Library) or OpenCV (for image handling).

Project Steps:

Data Collection: Choose an image that you want to compress and reconstruct. You can use any image file format (e. g., JPG, PNG).

Image to Matrix Conversion: Convert the image into a grayscale matrix. If you are using Python, then you can use a library like PIL or OpenCV for this purpose.

SVD: Apply singular value decomposition (SVD) to the grayscale image matrix. You will decompose it into three matrices: U, Σ (diagonal matrix of singular values), and V^T. You can use the *numpy.linalg.svd* function for this.

Dimension Reduction: Determine how many singular values to retain based on a certain compression ratio or quality threshold. Typically, you can retain the top k singular values and their corresponding columns in matrices U and V^T.

Compression: Use the retained matrices (U_k, Σ_k, V_k^T) to approximate the original matrix. This approximation is a compressed version of the image with reduced dimensions. Calculate the compressed image using the formula: Compressed_Image = $U_k\Sigma_k V_k^T$.

Reconstruction: Reconstruct the compressed image using the compressed matrices. This will be an approximation of the original image.

Evaluation: Compare the original image with the reconstructed image in terms of image quality (e. g., using mean squared error, peak signal-to-noise ratio) and compression ratio (original image size vs. compressed image size).

Visualization: Display the original image, compressed image, and reconstructed image for visual comparison.

Optional Extensions: Implement a user interface to select the compression ratio or quality threshold interactively. Experiment with different images and compression settings to observe the impact on image quality and file size.

9.8 Technology-aided problems and answers

In Exercises 1–3, orthogonally diagonalize the given symmetric matrix.

1. $\begin{bmatrix} -1 & -1 & 1 \\ -1 & 2 & 4 \\ 1 & 4 & 2 \end{bmatrix}$.

2. $\begin{bmatrix} 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \end{bmatrix}$.

3. $\begin{bmatrix} 1 & 1 & 1 & 9 \\ 1 & 1 & 9 & 1 \\ 1 & 9 & 1 & 1 \\ 9 & 1 & 1 & 1 \end{bmatrix}$.

In Exercises 4–7, let A be the given matrix.

(a) Find numerically an SVD of the matrix A.

(b) Verify that $U\Sigma V^T = A$.

(c) Estimate the rank of the matrix.

4. $\begin{bmatrix} 4.9 & 6.3 & 5.7 & -5.9 \\ 4.5 & -8.0 & -9.3 & 9.2 \end{bmatrix}$.

5. $\begin{bmatrix} -8.5 & -5.5 \\ -3.7 & -3.5 \\ 9.7 & 5.0 \\ 7.9 & 5.6 \end{bmatrix}$.

6. $\begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\ \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} \end{bmatrix}$.

In Exercises 8–10, let A be the given matrix. Find numerically the pseudoinverse A^+ of A. Verify the Moore–Penrose properties for (A, A^+).

8. $\begin{bmatrix} -8.4 & 1.9 & -5.0 & 8.8 & -5.3 \\ 8.5 & 4.9 & 7.8 & 1.7 & 7.2 \end{bmatrix}$.

9. $\begin{bmatrix} 1 & 4 & 69 \\ 2 & -3 & 28 \\ -3 & 2 & -37 \\ 4 & 2 & -59 \end{bmatrix}$.

10. $\begin{bmatrix} -9.9 & -8.5 \\ -8.6 & 3.0 \\ 8.0 & 7.2 \\ 6.6 & -2.9 \\ -9.1 & -5.3 \end{bmatrix}$.

11. Find a Schur decomposition for $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

In Exercises 12–13, let A be the given matrix. Find numerically a polar decomposition $A = PQ$ of A. In each case, prove that Q is orthogonal and P is positive semidefinite.

12. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$.

13. $\begin{bmatrix} 0.64 & 0.64 & 0.64 & 0.64 \\ 0.64 & 0.64 & 0.64 & 0.64 \\ 0.64 & 0.64 & 0.64 & 0.64 \\ 0.64 & 0.64 & 0.64 & 0.64 \end{bmatrix}$.

14. Let

$$f(x) = |x|.$$

(a) Find either symbolically or numerically the Fourier coefficients a_0, a_1, a_2, a_3 and b_1, b_2, b_3 for f on $[-\pi, \pi]$.

(b) Plot in the same graph f and the approximation f_{pr} obtained by using the coefficients calculated in Part (a).

15. Repeat Exercise 14 by using the function

$$f(x) = x^2 + \sin(x).$$

9.8.1 Selected solutions with Mathematica

```
(* Exercises 1--3. *)
S={{-1,-1,1},{-1,2,4},{1,4,2}}
eigsys = Eigensystem[S]         (*  *)
D1=DiagonalMatrix[eigsys[[1]]]      (*The eigenvalues on the  *)
eves=eigsys[[2]]             (*diagonal of D1.  Eigenvectors. *)
Q = Transpose[Orthogonalize[eves]]   (* Orthonormalization of *)
Transpose[Q] . S . Q          (*eigenvectors. Checking for D1 *)
Transpose[Q] . Q             (* Checking Q for orthogonality. *)
(*Likewise with the other two matrices: 6,6,6,6... 1,1,1,9 ...*)
(* Exercises 4--7. *)
A ={{4.9,6.3,5.7,-5.9},{4.5,-8.0,-9.3,9.2}}
{u,s,v}=SingularValueDecomposition[A]      (*s gives us rank 2*)
Transpose[u] . u  (* Check u and v for orthogonality. *)
Transpose[v] . v
u . s . Transpose[v] // MatrixForm(* Verification, got A back.*)
(* Likewise with 5--7. *)
(* Exercises 8--10. *)
A={{-8.4,1.9,-5.0,8.8,-5.3},{8.5,4.9,7.8,1.7,7.2}}
psA = PseudoInverse[A]     (* pseudoinverse one step.*)
A . psA . A - A                     (* Checking all  *)
psA . A . psA - psA   (* Moore-Penrose  conditions.  *)
Transpose[A . psA] - A . psA     (*  All matrices    *)
Transpose[psA . A] - psA . A          (* are zero.   *)
(* Likewise with 9--10. *)
(* Exercise 11. *)
B ={{1,2,3},{4,5,6},{7,8,9}}
Eigenvalues[B] (* First we check the eigenvalues. All real. OK.*)
sd = SchurDecomposition[N[B]];       (* Schur Decomposition.   *)
Q=sd[[1]]                            (* Q and *)
T=sd[[2]]                            (* T. *)
Transpose[Q] . Q                     (* Q is orthogonal. *)
Q . T . Transpose[Q]                 (* Checking and got B.*)
(* Exercises 12--13. *)
A ={{1,1,1,1},{2,2,2,2},{3,3,3,3},{4,4,4,4}}
{U,S,V}=SingularValueDecomposition[A] // N
P = U . S. Transpose[U]   (* Define P. *)
Q=U . Transpose[V]        (* Define Q. *)
Transpose[Q] . Q          (* Q is orthogonal. *)
Eigenvalues[P]            (* P is semidefinite. *)
P . Q                     (* PQ is A*)
(* Likewise with 13. *)
(* Exercises 14--15. *)
ff = FourierSeries[Abs[x], x, 3]
(* Fourier Series up the third terms in complex form. *)
Plot[{ff, Abs[x]}, {x, -Pi, Pi}]  (* Plotting FS and |x|. *)
(* Likewise for 15. *)
```

9.8.2 Selected solutions with MATLAB

```
% Exercises 1--3.
S=[-1 -1 1; -1 2 4; 1 4 2]
[Q,D1]=eig(S)     % The orthogonal normalization is done in 1 step!
% D1 is diagonal with the eigenvalues on the diagonal.
Q' * Q            % and Q is orthogonal.
Q' * S * Q        % The product is D as it should.
% Likewise with the other two matrices: 6,6,6,6... 1,1,1,9 ...
% Exercises 4--7.
A =[4.9 6.3 5.7 -5.9; 4.5 -8.0 -9.3 9.2]
[U,S,V] = svd(A)                % One step SVD.       Checking...
U' * U, V' * V                  % U and V are orthogonal.
U*S*V'                          % The product is A.
% Likewise with 5--7.
% Exercises 8--10.
A=[-8.4 1.9 -5.0 8.8 -5.3; 8.5 4.9 7.8 1.7 7.2]
psA = pinv(A)                   % One step computation of the pseudoinverse.
A * psA * A - A                 % Checking all
psA * A * psA - psA             % Moore-Penrose
(A * psA)' - A * psA            % conditions. All matrices
(psA * A)' - psA * A            % are approximately zero.
% Likewise with 9--10.
% Exercise 11.
B =[1 2 3; 4 5 6; 7 8 9]
eig(B)                  % First we check the eigenvalues. All real. OK.
[Q, T] = schur(B)               % Schur Decomposition, Q and T.
Q' * Q                          % Q is orthogonal.
Q * T * Q'                      % Checking and got B.
% Exercises 12--13.
A =[1 1 1 1; 2 2 2 2; 3 3 3 3; 4 4 4 4]
[U,S,V] = svd(A)                % One step SVD.
P = U*S*U'   % Define P.
Q=U*V'       % Define Q.
Q'*Q         % Q is orthogonal.
eig(P)       % P is semidefinite.
P*Q          % PQ is A.
% Exercises 14--15.
% Define the Fourier series up to a_3 and b_3.
function A=fs_abs(x)
L=pi;
fa0 =@(x) abs(x);
fa1 =@(x) abs(x).*cos(x);
fa2 =@(x) abs(x).*cos(2*x);
fa3 =@(x) abs(x).*cos(3*x);
fb1 =@(x) abs(x).*sin(x);
fb2 =@(x) abs(x).*sin(2*x);
fb3 =@(x) abs(x).*sin(3*x);
a0=(1/(2*L))*integral(fa0,-L,L)
```

```
a1=(1/L)*integral(fa1,-L,L);
a2=(1/L)*integral(fa2,-L,L);
a3=(1/L)*integral(fa3,-L,L);
b1=(1/L)*integral(fb1,-L,L);
b2=(1/L)*integral(fb2,-L,L);
b3=(1/L)*integral(fb3,-L,L);
A=a0+a1*cos(x)+a2*cos(2*x)+a3*cos(3*x)+...
b1*sin(x)+b2*sin(2*x)+b3*sin(3*x);
end
% Plot the Fourier polynomial and |x}| together
function abs_plot()
N=100;
x=linspace(-pi,pi,N);
y=fs_abs(x);
plot(x,abs(x),x,y);
end
% After saving the above function files we type
abs_plot()
% Likewise with 15.
```

9.8.3 Selected solutions with Maple

```
# Exercises 1--3.
with(LinearAlgebra);
S := Matrix([[-1,-1,1],[-1,2,4],[1,4,2]]);
v, e := Eigenvectors(S);   # Eigenvalues and eigenvectors.
D1 := DiagonalMatrix(v); # Eigenvalues on diagonal.
# Then GramSchmidt on eigenvectors, Then make them unit.
L1:=GramSchmidt([seq(Column(e,i),i=1..nops(e))]);
L2:= [seq(Normalize(L1[i],Euclidean),i=1..nops(L1))];
Q:=<L2[1]|L2[2]|L2[3]>;    # Q
Transpose(Q) . Q;          # Checking that Q is orthogonal.
Transpose(Q) . S. Q;       # Q'SQ is D1 as expected.
# Likewise with the other two matrices: 6,6,6,6... 1,1,1,9 ...
# Exercises 4--7.
with(LinearAlgebra);
A :=Matrix([[4.9,6.3,5.7,-5.9],[4.5,-8.0,-9.3,9.2]]);
U, s, Vt := SingularValues(A, output = ['U', 'S', 'Vt']);
S := DiagonalMatrix(s[1 .. 2], 2, 4);
Transpose(U) . U;   # U and V are orthogonal.
Transpose(Vt) . Vt;
U . S . Vt; # The product is A.
# Exercises 8--10.
A:=Matrix([[-8.4,1.9,-5.0,8.8,-5.3],[8.5,4.9,7.8,1.7,7.2]]);
psA:=MatrixInverse(A, method=pseudo); # Pseudoinverse
A . psA . A - A;                 # Checking all
psA . A . psA - psA;   # Moore-Penrose  conditions.
```

```
Transpose(A . psA) - A . psA ;    #  All matrices
Transpose(psA . A) - psA . A;          # are zero.
# Likewise with 9--10.
# Exercise 11.
B := Matrix([[1,2,3],[4,5,6],[7,8,9]]);
Eigenvalues(B); # First we check the eigenvalues. All real. OK.
T, Q := SchurForm(B, output = ['T', 'Z']); # Schur decomposition.
Transpose(Q).Q;            # Q is orthogonal.
Q . T . Transpose(Q);      # Checkin and got B.
# Exercises 12--13.
A:=Matrix([[1,1,1,1],[2,2,2,2],[3,3,3,3],[4,4,4,4]]);
U, S, Vt := SingularValues(A, output = ['U', 'S', 'Vt']); # SVD
P:=U. DiagonalMatrix(S) . Transpose(U); # Define P.
Q:=U.Vt;                                # Define Q.
Transpose(Q) . Q;              # Q is orthogonal.
Eigenvalues(P);                # P is semidefinite.
P . Q;                         # PQ is A.
# Exercises 14--15.
# Write functions for the Fourier coefficients, then the series.
a := proc(f,n)
int(f(x)*cos(n*Pi*x/L),x=-L..L)/L;
end;
b := proc(f,n)
int(f(x)*sin(n*Pi*x/L),x=-L..L)/L;
end;
FSS := proc(f,n)
a(f,0)/2+sum(a(f,k)*cos(k*Pi*x/L)+b(f,k)*sin(k*Pi*x/L),k=1..n);
end;
f:=x->abs(x); L:=Pi;
fss:=FSS(f,3);
plot({f(x),fss},x=-Pi..Pi);
# Likewise with 15.
```

A Introduction to complex numbers

Today most areas of mathematics, physics, and engineering use complex numbers. Complex numbers were discovered by Cardano and first mentioned in his book *Ars magna* (AD 1545). However, it was Gauss who according to G. H. Hardy, “was the first mathematician to use complex numbers in a really confident and scientific way”.

A.1 Arithmetic with complex numbers

The *imaginary unit* i or $\sqrt{-1}$ is defined by the property

$$i^2 = -1.$$

Hence

$$i^3 = -i, \quad i^4 = 1, \quad i^5 = i.$$

Example A.1.1. Compute i^{1246}.

Solution.

$$i^{1246} = i^{311\cdot 4+2} = \left(i^4\right)^{311} i^2 = 1^{311}(-1) = -1.$$

A *complex number* z is an expression of the form $z = a + bi$, where both a and b are real numbers. The set of all complex numbers is denoted by $\mathbf{C}$. The *real part* $\mathrm{Re}(z)$ of z is a. The *imaginary part* $\mathrm{Im}(z)$ of z is b. If $b = 0$, then z is a real number. If $a = 0$, then z is *pure imaginary*. The *complex conjugate* of z is

$$\bar{z} = a - ib.$$

Example A.1.2. We have

$$\mathrm{Re}(1-2i) = 1, \quad \mathrm{Im}(5-2i) = -2, \quad \overline{1-i} = 1+i, \quad \overline{-3} = -3.$$

Two complex numbers are *equal* if their respective real and imaginary parts are equal. For example, $5 + xi = y - 4i$ if and only if $y = 5$ and $x = -4$.

The *absolute value* $|z|$ of a complex number $z = a + ib$ is the nonnegative real number

$$|z| = \sqrt{a^2 + b^2}.$$

Example A.1.3. We have

$$|-2+3i| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}.$$

https://doi.org/10.1515/9783111331850-010

The *sum*, *difference*, and *product* of complex numbers is as in real numbers, with the following provisions: All powers of i are calculated; The terms are collected so that the final result in the form $a + ib$ for real a and b.

Example A.1.4. We have

$$(1 - 2i) - (2 + 3i)(-1 + i) = (1 - 2i) - (-5 - i) = 6 - i.$$

The *quotient* z/w of two complex numbers $z = a + bi$ and $w = c + di$ with $c + di \neq 0$ is the number

$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i.$$

Example A.1.5. We have

$$\frac{2 + 3i}{1 + 2i} = \frac{(2 + 3i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{8 - i}{5} = \frac{8}{5} - \frac{1}{5} i.$$

The following theorem summarizes the basic properties of complex conjugation. Its proof is left as an exercise.

Theorem A.1.6. Let z and w be complex numbers. Then

1. $z + \overline{z} = 2\mathrm{Re}(z)$;
2. $z - \overline{z} = 2\mathrm{Im}(z)i$;
3. $z\overline{z} = |z|^2$;
4. z is real if and only if $\overline{z} = z$;
5. z is pure imaginary or zero if and only if $\overline{z} = -z$;
6. $\overline{z + w} = \overline{z} + \overline{w}$;
7. $\overline{z - w} = \overline{z} - \overline{w}$;
8. $\overline{zw} = \overline{z}\overline{w}$;
9. $\overline{\frac{z}{w}} = \frac{\overline{z}}{\overline{w}}$.

A.2 Geometric interpretation of complex numbers

Every complex number $z = a + ib$ can be represented by the vector (or point) (a, b) in the plane. The x- and y-axes in this context are called the *real* and *imaginary* axes, respectively. We have the following geometric interpretations (Figure A.1).

1. The opposite $-z$ of z is the reflection of z with respect to the origin.
2. The conjugate $\overline{z}$ is the reflection with respect to the real axis.
3. Addition of two complex numbers corresponds to vector addition in $\mathbf{R}^2$.
4. Scalar multiplication by a *real* number corresponds to scalar multiplication in $\mathbf{R}^2$.
5. The absolute value $|z|$ is the length of a vector z.

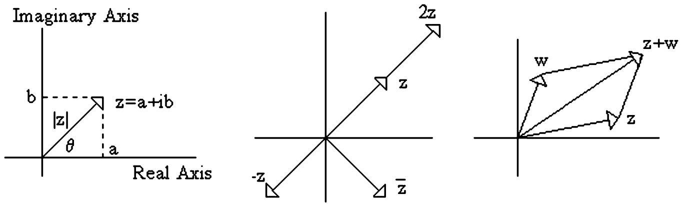

Figure A.1: Complex numbers as 2-vectors.

The angle θ between the positive real axis and the vector (a, b) representing $z = a + ib$ is called the *argument* of z. Because

$$a = |z|\cos\theta\,, \quad b = |z|\sin\theta,$$

we have

$$z = |z|(\cos\theta + i\sin\theta). \tag{A.1}$$

Equation (A.1) is called a *polar representation* of z.

Example A.2.1. Find the polar representation of $-1 + i$.

Solution. First, $|-1 + i| = \sqrt{2}$. The argument of $-1 + i$ can be computed from

$$\sqrt{2}\cos\theta = -1, \quad \sqrt{2}\sin\theta = 1,$$

which imply $\theta = 3\pi/4$. Hence

$$-1 + i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right).$$

Polar representations of complex numbers are very useful, especially in questions related to complex multiplication and division. For example, if $w = |w|(\cos\phi + i\sin\phi)$, then the product zw and the quotient z/w have polar representations:

$$\begin{aligned} zw &= |z||w|(\cos(\theta+\phi) + i\sin(\theta+\phi)), \qquad \text{(A.2)} \\ \frac{z}{w} &= \frac{|z|}{|w|}(\cos(\theta-\phi) + i\sin(\theta-\phi)). \end{aligned}$$

These identities can be proved by using the standard trigonometric identities expressing the sine and cosine of the sum or difference of two angles. Also, we may easily compute the polar representations of powers by iterating (A.2) with $w = z$. We get

$$z^n = |z|^n(\cos n\theta + i\sin n\theta). \tag{A.3}$$

Example A.2.2. Write $(-1 + i)^{10}$ in the form $a + ib$.

Solution. By equation (A.3) we have

$$\begin{aligned}(-1+i)^{10} &= (\sqrt{2})^{10}\left(\cos\left(10\cdot\frac{3\pi}{4}\right)+i\sin\left(10\cdot\frac{3\pi}{4}\right)\right)\\ &= 32\left(\cos\frac{15\pi}{2}+i\sin\frac{15\pi}{2}\right)\\ &= -32i.\end{aligned}$$

B Uniqueness of RREF

We prove that the reduced row echelon form of any matrix is unique.

Theorem 1.2.6, Section 1.2. *Every matrix is row equivalent to a unique matrix in reduced row echelon form.*

Proof. Let A be any $m \times n$ matrix. Then A has at least one reduced echelon form, computed by the Gauss elimination process.

Let N be another reduced echelon form of A. We prove that $M = N$. Firstly, M is row equivalent to N, because M is row equivalent to A and A is row equivalent to N. By Theorem 4.6.4 the columns of M and N satisfy the same linear dependence relations. Let M have k pivot columns. These columns are precisely $\mathbf{e}_1, \dots, \mathbf{e}_k$, with each $\mathbf{e}_i$ in $\mathbf{R}^m$, because M is in reduced echelon form. Moreover, a column of M (and of N) is a pivot column if and only it is *not* a linear combination of the columns to the left of it. Let $\mathbf{m}_i$ be the ith column of M.

Case 1. Let $\mathbf{m}_i$ be a pivot column. Then $\mathbf{m}_i = \mathbf{e}_j$ for some j, and $\mathbf{m}_i$ is not a linear combination of the preceding columns. Hence the same is true for the ith column $\mathbf{n}_i$ of N, because the columns of M and N satisfy the same dependence relations. So $\mathbf{n}_i$ is a pivot column of N, and because it is the jth pivot column, it follows that $\mathbf{n}_i = \mathbf{e}_j$. Therefore $\mathbf{m}_i = \mathbf{n}_i$.

Case 2. Let $\mathbf{m}_i$ be a nonpivot column. Then $\mathbf{m}_i$ is a linear combination of the preceding pivot columns by Theorem 4.6.6. So the same is true for the ith column $\mathbf{n}_i$ of N, because the columns of M and N satisfy the same dependence relations. By case 1 the pivot columns of M and N are the same, and therefore we must have $\mathbf{m}_i = \mathbf{n}_i$.

We conclude that M and N have the same columns. So $M = N$, as stated. □

https://doi.org/10.1515/9783111331850-011

Answers to selected exercises

Chapter 1

Section 1.1

1. (a), (c) linear. (b) nonlinear. (a) homogeneous. (c) nonhomogeneous.
General solution of (a) $x_1 = -4r + \frac{s}{2}, x_2 = r, x_3 = s$. Particular solutions of (a) $x_1 = 0, x_2 = 0, x_3 = 0$ and $x_1 = -4, x_2 = 1, x_3 = 0$.
General solution of (c) $x_1 = r, x_2 = 1 - s - t + u, x_3 = s, x_4 = t, x_5 = u$. Particular solutions of (c) $x_1 = 0, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0$ and $x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 1, x_5 = 1$.

3.
(a) If $a = 0$, then infinitely many. If $a = 4$, then no solutions. If $a \neq 0, 4$, then one solution,
(b) If $a = 0$, then no solutions. If $a \neq 0$, then infinitely many.

5. (a) False, (b) True, (c) False, (d) False.

7. $x_1 = 5, x_2 = 10$.

9. General solution: $(-2t - 2r - s, t, -1 + 2r, s/2, s, r), r, s, t \in \mathbf{R}$.

11.
(a) $x_1 - x_2 + x_3 - 5x_4 + 6x_5 - x_6 = 1, x_6 - x_5 = 0, 2x_5 - 2x_3 = 0$.
(b) $x_1 - x_2 + x_3 - 5x_4 + 6x_5 - x_6 = 0, x_6 - x_5 = 0, 2x_5 - 2x_3 = 0$.
(c) Exchange Rows 2 and 3.

13. $x_6 = r, x_5 = r, x_3 = r, x_4 = s, x_2 = t, x_1 = t - 6r + 5s, r, s, t$.

15. $x = -1, y = 2, z = 2$.

17. No solutions.

19. $(37/4, 17/4, 11/4)$.

21. No solutions.

23. (a) False. (b) False. (c) True. (d) False.

25. $\theta = \pi/4 + 2k\pi$, where k is any integer.

27. Direct substitution verifies the claims.

29. *Hint:* Eliminate x_1 by multiplying the top row by $-a_{21}$ and the bottom row by a_{11} and add to replace the second row. The coefficient of x_2 of the second row is $a_{11}a_{22} - a_{12}a_2$.

31. 8 solutions $(\pm\sqrt{2}, \pm 1, \pm 1)$.

33. $(1, i)$.

35. $2x^2 - x + 3$.

37. $A = \frac{1}{2}, B = -1, C = \frac{1}{2}$.

https://doi.org/10.1515/9783111331850-012

Section 1.2

1.

(a) Not echelon form.

(b) Reduced row echelon form.

3. Row echelon form but not reduced row echelon form.

5. $a = 1$, $d = 0$, and b, c are any real numbers.

7.

(a) $R_i - cR_j \to R_i$ reverses the effect of $R_i + cR_j \to R_i$.

(b) $c^{-1}R_i \to R_i$ reverses the effect of $cR_i \to R_i$.

(c) $R_i \leftrightarrow R_j$ reverses the effect of $R_i \leftrightarrow R_j$.

9. Let $A \sim B$ and $B \sim C$. Let $O_1, \dots, O_k$ be a sequence of operations that yields B from A, and let $P_1, \dots, P_r$ be a sequence of operations that yields C from B. Then the sequence $O_1, \dots, O_k, P_1, \dots, P_r$ yields C from A. Hence $A \sim C$.

11. Both A and C have I_3 as a reduced row echelon form. Hence $A \sim I$ and $C \sim I$. Therefore $A \sim I$ and $I \sim C$ by Exercise 8. Hence $A \sim C$ by Exercise 7.

13. If $a \neq 0$, then A reduces as $A \sim \begin{bmatrix} a & b \\ 0 & \frac{ad-cb}{a} \end{bmatrix} \sim \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \sim I.$

The second equivalence holds because $ad - bc \neq 0$. If $a = 0$, then switch rows and repeat.

15. True.

17. $\begin{bmatrix} 1 & 4 & 0 & 5 & 0 & 6 \\ 0 & 0 & 1 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$

19. $\begin{bmatrix} 1 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$

21.

(a) R,

(b) $[\ R \quad R\ ]$,

(c) $[\ I_n \quad A\ ]$,

(d) $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$.

23. $x = 1, y = 2, z = -2, w = -4$.

25. $x = r_1 - 2, y = -r_2 + 1, z = r_1 - r_2 - 4, w = r_2, t = r_1, r_1, r_2 \in \mathbf{R}$.

27.

(a) The last column is nonpivot, so there are solutions. Because the third column is nonpivot, there are infinitely many solutions.

(b) The last column is pivot, so there are no solutions.

29.

(a) Infinitely many solutions.

(b) One solution, the trivial solution.

31.

(a) No solutions.

(b) No solutions.

(c) If the last column is pivot, then there are no solutions. Otherwise, there is exactly one solution.

(d) If the last column is pivot, then there are no solutions. Otherwise, there are infinitely many solutions.

33. If the last column of $[A : \mathbf{b}]$ is a pivot column, then the system is inconsistent. Otherwise, the system has infinitely many solutions because it has free variables, since $m < n$.

35.

(a) If $a = 6$, then there are infinitely many solutions. If $a \neq 6$, then there is exactly one solution, namely, $x = 2, y = 0$.

(b) if $a = 8$, then there are infinitely many solutions. If $a \neq 8$, then there are no solutions.

37. False.

39. $\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{n(n+1)(2n+!)}{6}$.

41. If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 magic square, then $a + b = 5, c + d = 5, a + c = 5, b + d = 5, a + d = 5, b + c = 5$. This system is inconsistent: Equations 1 and 3 imply $b = c$. Equation 6 implies $2b = 5$. But then b would not be an integer.

Section 1.3

1. $D = -10P + 0.05I + 2C$.

3. The volumes of the solutions containing A, B, and C are $2.0\ \text{cm}^3$, $3.5\ \text{cm}^3$, and $1.8\ \text{cm}^3$.

5. $i_1 = \frac{30}{13} \simeq 2.31$, $i_2 = \frac{18}{13} \simeq 1.38$, $i_3 = \frac{12}{13} \simeq 0.92$, $i_4 = \frac{6}{13} \simeq 0.46$, $i_5 = \frac{6}{13} \simeq 0.46$ Amperes.

7. $x_1 = \frac{9}{8}, x_2 = \frac{11}{8}, x_3 = \frac{9}{8}, x_4 = \frac{11}{8}$.

9. $x_1 = 150 + r, x_2 = 200 + r, x_3 = 900 - r, x_4 = r, x_5 = 600, r \in \mathbf{R}$.

11. $-3x + 2y + z = -2$.

13. $3A - B = 1000$.

15. We get infinitely many solutions that can expressed as $x = \frac{33}{47}r, y = \frac{13}{47}r, z = \frac{1}{47}r, w = r, r \in \mathbf{R}$. Fibonacci found the particular solution $w = 47, x = 33, y = 13$, and $z = 1$, which is obtained with $r = 47$.

Section 1.4

1. $5x + y = 14, x - 2y = -6$.

3. $x = 1.98, y = 3.96$. The exact solution is $x = 2, y = 4$.

5. $x = 1.0011, y = 4.9990, z = -2.0022$.

7. $x = 2.0176, y = -1.9952, z = 2.9936$.

9. $x = 1.9999, y = -1.9996, z = 2.9999$.

11.

(a) The first five Gauss–Seidel iterates are $(2, 0)$, $(2, -2)$, $(0, -2)$, $(0, 0)$, $(2, 0)$. Because the first and last iterates are identical, these values are repeated as k grows larger. Thus the iteration diverges.

(b) The fourth Gauss–Seidel iterate is $(-0.5977, 0.5977)$, and the fifth one is $(-0.6006, 0.6006)$. So the iteration converges to $(-0.6, 0.6)$ to at least 2 decimal places.

13. $x = 3.1, y = 1.1$.

15. The solution of the modified system is $x' = 1.2, y = 4, z = -7$. Hence the solution of the given system is $x = 1200, y = 4, z = -7$.

Chapter 2

Section 2.1

1. For A: The size is 3×2, the $(2, 2)$ entry is 3, and the $(3, 1)$ entry is -2.
For B: The size is 2×3, the $(2, 2)$ entry is 2, and the $(2, 3)$ entry is 1.

3. (a) $\begin{bmatrix} -9 & 9 \\ 9 & -9 \\ -9 & 9 \end{bmatrix}$. (b) Undefined. (c) $\begin{bmatrix} -12 & 26 \\ -40 & -18 \end{bmatrix}$.

5. (a) $\begin{bmatrix} -\frac{3}{2} & \frac{5}{2} \\ 2 & -5 \end{bmatrix}$. (b) $\begin{bmatrix} \frac{1}{3} & -\frac{4}{3} \\ -\frac{13}{3} & \frac{8}{3} \end{bmatrix}$.

7. Work with the ith components of the vectors. The corresponding properties hold for real numbers.

9. Each side is $\begin{bmatrix} 5 & -1 & -3 \\ -3 & -1 & 5 \end{bmatrix}$.

11.

(a) $\text{tr}(A + B) = (a_{11} + b_{11}) + \cdots + (a_{nn} + b_{nn})$. This is $\text{tr}(A) + \text{tr}(B)$.

(b) $\text{tr}(cA) = ca_{11} + \cdots + ca_{nn}$. This is $c\text{tr}(A)$.

(c) A and A^T have the same diagonal and thus the same trace.

13. (a) and (b) $A+B$ and cA have zeros below the diagonal. These entries were sums of zeros or multiples of zero.
(c) $A + B$ and cA have zeros off the main diagonal.

15. Only C is skew-symmetric.

17. *Hint:* $(A + B)^T = A^T + B^T = -A - B$.

19. *Hint:* $A^H = A$ implies $\overline{A} = A^T$. But A and A^T have the same diagonal. Hence the same is true for $\overline{A}$ and A. Now notice that $\overline{z} = z$ implies that z is real.

21. *Hint:* $(\overline{A+B})^T = (\overline{A} + \overline{B})^T = (\overline{A})^T + (\overline{B})^T$ and $(\overline{cA})^T = \overline{c}(\overline{A})^T$. Then use the assumptions. cA is not Hermitian for nonreal c.

23. (a) True. (b) False. (c) True.

25. *Hint:* $(\overline{A+B})^T = (\overline{A} + \overline{B})^T = (\overline{A})^T + (\overline{B})^T$ and $(\overline{cA})^T = \overline{c}(\overline{A})^T$. Then use the assumptions. cA is not skew-Hermitian for nonreal c.

27. For each matrix, verify that $(\overline{A})^T = A$.

29. $(\frac{2}{3}, \frac{5}{3})$.

Section 2.2

1. $A\mathbf{u} = \begin{bmatrix} -10 & -4 & 23 \end{bmatrix}^T$. $A\mathbf{v}$ and $A\mathbf{w}$ are undefined.

3. $\mathbf{uu}$ and $\mathbf{uv}$ are undefined. $\mathbf{u}^T\mathbf{u} = [17]$. It is also acceptable to write $\mathbf{u}^T\mathbf{u} = 17$.

5. (a) $\begin{bmatrix} -2 & 9 \end{bmatrix}^T$. (b) $\begin{bmatrix} 30 & 49 \end{bmatrix}^T$.

7. We get the matrix with columns the vectors of the combination times the vector with components the coefficients $-3, 1, -2, 1$.

9. We get the product $r\mathbf{w}$, where r is the sum of the components of the vector, and $\mathbf{w}$ is one of the equal columns of A.

11. $[r, -r]^T, r \in R$.

13. $[3, -7]^T$.

15. All scalar multiples of $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

17. $\begin{bmatrix} 19-4i & -2+i \end{bmatrix}^T$.

19. $\begin{bmatrix} 16 & 26 \end{bmatrix}^T$.

21. $\begin{bmatrix} \frac{4}{3} & \frac{1}{3} \end{bmatrix}^T$.

23. $[-2r-4, r]^T, r \in R$.

25. The vector $(5s+9r+7, -6s-2r-2, s, r)$, $s, r \in R$.

27.
(a) $n = m = 2$.
(b) Domain and codomain: $\mathbf{R}^2$.
(c) $[-2r, r]^T, r \in R$.

29.
(a) $n = 2, m = 3$.
(b) Domain: $\mathbf{R}^2$. Codomain: $\mathbf{R}^3$.
(c) $[0, 0]^T$.

31.
(a) $n = 4, m = 2$.
(b) Domain: $\mathbf{R}^4$. Codomain: $\mathbf{R}^2$.
(c) $(-s + 2r, -s, s, r),\ s, r \in R$.

33. $n = 7$ and $m = 4$.

35. $\begin{bmatrix} \frac{7}{2} & \frac{1}{2} \\ 3 & -3 \end{bmatrix}$.

37. $[40, -11]^T$.

39.
(a) $|x_2|$ makes it nonlinear.
(b) T does not map zero to zero.

41. No: The codomain is $\mathbf{R}^2$. The range consists of the multiples of $\begin{bmatrix} 1 & -3 \end{bmatrix}^T$.

43. Yes. Both domain and codomain are $\mathbf{R}^3$.

45. Under this assumption, $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbf{R}^m$, with A of size $m \times n$. Hence the range and codomain are $\mathbf{R}^m$.

47. $$\begin{bmatrix} x' \\ y' \\ z' \\ t' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}.$$

Section 2.3

1. (a) False. (b) True. (c) True. (d) True.

3. (a) Yes. (b) Yes. (c) Yes. (d) Yes. (e) Yes. (f) No. (g) Yes.

5. (a) Yes. (b) No. (c) No.

7. (a) No. (b) No.

9. (a) Yes. (b) No.

11. $a \neq 0$ and $b \neq 0$.

13. *Hint:* Let V_1 and V_2 be two sets. Clearly, $V_2 \subseteq V_1$. To prove $V_1 \subseteq V_2$, notice that $\mathbf{u} = \frac{1}{2}(\mathbf{u}+\mathbf{v}) + \frac{1}{2}(\mathbf{u}-\mathbf{v})$ and $\mathbf{v} = \frac{1}{2}(\mathbf{u}+\mathbf{v}) - \frac{1}{2}(\mathbf{u}-\mathbf{v})$.

15. Each span equals $\mathbf{R}^2$.

17.
(a) No. There are 9 columns, but 10 pivots are needed to span $\mathbf{R}^{10}$.
(b) Yes. By Part (a) not all of $\mathbf{R}^{10}$ is spanned.

19. 35.

21. All $x \neq -1$.

23. The region in the first quadrant between the positive x-axis and the line $y = x$.

Section 2.4

1. Yes.

3. No.

5. $3\mathbf{v}_1 + \mathbf{v}_2 - 4\mathbf{v}_3 - 5\mathbf{v}_4 = \mathbf{0}$.

7. Yes.

9. No.

11. No.

13.
(a) Linearly dependent only for $a = -1, 2$.
(b) Not linearly dependent for all real values of a.

15. $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T$.

17. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

19. (a) False. (b) True. (c) False. (d) True.

21. $c_1 = -1$, $c_2 = -1$, $c_3 = -2$.

23. Let G be a subset of S. We take a linear combination of the vectors of G and set it equal to zero. Then we add the remaining vectors of S with zero coefficients. All c_i are zero, because S is linearly independent. Hence G is linearly independent.

25. Since the columns are linearly independent, every column is a pivot column. Hence, if the system is consistent, then it has only one solution, because there are no free variables. So any such system has at most one solution.

27. $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2$.

29. *Hint:* There are c_i not all zero such that $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. Then apply T to both sides to get $c_1T(\mathbf{v}_1) + \cdots + c_kT(\mathbf{v}_k) = \mathbf{0}$.

31. *Hint:* Set a linear combination of the pivot columns equal to zero. Since the matrix is RREF, the leading ones occur in different components of the columns.

Section 2.5

1.
(a) -20.
(b) 0.

(c) $\begin{bmatrix} -9 & -18 & 9 & 9\sqrt{3} \end{bmatrix}^T$.
(d) $\begin{bmatrix} -3+20\sqrt{2} & 6-15\sqrt{2} & -6+25\sqrt{2} \end{bmatrix}^T$.
(e) $3-5\sqrt{2}$.
(f) $\begin{bmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{\sqrt{3}} \end{bmatrix}^T$.

3. $\begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}^T$

5. $\begin{bmatrix} \frac{18\sqrt{2}}{5} & -\frac{27}{5\sqrt{2}} & \frac{9}{\sqrt{2}} \end{bmatrix}^T$.

7. $x = 13$.

9.
(a) $\mathbf{u}_{\mathrm{pr}} = (-\frac{33}{35}, -\frac{99}{35}, \frac{33}{7})$,
(b) $\mathbf{u}_{\mathrm{pr}} = (0, 0, -\frac{3}{10}, \frac{9}{10})$.

11.
(a) $33 \le \sqrt{37}\sqrt{35}$,
(b) $3 \le \sqrt{6}\sqrt{10}$.

15. *Hint:* Expand the left-hand side as $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}$.

17. *Hint:* Use $(\mathbf{u}\pm\mathbf{v})\cdot(\mathbf{u}\pm\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \pm 2\mathbf{u}\cdot\mathbf{v}$.

19. $\mathbf{u}\cdot\mathbf{v} = 3$.

21. *Hint:* $\mathbf{u}\cdot(c\mathbf{v} + d\mathbf{w}) = c\mathbf{u}\cdot\mathbf{v} + d\mathbf{u}\cdot\mathbf{w}$.

23. The plane $x + y + z = 0$.

25. For each vector $\mathbf{e}_i$, let $u_i = T(\mathbf{e}_i)$ and $\mathbf{u} = (u_1, \dots, v_n)$. For any vector $\mathbf{v} = (v_1, \dots, v_n)$ of $\mathbf{R}^n$, we have $T(\mathbf{v}) = v_1 T(\mathbf{e}_1) + \cdots + v_n T(\mathbf{e}_n) = \mathbf{u}\cdot\mathbf{v}$.

27. For the xy-plane: $(16, 4, 0)$. For the xz-plane: $(12, 0, 4)$. For the yz-plane: $(0, -12, 16)$.

29. (a) l_1 and l_3. (b) l_2 and l_3. (c) l_3 and l_4.

31. Equating the corresponding coordinates, we see that the resulting system is inconsistent. So the lines do not intersect. They are not parallel because their direction vectors are not proportional.

33. $x = 5 - 3t, y = -4 + 2t, z = 2 - t$.

35. $x + 2y + z = -1$.

37. $-3(x+4) + 2(y-2) + (z-7) = 0$.

39. $x - 6y + 2z = 21$.

41. $-x_1 + 3x_2 - 2x_3 + 8x_4 + 4x_5 + 3 = 0$.

43. *Hint:* A line through the origin is of the form $c\mathbf{v}$ for a fixed vector $\mathbf{v}$ and all scalars c. Then $T(c\mathbf{v}) = cT(\mathbf{v})$ is either *0* or another line through the origin. A line not through the origin is of the form $\mathbf{u} + l$, where $\mathbf{u}$ is some vector, and l is a line through the origin.

45. The transformation with matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ maps the x-axis to the line $y = x$ and the line $y = -x$ to the origin.

47. *Hint:* For $\mathbf{v}_1, \mathbf{v}_2$ in $\mathbf{R}^k$ with $k \geq 2$, we let $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. We take the dot product on both sides with $\mathbf{v}_1$ to get $c_1\mathbf{v}_1 \cdot \mathbf{v}_1 + c_2\mathbf{v}_2 \cdot \mathbf{v}_1 = 0$ or $c_1\mathbf{v}_1 \cdot \mathbf{v}_1 = 0$ by orthogonality. But $\mathbf{v}_1 \cdot \mathbf{v}_1$ is not zero, because $\mathbf{v}_1$ is not a zero vector by linear independence. So $c_1 = 0$. Likewise, $c_2 = 0$. The general case is proved similarly.

Section 2.6

1.

(a) $\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Rotation by 180°.

(b) $\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. None.

3.

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \end{bmatrix}$. Shear by a factor of 3 along the opposite x-direction.

(b) $\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \end{bmatrix}$. None.

5.

(a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. None.

(b) $\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. None.

7. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Projection onto the x-axis.

(b) $\begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix}$. None.

9. $T(x,y) = (-y,-x)$ and $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

11. $R(x,y) = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}$ and $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

13. $\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$.

15.

(a) Origin, $\begin{bmatrix} -x \\ -y \\ -z \end{bmatrix}$.

(b) xy-plane, $\begin{bmatrix} x \\ y \\ -z \end{bmatrix}$.

(c) x-axis, $\begin{bmatrix} x \\ -y \\ -z \end{bmatrix}$.

(d) bisecting the *xy*-plane, $\begin{bmatrix} y \\ x \\ -z \end{bmatrix}$.

(e) plane $y = x$, $\begin{bmatrix} y \\ x \\ z \end{bmatrix}$.

17.

(a) *xy*-plane, $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$.

(b) *x*-axis, $\begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$.

19. $T(\mathbf{0}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

21. $T(\mathbf{0}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $T(\mathbf{e}_2) = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, $T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $T(\mathbf{e}_4) = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

23. $\begin{bmatrix} -1 & 3 & 0 \\ 1 & 0 & -1 \\ 1 & -5 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

25. $A = \begin{bmatrix} 0 & 5 \\ 2 & -8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

27. $A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

29. *Hint:* $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ implies $L(\mathbf{x}) = T(\mathbf{x}) - \mathbf{b} = A\mathbf{x}$.

31. Any straight line of the plane can be written in the form $r\mathbf{a} + \mathbf{c}$, where $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{c}$ are fixed vectors, and r is any scalar. Then $T(r\mathbf{a} + \mathbf{c}) = A(r\mathbf{a} + \mathbf{c}) + \mathbf{b} = r(A\mathbf{a}) + (A\mathbf{c} + \mathbf{b})$. But for the fixed vectors $A\mathbf{a}$ and $A\mathbf{c} + \mathbf{b}$, the vector $r(A\mathbf{a}) + (A\mathbf{c} + \mathbf{b})$ either runs through the vectors of a straight line if $A\mathbf{a} \neq \mathbf{0}$, or it is a point if $A\mathbf{a} = \mathbf{0}$.

33. We pick two points on the line, say, $(-1, 0)$ and $(0, 1)$, and compute their images. $T(-1, 0) = (-2, 0)$ and $T(0, 1) = (-4, 5)$. Because the images are distinct, the given line maps onto the line through $(-2, 0)$ and $(-4, 5)$. Then we can sketch.

Section 2.7

1. Averaging once: $(1, \frac{5}{2}, 5, \frac{9}{2}, \frac{5}{2}, 6, 5, \frac{11}{2})$. Then again: $(\frac{1}{2}, \frac{7}{4}, \frac{15}{4}, \frac{19}{4}, \frac{7}{2}, \frac{17}{4}, \frac{11}{2}, \frac{21}{4})$. Then we plot.

3. $\begin{bmatrix} Y_{k+1} \\ A_{k+1} \end{bmatrix} = \begin{bmatrix} 2 & 10 \\ \frac{4}{5} & 0 \end{bmatrix} \begin{bmatrix} Y_k \\ A_k \end{bmatrix}$. Initial condition: $\begin{bmatrix} Y_0 \\ A_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. After 3 time units: $\begin{bmatrix} Y_3 \\ A_3 \end{bmatrix} = \begin{bmatrix} 16000 \\ 2560 \end{bmatrix}$.

5. $\begin{bmatrix} Y_{k+1} \\ A_{k+1} \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} Y_k \\ A_k \end{bmatrix}$. Initial condition: $\begin{bmatrix} Y_0 \\ A_0 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. After 3 time units: $\begin{bmatrix} Y_3 \\ A_3 \end{bmatrix} = \begin{bmatrix} 31400 \\ 3050 \end{bmatrix}$.

7. $\begin{bmatrix} Y_{k+1} \\ A_{k+1} \end{bmatrix} = \begin{bmatrix} 3 & 12 \\ \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} Y_k \\ A_k \end{bmatrix}$. Initial condition: $\begin{bmatrix} Y_0 \\ A_0 \end{bmatrix} = \begin{bmatrix} 300 \\ 100 \end{bmatrix}$. After 3 time units: $\begin{bmatrix} Y_3 \\ A_3 \end{bmatrix} = \begin{bmatrix} 30900 \\ 2500 \end{bmatrix}$.

9. $\begin{bmatrix} A_{k+1} \\ B_{k+1} \\ C_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{5}{2} & \frac{3}{2} \\ \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix} \begin{bmatrix} A_k \\ B_k \\ C_k \end{bmatrix}$. Initial condition: $\begin{bmatrix} A_0 \\ B_0 \\ C_0 \end{bmatrix} = \begin{bmatrix} 4800 \\ 4800 \\ 4800 \end{bmatrix}$. After 6 weeks: $\begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix} = \begin{bmatrix} 15975 \\ 2625 \\ 1700 \end{bmatrix}$.

Chapter 3

Section 3.1

1.

(a) $\begin{bmatrix} 7 & 14 & -35 & 21 \\ 4 & 8 & -20 & 12 \end{bmatrix}$.

(b) Impossible.

(c) $\begin{bmatrix} 5 & 10 & -3 & 4 \\ 4 & 8 & 12 & 16 \end{bmatrix}$.

(d) $\begin{bmatrix} -a & 0 & -b \\ 2a & -5c & 2b-5d \\ -3a & 4c & 4d-3b \end{bmatrix}$.

3. 1.

5. −35.

7. $\begin{bmatrix} 8 & 8 \\ -40 & -16 \end{bmatrix}$.

9. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

11.

(a) F replaces the first row with a zero row and moves rows 1, 2, 3 down by one position.

(b) F replaces the last column with a zero column and moves columns 2, 3, 4 to the left by one position.

13. Just compute to verify.

15. If A has size $m \times n$ and B has size $k \times r$, then AB is defined only if $k = n$, and BA is defined only if $r = m$. So B has size $n \times m$. Now AB has size $m \times m$, and BA has size $n \times n$.

17. (a) False. (b) True.

19. Straightforward verification.

23. $A = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$.

25. $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

27. $A = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$.

29. *Hint:* If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is such $B^2 = A$, then check that the system $a^2+bc = 0$, $ab+bd = 0$, $ac+cd = 1$, $bc + d^2 = 0$ has no solutions.

31. $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}$.

33. (a) *Hint:* Expand the square. Two of the terms are $AB + BA$. Use the assumption.

35. A and B must commute.

37. *Hint:* $A^T = A$ implies $(A^T)^2 = A^2$ or $A^TA^T = A^2$ or $(A^2)^T = A^2$. (We used $(CB)^T = B^TC^T$.)

39. False.

41. $(ABA)^H = A^HB^HA^H = ABA$. (We used $(CB)^H = B^HC^H$.)

43. False.

45. Compute the indicated.

47. *Hint:* $(A^T)^2 = A^TA^T = (A^2)^T$. (We used $(CB)^T = B^TC^T$.)

49. *Hint:* The assumption implies that $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution. Left multiply by A.

51. (a) *Hint:* Left multiply $B\mathbf{x} = \mathbf{0}$ by A to get $\mathbf{x} = \mathbf{0}$.

53. *Hint:* Use Exercise 52.

55. *Hint:* If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the trace of A^TA is $a^2 + b^2 + c^2 + d^2$. Set it equal to 0. This statement is true for a matrix A of any size.

57. The columns of the product $\begin{bmatrix} 0 & a & b & a+b \\ 0 & c & d & c+d \end{bmatrix}$ indicate the coordinates of the images of the vertices.

59. $A^2 = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 4+4i & 4-4i \\ 4-4i & 4+4i \end{bmatrix}$, $A^4 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$.

61. *Hint:* For $n = 2$, if A is the matrix, then A^2 is

$$\begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & -2\sin(\theta)\cos(\theta) \\ 2\sin(\theta)\cos(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{bmatrix}.$$

Then use the half-angle formulas.

Section 3.2

1. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & -\frac{5}{8} \\ -\frac{1}{2} & \frac{7}{8} \end{bmatrix}$.

3. We need the inverse of the given matrix: $\begin{bmatrix} 4 & -\frac{3}{2} \\ -1 & \frac{1}{2} \end{bmatrix}$.

5. A is invertible because $a^2 + b^2 = 1 \neq 0$. $A^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

7. $(2A)^3 = \begin{bmatrix} 8 & 8 \\ -40 & -16 \end{bmatrix}$. Hence $(2A)^{-3} = \begin{bmatrix} -\frac{1}{12} & -\frac{1}{24} \\ \frac{5}{24} & \frac{1}{24} \end{bmatrix}$.

9. (a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. (b) $\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$.

11. (a) $\begin{bmatrix} 3 & -2 \\ -5 & 3 \end{bmatrix}$. (b) The matrix is noninvertible.

13. (a) $\begin{bmatrix} -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ 0 & -1 & 1 \\ \frac{3}{5} & \frac{6}{5} & -\frac{7}{5} \end{bmatrix}$. (b) $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{4}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

15. $\begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$.

17. $\begin{bmatrix} a - 2a^2 - a^3 \\ a - a^2 \\ a - a^2 - a^3 \end{bmatrix}$.

19. The matrix is invertible if and only if $a \neq 2$.

21. $\begin{bmatrix} 5 & 7 \\ -2 & -3 \end{bmatrix}$.

23. It appears that the general inverse B_n has 1s on the diagonal and –1s on the superdiagonal above it and zeros elsewhere. Verify that $A_n B_n = I_n$.

25. $A = I_2$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

27. *Hint:* Left multiply $AB = BA$ by A^{-1}. Then right multiply by the same.

29. The reduction of $[A : I]$ fails to produce $[I : B]$.

31. Left multiply by A.

33. $(I-A)(I+A)=I^2-A^2=I$.

35. $A(A+2I)=A^2+2A=I$.

41. $(A^{-1})^T=(A^T)^{-1}=A^{-1}$.

43. $A^H(A^{-1})^H=(A^{-1}A)^H=I^H=I$. (We used $(CB)^H=B^HC^H$.)

45. Q and $Q'=\begin{bmatrix}1&0\\0&1\\0&1\end{bmatrix}$.

47. $n\times m$.

49. If C is a left inverse, then $CA=I$ Hence $(CA)^T=I$ or $A^TC^T=I$.

51. *Hint:* The equivalences of (b), (c), and (d) are known. It is sufficient to show that (a) is equivalent to (c). If C is a left inverse, then $CA=I$. Now

$$A\mathbf{x}=\mathbf{0}\Rightarrow CA\mathbf{x}=\mathbf{0}\Rightarrow\mathbf{x}=\mathbf{0}.$$

Section 3.3

1. A is elementary and corresponds to $R_1+R_2\to R_1$. B is not elementary.

3. Only F is elementary. The operation is $R_1-9R_2\to R_1$.

5. $-I_n$ is not elementary for $n\geq 2$. More than one row needs scaling.

7. $R_1-5R_3\to R_1$ for L and $-R_1+R_3\to R_3$ for M.

9. $R_1+5R_3\to R_1$ and $R_1+R_3\to R_3$.

11. (a) $\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$. (b) $\begin{bmatrix}1&0&-2\\0&1&0\\0&0&1\end{bmatrix}$. (c) $\begin{bmatrix}1&0&0\\0&-2&0\\0&0&1\end{bmatrix}$. (d) $\begin{bmatrix}1&0&0\\0&1&0\\5&0&1\end{bmatrix}$.

13. $\begin{bmatrix}1&0\\-\frac{1}{2}&1\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}2&0\\0&1\end{bmatrix}$ is another factorization of A.

15. $E_1=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$, $E_2=\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}$.

17. This is true, because the coefficient matrix is invertible:
$\begin{bmatrix}1&-1\\1&2\end{bmatrix}^{-1}=\begin{bmatrix}\frac{2}{3}&\frac{1}{3}\\-\frac{1}{3}&\frac{1}{3}\end{bmatrix}$.

19. $A=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}c_3&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&c_2&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&c_1\end{bmatrix}$.

21. (a) True. (b) True.

23. *Hint:* Interchanging the same rows of I twice yields I.

25. The coordinates i and j of the vector are interchanged.

27. *Hint:* Each permutation matrix is the product of elementary permutation matrices.

Section 3.4

1. First, we solve the system $y_1 = -11$, $-3y_1 + y_2 = 32$ to get $y_1 = -11\, y_2 = -1$; then we solve $4x_1 + x_2 = -11$, $-x_2 = -1$ to get $x_1 = -3$, $x_2 = 1$.

3. First, we solve the system $y_1 = 6$, $3y_1 + y_2 = 22$, $-4y_1 + 2y_2 + y_3 = -13$ to get $y_1 = 6, y_2 = 4, y_3 = 3$; then we solve $4x_1 + x_2 + x_3 = 6$, $5x_2 - x_3 = 4$, $3x_3 = 3$ to get $x_1 = 1$, $x_2 = 1$, $x_3 = 1$.

5. $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix}$.

7. $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 1 & 2 & 1 \\ 0 & 2 & -1 & 2 & 2 \\ 0 & 0 & 0 & 2 & -1 \end{bmatrix}$.

9. Using $LU = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 0 & -1 \end{bmatrix}$, we get $\mathbf{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

11. Using LU $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 5 & -1 \\ 0 & 0 & 3 \end{bmatrix}$, we get $\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$.

13. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -6 \end{bmatrix}$.

15. A $PA = LU$ factorization is

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix}.$$

So $PA\mathbf{x} = P\mathbf{b} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$ yields $\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$.

17. A $PA = LU$ factorization is given by

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}\begin{bmatrix} 2 & -5 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{bmatrix}.$$

So $PA\mathbf{x} = P\mathbf{b} = \begin{bmatrix} -8 \\ 4 \\ 2 \end{bmatrix}$ yields $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

19. By Exercise 18 the product AB of lower triangular matrices is lower triangular. It suffices to show that the (i, i) entry c_{ii} of AB is 1. We have

$$c_{ii} = \sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{i-1} a_{ik}b_{ki} + a_{ii}b_{ii} + \sum_{k=i+1}^{n} a_{ik}b_{ki} = 0 + 1 + 0 = 1,$$

because $b_{ki} = 0$ for $k < i$ and $a_{ik} = 0$ for $k > i$ and $a_{ii} = b_{ii} = 1$.

21. Let A be an $n \times n$ invertible lower triangular matrix. Let B be its inverse. We prove that B is lower triangular. The inverse of A^T is B^T. Because A^T is upper triangular, its inverse B^T is upper triangular (in each stage of the reduction of $[A^T : I]$, there are only zeros below the main diagonal of the left side). Hence B is lower triangular.

23. Let A be invertible, and let $A = LU$. We prove that an LU factorization is unique. We prove that if $L'U'$ is another LU factorization, then $L = L'$ and $U = U'$. First, we note that L has to be invertible. By Exercise 22, L^{-1} is unit lower triangular. Therefore

$$A = LU = L'U' \Rightarrow U = \left(L^{-1}L'\right)U'.$$

Now $L^{-1}L'$ is unit lower triangular by Exercise 18. The product of the lower triangular $L^{-1}L'$ with the upper triangular U' cannot be upper triangular $= U$, unless $L^{-1}L'$ is a diagonal matrix. But because it is unit triangular, it has to be the identity. Hence $L^{-1}L' = I \Rightarrow L = L'$. Therefore

$$A = LU = LU' \Rightarrow L^{-1}LU = L^{-1}LU' \Rightarrow U = U'.$$

25. Perform the multiplication.

Section 3.5

1. $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}.$

3. Perform the operations.

5. Perform the operations.

7. *Hint:* (a) and (b) Only the main diagonals and the subdiagonals are possibly nonzero.

(c) It may not be. For example, if $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then M^2 is not tridiagonal.

9. $\begin{bmatrix} 1 & -a & a^2 & -a^3 \\ 0 & 1 & -a & a^2 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

Section 3.6

1. Only A is stochastic.

3. E is doubly stochastic. F is stochastic.

5. Yes. Powers of stochastic are stochastic.

7. For A, we have $x = 0.8, y = 0.8$. For B, we have $x = \frac{3}{4}, y = \frac{1}{4}$.

9. (a) False. (b) False.

13. $(I - C)^{-1} = \begin{bmatrix} 2.5 & 2.5 \\ 0.625 & 3.125 \end{bmatrix}$. So C is productive. The production vector is $(I - C)^{-1}\mathbf{d} = \begin{bmatrix} 75 \\ 68.75 \end{bmatrix}$.

15. We need to solve $(I - C)\mathbf{x} = \mathbf{0}$. $I - C$ reduces to $\begin{bmatrix} 1 & 0. & -0.666 \\ 0 & 1 & -0.555 \\ 0 & 0 & 0 \end{bmatrix}$. The equilibrium output is then $(0.666r, 0.555r, r)$, $r \in \mathbf{R}$.

17. After one year, (1.525, 1.975) in millions. After three years, (1.362, 2.137). For example, after three years, about 2.137 million will likely be in suburbia.

Section 3.7

1. There are 3 walks from 5 to 1. The $(5,1)$-entry of $A(G)^3$ is 3.

3. There are 16 walks from C to C. The $(3,3)$-entry of $A(G)^4$ is 16.

5. There are 3 walks from A to B. The (A,B)-entry of $A(G)^3$ is 3.

7. There are 2 walks from 4 to 4. The $(4,4)$-entry of $A(G)^3$ is 2.

9. The dominance matrix $A(D)$ shows that in one-stage dominance, Individuals 2, 3, and 5 are equally influencial, each dominating 3 individuals. For the two-stage dominance, we see from $A(D)^2$ that Individual 2 has the highest sum of row entries.

Chapter 4

Section 4.1

1. The axioms hold because they hold for the real entries.

3. Closed under the operations:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & -(a_1 + a_2) \end{bmatrix}$$

and

$$r\begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix} = \begin{bmatrix} ra_1 & rb_1 \\ rc_1 & -(ra_1) \end{bmatrix}.$$

5. No. Addition is not closed:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

7. Yes, it is closed under the operations and the axioms are satisfied.

9. No. Axiom M3 fails: On one hand

$$2 \odot (1,1,1) = (1,1,2),$$

and on the other hand

$$(1+1) \odot (1,1,1) = 1 \odot (1,1,1) + 1 \odot (1,1,1) = (1,1,1) + (1,1,1) = (2,2,2).$$

11. Yes. S consists of the constant polynomials. These are closed under the operations.

13. No. S does not include the zero polynomial.

15. No. $p = 1$ and $q = x$ are in S, but $p + q = 1 + x$ is not.

17. False. $(x + x^2) + (-x^2) = x$ is not of even degree.

19. No. $2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is not in the set.

21. Yes. The set is closed under the operations.

23. Yes. The set is closed under the operations.

25. No. The sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not invertible.

27. Yes, closed under the operations: $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B) = 0 + 0 = 0$ and $\mathrm{tr}(cA) = c\,\mathrm{tr}(A) = c0 = 0$.

29. Yes, closed under the operations: For skew-symmetric A and B,

$$(A + B)^T = A^T + B^T = -A - B = -(A + B),$$

and $(cA)^T = cA^T = c(-A) = -(cA)$.

31. Yes, closed under the operations: Let B_1 and B_2 be in the set. Then

$$\begin{aligned} &[A, B_1 + B_2] = A(B_1 + B_2) - (B_1 + B_2)A = AB_1 + AB_2 - B_1A - B_2A \\ &[A, B_1] + [A, B_2] = 0 + 0 = 0. \end{aligned}$$

Similarly, $[A, cB_1] = 0$.

33. Yes. If f_1 and f_2 are odd, then the sum is odd:

$$(f_1 + f_2)(-x) = f_1(-x) + f_2(-x) = -f_1(x) - f_2(x) = -(f_1 + f_2)(x).$$

Similarly, cf_1 is odd.

35.

(a) Yes. For $f_1, f_2 \in S$,

$$(f_1 + f_2)(0) = f_1(0) + f_2(0) = f_1(1) + f_2(1) = (f_1 + f_2)(1).$$

Similarly, $(cf_1)(0) = (cf_1)(1)$.

(b) Yes. For $f_1, f_2 \in S$,

$$(f_1 + f_2)(3) = f_1(3) + f_2(3) = 0 + 0 = 0.$$

Similarly, $(cf_1)(3) = 0$.

(c) No. The zero function is not in S.

37. Part 2: *Hint:* $c\mathbf{0} + c\mathbf{0} = c(\mathbf{0} + \mathbf{0}) = c\mathbf{0}$. Add $-c\mathbf{0}$.
For Part 3: If $r = 0$, then true. If $r \neq 0$, then

$$r\mathbf{u} = r\mathbf{v} \Rightarrow r^{-1}(r\mathbf{u}) = r^{-1}(r\mathbf{v}) \Rightarrow \left(r^{-1}r\right)\mathbf{u} = \left(r^{-1}r\right)\mathbf{v} \Rightarrow 1\mathbf{u} = 1\mathbf{v} \Rightarrow \mathbf{u} = \mathbf{v}.$$

39. The union of the x-axis with the y-axis is not a subspace of $\mathbf{R}^2$. $(1,0)+(0,1) = (1,1)$ is not in the union.

41. The sum is nonempty, and for $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ in $V_1 + V_2$,

$$\mathbf{u} + \mathbf{v} = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) \in V_1 + V_2.$$

Likewise, for scalar multiplication.

43. This is $\mathbf{R}^3$, because $(a, b, c) = (a, b, 0) + (0, 0, c)$, and the intersection is the zero subspace.

45. The sum of two continuous functions is continuous. Also, a scalar times a continuous function is continuous.

47. $y_1 = 1 + x$ and $y_2 = 1 + x + e^x$ are both solutions. However, the difference $y_2 - y_1 = e^x$ is not a solution.

Section 4.2

1. (a) Yes. (b) No. (c) No.

3. No.

5. Yes.

7. Yes.

9. No.

11. Yes.

13. Yes.

15. Each polynomial in P_n is a linear combination in $1, x, \dots, x^n$.

17. Any $D \in D_n$ can be written as

$$D = a_{11}E_{11} + \cdots + a_{nn}E_{nn},$$

where a_{ii} are the diagonal entries of D.

19. $a \neq 0$ and $b \neq 0$.

21. $\mathbf{u} \pm \mathbf{v}$ are in $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$, so the second set is a subset of the first. Conversely,

$$\mathbf{u} = \frac{1}{2}\big((\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v})\big), \quad \mathbf{v} = \frac{1}{2}\big((\mathbf{u} + \mathbf{v}) - (\mathbf{u} - \mathbf{v})\big)$$

show that the first set is a subset of the second.

23. Yes. A linear combination of the polynomials leads to a polynomial with coefficients $c_2 + c_3$, $-2c_1 + c_2 - c_3$, and $c_1 + c_2$. When each is set equal to zero, the linear system has only the trivial solution.

25. Yes. A linear combination set equal to zero yields the system $c_1+c_2 = 0$, $c_1a+c_2b = 0$, $c_1a+c_2b+c_3 = 0$, which reduces to $c_3 = 0$, $c_1 + c_2 = 0$, $c_1a + c_2b = 0$. The last two equations imply $c_1 = c_2 = 0$ since $a \neq b$.

27. $a = -1$ or $a = 2$.

29. The sum of the elements equals the zero vector.

31. A linear combination equals zero leads to

$$(c_1 + c_3)\mathbf{v}_1 + (c_2 - c_1)\mathbf{v}_2 + (c_3 - c_2)\mathbf{v}_3 = \mathbf{0},$$

which implies $c_1 + c_3 = 0$, $c_2 - c_1 = 0$, $c_3 - c_2 = 0$, since $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent. The system has only the trivial solution.

33. $c_1 = -1$, $c_2 = -1$, $c_3 = -2$.

35. We may assume, possibly by renaming that the subset is $\{\mathbf{v}_1, \dots, \mathbf{v}_i\}$. Suppose

$$c_1\mathbf{v}_1 + \cdots + c_i\mathbf{v}_i = \mathbf{0}.$$

Then

$$c_1\mathbf{v}_1 + \cdots + c_i\mathbf{v}_i + 0\mathbf{v}_{i+1} \dots 0\mathbf{v}_k = \mathbf{0},$$

which implies that the coefficients are zero by the linear independence of S.

39. No. For example, $\{x + 1, 1\}$ and $\{x - 1, 1\}$ are each linearly independent. However, $\{x + 1, x - 1, 1\}$ is linearly dependent.

41. Any element of $V_1 + V_2$ is a sum of a linear combination in the $\mathbf{u}_i$s and in the $\mathbf{v}_j$s. Thus it is a linear combination in $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}_1, \dots, \mathbf{v}_n\}$.

43. If we take a linear combination

$$c_1 f + c_2 g = \mathbf{0},$$

then

$$c_1 f(x) + c_2 g(x) = 0$$

for all $x \in [0, 2\pi]$. Hence

$$c_1 f(0) + c_2 g(0) = c_1(-0.5) + c_2 0 = 0.$$

Hence $c_1 = 0$. So $c_2 g = \mathbf{0}$ or $c_2 g(x) = 0$ for all $x \in [0, 2\pi]$. But then $c_2 g(1) = 0$. So $c_2 = 0$, since $g(1) \neq 0$.

Section 4.3

1. No. Linearly dependent.

3. Yes. Four linearly independent matrices.

5. True. The matrix $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$ has 3 pivots.

7. True since

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has 4 pivots.

9. (a) $\mathcal{B} = \{2 + x + 2x^2, x^2, 1 - x - x^2\}$. (b) $V = P_2$.

11. (a) $\mathcal{B} = \{x^2, 1 + x, -1 + x^2\}$ and (b) $V = P_2$.

13. (a) $\mathcal{B} = \{-x + x^2, -5 + x, -x^2\}$. (b) $V = P_2$.

15. $\{-x + x^2, x + x^2, 1\}$.

17. $\{1 + x, -1 + x^2, x^2\}$.

19. $\{-x + x^2, -5 + x, x^2\}$.

21. True.

23. $\dim(V) = 2$.

25. $\dim(V) = 3$.

27. $\dim(V) = 2$.

29. $\dim(V) = 2$.

31. $\dim(V) = 3$.

33. $\dim(V) = 3$.

35. $\dim(V) = 2$.

37. (a) False. (b) True. (c) True. (d) False. (e) True.

39. From the graph $g = -f$. The set is linearly dependent. A basis is $\{f\}$. The dimension is 1.

41. Linearly dependent set, because $\sin(2x) = 2\sin(x)\cos(x)$ for all x. A basis is $\{\sin(x)\cos(x)\}$. The dimension is 1.

43. The dimension is $n - 1$. The solution space has $n - 1$ free variables.

47. *Hint:* Check that the union of bases of V_1 and of V_2 yields a basis for $V_1 \oplus V_2$.

49. The dimensions are
(a) mn,
(b) $(m + 1)(n + 1)$,
(c) $mnkr$.

Section 4.4

1. $p = -3 + 24x$.

3. $p = (2a - 3b) + (2a + 3b)x$.

5. $[p]_{\mathcal{B}} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$.

7. $[p]_{\mathcal{B}} = \begin{bmatrix} a - b \\ a + b \end{bmatrix}$.

9. $\begin{bmatrix} 1 & -2 & 1 & -1 \end{bmatrix}^T$.

13. $\begin{bmatrix} 3 & 2 \\ -2 & -1 \end{bmatrix}$.

15. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

17. $\begin{bmatrix} 1 & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}$.

19. $\begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{8} \end{bmatrix}$.

21. $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

23.

(a) A linear combination of the vectors in $\mathcal{B}$ set to zero yields the system $3c_1 + c_2 = 0$, $5c_1 - 9c_2 = 0$ by the linear independence of $\mathcal{A}$. So $c_1 = c_2 = 0$, and $\mathcal{B}$ is linearly independent.

(b) $\frac{1}{32}\begin{bmatrix} 9 & 1 \\ 5 & -3 \end{bmatrix}$.

(c) $\begin{bmatrix} 3 & 1 \\ 5 & -9 \end{bmatrix}$.

25. The transition matrix is $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, and the new coordinates of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ are given by $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

27. The matrix is $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the vector is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Section 4.5

1.

(a) Basis: $\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$. The nullity is 1.

(b) Basis: $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$. The nullity 1.

3.

(a) Basis: $\left\{\begin{bmatrix}-2\\1\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}3\\0\\0\\1\\0\\0\end{bmatrix}\right\}$. The nullity is 3.

(b) Basis: Empty. The nullity is 0.

5.

(a) Basis: $\left\{\begin{bmatrix}-1\\1\\1\\0\end{bmatrix},\begin{bmatrix}2\\1\\0\\1\end{bmatrix}\right\}$. The nullity is 2.

(b) Basis: $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. The nullity is 1.

7. Basis: $\left\{\begin{bmatrix}-1\\-1\\0\\0\\1\end{bmatrix},\begin{bmatrix}-4\\-2\\1\\0\\0\end{bmatrix}\right\}$. The nullity is 2.

Let N be the nullity, let P be the number of pivot columns, and let C be the number of columns of a matrix. Then

9.

(a) $N = 1, P = 1, 1 + 1 = 2 = C$;

(b) $N = 0, P = 2, 0 + 2 = 2 = C$.

11. $N = 3, P = 2, 3 + 2 = 5 = C$.

13. If $\mathbf{x} \in \text{Null}(A)$, then $A\mathbf{x} = \mathbf{0}$. Hence $BA\mathbf{x} = B\mathbf{0} = \mathbf{0}$. Therefore $\text{Null}(A) \subseteq \text{Null}(BA)$.
If $\mathbf{x} \in \text{Null}(BA)$, then $BA\mathbf{x} = \mathbf{0}$. Hence $B^{-1}BA\mathbf{x} = B^{-1}\mathbf{0} = \mathbf{0}$. So $A\mathbf{x} = \mathbf{0}$. Therefore $\text{Null}(BA) \subseteq \text{Null}(A)$.

Section 4.6

1. **a**, **b**, **c** are in the column space.

3. Only **u** and **w** are in the column space.

5. None of the vectors is in the column space.

7. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$

9. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \\ 2 \end{bmatrix} \right\}.$

11. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \\ 2 \end{bmatrix} \right\}.$

13. The column space is $\mathbf{R}^2$.

15. The null space is spanned by $\left\{ \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}^T \right\}$, and the column space is spanned by the set $\left\{ \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & -2 \end{bmatrix}^T \right\}$.

17. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

19. $\left\{ \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$

21. $\left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$

23. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$

25. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

27. Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 3 \end{bmatrix} \right\}$. The rank is 2.

29. Basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. The rank is 3.

31. $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

33. $\left\{\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}\right\}$.

35. $B \sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so the rank of B is 2. And $[B : \mathbf{b}] \sim \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ so the rank of $[B : \mathbf{b}]$ is again 2.

37. Yes. A has 450 columns and nullity 50, so its rank is 400, by the Rank Theorem. Hence, the column space has dimension 400. Therefore, the column space is all of $\mathbf{R}^{400}$. So, every 400–vector $\mathbf{b}$ is spanned by the columns of A. Therefore, $A\mathbf{x} = \mathbf{b}$ is consistent for all 400–vectors $\mathbf{b}$.

Section 4.7

1. (a) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, (c) $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

3. $A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. $A^3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

5. No, since $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$.

7. $\{(1,1,1)\}$ is a basis of the null space of A over Z_2. The only elements of the null space over Z_2 are $(1,1,1)$ and $(0,0,0)$. Over $\mathbf{R}$, the null space of A is $\{\mathbf{0}\}$, so the only basis is the empty set.

9. Since the matrices commute, $AB = BA$. Hence

$$\begin{aligned}(A+B)^2 &= (A+B)(A+B)\\ &= A^2 + AB + BA + B^2\\ &= A^2 + 2AB + B^2\\ &= A^2 + B^2,\end{aligned}$$

since $2AB$ is the zero matrix over Z_2.

11. 0101010.

13. (a) $(1,1,1,1)$, (b) $(0,1,1,1)$.

15. (a) $(0,1,1,0)$, (b) $(0,1,1,0)$.

17. (a) $(1,1,1,0)$, (b) $(1,1,1,0)$.

Chapter 5

Section 5.1

1. Linear.

3. Nonlinear.

5. Nonlinear.

9. $\begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y + 2z \\ x - y - z \end{bmatrix}$ and $\begin{bmatrix} -\frac{95}{2} \\ 0 \end{bmatrix}$.

11. $(-a - b + c) + (2a + \frac{1}{2}b + \frac{1}{2}c)x$.

13. True.

15. Because $1+x$ and $3+3x$ are linearly dependent, there is freedom to choose values on polynomials that are not multiples of $1+x$. For example, $T_1(a+bx) = a\begin{bmatrix} 1 \\ 0 \end{bmatrix}+b\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $T_2(a+bx) = a\begin{bmatrix} 0 \\ 1 \end{bmatrix}+b\begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

17. $T(a + bx) = \begin{bmatrix} a \\ a \end{bmatrix}$, and $p = 1$, $q = 1 + x$.

19. They are of the form $T(x) = cx$ for some fixed scalar c.

Section 5.2

1. Kernel basis: $\{1 + x + x^2\}$.
Range basis: $\{1 - x^2, -1 + x\}$.
Nullity is 1 and rank is 2.
Dimension theorem: $1 + 2 = \dim(P_2)$.

3. Kernel basis: $\{2 + x\}$.
Range basis: $\{x + 2x^2\}$.
Nullity is 1 and rank is 1.
Dimension theorem: $1 + 1 = \dim(P_2)$.

7. Kernel basis: the empty set.
Range basis: $\{1, x, x^2\}$.
Nullity is 0 and rank is 3.
Dimension theorem: $0 + 3 = \dim(P_2)$.

9. $\dim(\text{Kernel}(T)) = 3$.

11. The nullity is 2. Hence the rank is $4 - 2 = 2$.

13. True. The system $a + b = 0$, $a - b = 0$ has only the trivial solution.

15. True. The system $a + b = r_1$, $a + c = r_2$ is consistent for all r_1, r_2.

17. The kernel is zero: $a - 2b = 0$, $-2a + b = 0$ has only the trivial solution. It is an isomorphism, because $T : P_1 \to P_1$.

19. The kernel is zero, but T is not onto since $(0,1,0)$ cannot be an image.

21. The kernel is $\{\mathbf{0}\}$, and each $\mathbf{v}$ is its own image.

23. Not isomorphism. The domain and codomain have different dimensions.

25. A plane is of the form $\mathbf{v}+\mathcal{P}$, where $\mathcal{P}$ is a plane through the origin. A linear transformation T maps $T(\mathbf{v}+\mathcal{P}) = T(\mathbf{v})+T(\mathcal{P})$. Since T is isomorphism and $\mathcal{P}$ is a plane through the origin, then $T(\mathcal{P})$ is a plane through the origin. Hence the image of $\mathbf{v}+\mathcal{P}$ is a plane.

27. T is one-to-one: If $T(\mathbf{x}_1) = T(\mathbf{x}_2)$, then $A\mathbf{x}_1+\mathbf{b} = A\mathbf{x}_2+\mathbf{b}$. So $A\mathbf{x}_1 = A\mathbf{x}_2$, since A is invertible. So $\mathbf{x}_1 = \mathbf{x}_2$. T is onto: For any $\mathbf{y}$, there is $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{y}-\mathbf{b}$, since A is invertible. Not an isomorphism: T is not linear.

Section 5.3

1. $\begin{bmatrix} 1 & -1 \\ 0 & -2 \\ 2 & 2 \end{bmatrix}$.

3. $\begin{bmatrix} 3 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}$.

5.

(a) $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(b) $A' = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

(c) Directly, $T(6x-2x^2) = 4+6x$.
Using A:

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 6 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 0 \end{bmatrix}.$$

So $4+6x$ in terms of the standard basis.
Using A': First, $6x-2x^2$ in terms of $\mathcal{B}'$,

$$6x-2x^2 = -2\left(-x+x^2\right)+0(1+x)+4(x).$$

Then the matrix product

$$\begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

to get the coordinates of the image with respect to $\mathcal{B}'$. So $T(6x-2x^2) = 0(-x+x^2)+4(1+x)+2(x) = 4+6x$.

7.

(a) $A = \begin{bmatrix} -1 & -1 \\ 1 & -1 \\ -3 & 1 \end{bmatrix}$.

(b) Directly: $T(5-2x) = 5 - 5x + 2x^2$.
Using A:

$$5 - 2x = \frac{3}{2}(1+x) - \frac{7}{2}(-1+x).$$

So

$$\begin{bmatrix} -1 & -1 \\ 1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ -\frac{7}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -8 \end{bmatrix}.$$

Hence

$$T(5-2x) = 2\left(-x + x^2\right) + 5(1+x) - 8x = 5 - 5x + 2x^2.$$

11. $T(a + bx + cx^2) = A + Bx + Cx^2$, where $A = 2a$, $B = -a + 3b + 2c$, and $C = c$.

13. $T(a + bx + cx^2 + dx^3) = A + Bx + Cx^2 + Dx^3$, where $A = -7a + 5b$, $B = \frac{13}{2}a + \frac{1}{2}b$, $C = \frac{9}{2}b - \frac{9}{2}a$, and $D = -3a - b$.

15. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$.

A is invertible, so T is an isomorphism.

17. $B = P^{-1}AP$ for some invertible P. Hence

$$A = PBP^{-1} = \left(P^{-1}\right)^{-1}BP^{-1}.$$

Hence A is similar to B.

19. If A is invertible, then $AB = A(BA)A^{-1}$. Hence AB is similar to BA. If B is invertible, then $AB = B^{-1}(BA)B$. Hence, again, AB is similar to BA.

Section 5.4

1. $(T + L)(a + bx) = (3a + 2b) - (a + b)x$, $(T + L)(-6 + 7x) = -4 - x$, and $-3T(a + bx) = -3b + 3ax$, $-3T(-6 + 7x) = -21 - 18x$.

3. $T(x, y) = (x, 2y)$, $L(x, y) = (y, x)$.

5. T is invertible, because its standard matrix $\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 2 & -1 & 0 \end{bmatrix}$ is invertible.

7. The kernel is zero, and it is $P_1 \to P_1$, so an isomorphism.

$$T^{-1}(a + bx) = \frac{1}{2}(b - a) + \frac{1}{2}(a + b)x.$$

17. $A^2 = A$ implies $A^{-1}A^2 = I$. Hence $A = I$.

23. $T^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y \\ -\frac{1}{2}x + \frac{1}{2}y \end{bmatrix}$. So, $T^{-2}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}y \\ -\frac{1}{2}x \end{bmatrix}$ and $T^{-2}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$.

Chapter 6

Section 6.1

1. (a) 0, (b) −18.

3. (a) 0, (b) 0.

5. (a) −120, (b) 0.

7. 0.

9. $C_{11} = d, C_{12} = -c, C_{21} = -b, C_{22} = a$.

11. $\det E_1 = \det E_2 = 1$.

13. $\det E_5 = \det E_6 = -1$.

19. $x = 3, x = 6$.

21. $x = 1, x = 3$.

23. The volume of the image cube is $V \times \det A = 1 \times 2 = 2$, where A is the volume of the unit cube.

Section 6.2

1. (a) 1000, (b) 0.

3. −1.

5. 1.

7. 24.

9. Interchange Rows 1 and 3.

11. Factor out 2 from Column 1. Then to Column 1 add 2 times Column 3. The new determinant has a repeated column, so it is zero. Then $2 \times 0 = 0$.

13. Add Row 1 to Row 2. The determinant is still 3. Then factor 3 out of Row 3 to get 9.

15. Factor out 2 out each row, total of $2^3 = 8$, then times the determinant 3.

17. Multiply Row 2 by −1, then Column 2 by −1.

19. −36.

21. −32.

23.

(a) Expanding each side yields

$$ka_1b_2c_3 - ka_1b_3c_2 - ka_2b_1c_3 + ka_3b_1c_2 + ka_2b_3c_1 - ka_3b_2c_1.$$

(b) Expanding each side yields

$$a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1.$$

27. $\det A = 3 \neq 0$, invertible.

29. $\det A = 0$, noninvertible.

31. $k = -2, 0, 2$.

33. $\det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B) = \det(B)^{-1}\det(A)\det(B) = \det(A)$.

35. $\det(A)^2 = \det(A)\det(A) = \det(A^T)\det(A) = \det(A^TA)$, and A^TA is symmetric.

37. If A is $n \times n$ and skew-symmetric, then $A^T = -A$. So $\det(A) = \det(A^T) = \det(-A) = (-1)^n\det(A) = -\det(A)$, since A is odd. Hence $\det(A) = 0$.

39. Adding Columns 1 and 2 produces a column of with entries $a + b$ for Column 2. This yields a factor of $(a + b)$ in the determinant, Then Column 3 minus Column 1 yields another factor of $(b - a)$.

41. Adding Columns 1 and 2 produces a new Column 2 with entries 0, $a + b$, $a^2 + ab$, and $a^3b + a^2b$. So $(a + b)$ can be factored out of Column 2. The first row consists of 1, 0, 0, 0. Cofactor expansion about Row 1 yields a 3×3 determinant, to which we repeat this process to get out another factor of (a + b) times a 2×2 determinant.

43. If we replace Row 3 by the sum of Rows 1 and 3 and, then again by the sum of Rows 2 and 3 the new Row 3 has all its entries equal to $a + b + c$. This is the common factor.

Section 6.3

1. $\frac{1}{-2}\begin{bmatrix} 5 & -4 \\ -3 & 2 \end{bmatrix}$.

3. $\frac{1}{1}\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

5. We have the following.

(a) The operations that produce $\det(A)$ are additions, subtractions, and multiplications but never divisions. So $\det(A)$ is an integer if we start with integer entries.

(b) $\operatorname{Adj}(A)$ has integer entries, because all cofactors have integer entries as additions, subtractions, and products of integers.

(c) If $\det(A)$ divides exactly all the entries of $\operatorname{Adj}(A)$, then the matrix $\frac{1}{\det(A)}\operatorname{Adj}(A) = A^{-1}$ has to have integer entries.

(d) If $\det(A) = \pm 1$, then A^{-1} has integer entries because $\frac{1}{\pm 1}\operatorname{Adj}(A)$ does.

7. $x = \frac{4}{4} = 1, y = \frac{0}{4} = 0, z = \frac{0}{4} = 0$.

9. $x_1 = bk_2 - dk_1$ and $ck_1 - ak_2$.

11. $x_5 = \frac{\det(A_5)}{\det(A)} = \frac{1}{5}$.

13. $\det(\mathrm{Adj}(A)) = 3^3 = 27$.

15. (a)
$$\begin{aligned}\mathrm{Adj}(AB) &= \det(AB)(AB)^{-1}\\ &= \det(A)\det(B)B^{-1}A^{-1}\\ &= \det(B)B^{-1}\det(A)A^{-1}\\ &= \mathrm{Adj}(B)\mathrm{Adj}(A)\\ &= \mathrm{Adj}(A)\mathrm{Adj}(B).\end{aligned}$$

Section 6.4

1. $(2,1,3,4)$ is odd with sign -1. $(1,4,2,3)$ is even with sign 1. $(1,5,2,4,3)$ is even with sign 1. $(1,4,3,5,2)$ is even with sign 1.

3. (a) $2(3)(4) = 24$, (b) $-2(3)(4) = -24$.

5. $1(2)(3)(4)(5) = 120$.

7. $$\begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 \end{bmatrix} \leftrightarrow (4,1,2,3), \quad \begin{bmatrix} 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 \end{bmatrix} \leftrightarrow (3,2,4,1).$$

9. $$\mathrm{sign}(p) = \begin{vmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 \end{vmatrix} = 1, \ \mathrm{sign}(q) = \begin{vmatrix} 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 \end{vmatrix} = -1.$$

23. $\det(\overline{A}^T) = \det(\overline{A}) = \overline{\det(A)}$.

Section 6.5

1. $x + 2y - 3 = 0$.

3. They are not: the determinant is -2.

5. $x^2 + y^2 + 2y = 0$ or $x^2 + (y+1)^2 = 1$. The center is at $(0,-1)$, and the radius is 1.

7. $y = 2x^2 - 3x + 4$.

9. $2x - y + 2z = 3$.

11. *Hint:* The sphere passing through (x_1, y_1, z_1), centered at (a,b,c) of radius r has the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

This expands as $A(x^2 + y^2 + z^2) + Bx + Cy + Dz + E = 0$, where $A = 1$.

15. First, we rewrite the system in unknown y and parametric coefficients in x:

$$y^2 + \left(x^2 - 1\right) = 0, \quad y^2 + \left(x^2 - 2x - 1\right) = 0.$$

The Sylvester resultant is

$$\begin{vmatrix} 1 & 0 & x^2 - 1 & 0 \\ 0 & 1 & 0 & x^2 - 1 \\ 1 & 0 & x^2 - 2x - 1 & 0 \\ 0 & 1 & 0 & x^2 - 2x - 1 \end{vmatrix} = 0.$$

Hence $4x^2 = 0$. So $x = 0$, and both equations yield $y = \pm 1$. Therefore we have two solutions: $x = 0, y = 1$ and $x = 0, y = -1$.

Chapter 7

Section 7.1

1. $A\mathbf{u} = (-5, 5) = 5\mathbf{u}$, the eigenvalue is 5, and $A\mathbf{v} = (0, 0) = 0\mathbf{v}$, the eigenvalue is 0.

3. The vectors $b\mathbf{u}$ and $c\mathbf{v}$ are eigenvectors of A that belong to *different* eigenvalues. Their sum is not necessarily an eigenvector of A.

5.

(a) This is a reflection about the x-axis. The eigenvectors are along the axes only. Along the x-axis the vectors remain the same, so their eigenvalue is 1; along the y-axis the vectors get reflected about the origin, so their eigenvalue is -1,

(b) This is a projection onto the x-axis. The eigenvectors are along the axes only. Along the x-axis the vectors remain the same, so they are eigenvectors with eigenvalue 1. Along the y-axis the vectors go to zero, so they are eigenvectors with eigenvalue 0.

7.

(a) The characteristic polynomial is $\lambda^2 - 5\lambda$. The eigenvalues are 5 and 0. The corresponding bases of eigenvectors are $\{(1, 1)\}$ and $\{(2, -3)\}$. We conclude that all multiplicities are 1.

(b) The characteristic polynomial is $\lambda^2 - 3\lambda - 54$. The eigenvalues are 9 and -6. The corresponding bases of eigenvectors are $\{(1, 1)\}$ and $\{(2, -3)\}$. All multiplicities are 1.

9.

(a) The characteristic polynomial is $-\lambda^3 + \lambda^2 + \lambda - 1$. The eigenvalues are 1 with basis of eigenvectors $\{(1, 0, 1), (0, 1, 0)\}$ and -1 with basis of eigenvectors $\{(-1, 0, 1)\}$. The algebraic and geometric multiplicity of 1 is 2. The algebraic and geometric multiplicity of -1 is 1.

(b) The characteristic polynomial is $(1 - \lambda)(2 - \lambda)(3 - \lambda)$. The eigenvalues are 1, 2, 3 with corresponding bases of eigenvectors $\{(1, 0, 0)\}$, $\{(1, 1, 0)\}$, $\{(0, 0, 1)\}$. All multiplicities are 1.

11.

(a) The characteristic polynomial is $-\lambda^3 + 6\lambda^2$. The eigenvalues are 0 with basis of eigenvectors $\{(-2, 1, 0), (-3, 0, 1)\}$ and 6 with basis of eigenvectors $\{(1, 1, 1)\}$. The algebraic and geometric multiplicity of 0 is 2. The algebraic and geometric multiplicity of 6 is 1.

(b) The characteristic polynomial is $-\lambda^3 + 2\lambda^2 + 4\lambda - 8$. The eigenvalues are 2 with basis of eigenvectors $\{(1,-2,0),(0,0,1)\}$ and -2 with basis of eigenvectors $\{(1,2,0)\}$. The algebraic and geometric multiplicity of 2 is 2. The algebraic and geometric multiplicity of -2 is 1.

13. The characteristic polynomial is $x^4 - 6x^3 + 12x^2 - 10x + 3$, and it factors as $(x-3)(x-1)^3$. The eigenvalues are $\lambda = 1, 3$.

15. $(1,1,1)$ is an eigenvector with eigenvalue $\lambda = a + b + c$.

17. If the main diagonal is replaced by $a - (a - b)$, then the characteristic matrix has all its entries equal to b, so it has zero determinant. Hence $a - b$ is an eigenvalue. A basis of the eigenspace is $\{(-1,0,1),(-1,1,0)\}$.

19. If 0 is an eigenvalue, then $A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ for a nonzero vector $\mathbf{v}$. Then $\mathbf{v} \in \text{Null(A)}$. Conversely, if a nonzero $\mathbf{v}$ is in the null space, then $A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$, and zero is an eigenvalue with eigenvector $\mathbf{v}$.

21. $\mathbf{x} = -\frac{3}{7}\mathbf{v}$.

23. *Hint*: First, show that if A is invertible, then $\lambda \neq 0$. Then left multiply $A\mathbf{v} = \lambda\mathbf{v}$ by $\lambda^{-1}A^{-1}$.

25. *Hint*: First, show that $\det(P^{-1}AP - \lambda I) = \det(A - \lambda I)$.

27. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

29. If λ_i are the diagonal entries, then $A\mathbf{e}_i = \lambda_i \mathbf{e}_i$.

31. The characteristic polynomial is $\lambda^2 - (a+d)\lambda + (ad - bc)$. This has real roots if and only if $(-(a+d))^2 - 4(ad - bc) \geq 0$. Rearrange.

33. *Hint*: 0 is the only eigenvalue of A. The geometric multiplicity of 0 is the dimension of $E_0 = \text{Null}(A)$ by Exercise 19.

35. By Exercise 34 the trace is the sum of the eigenvalues, $111 + (-222) = 222 + \lambda_2$. So $\lambda_2 = -333$. For the determinant, use the second part of Exercise 34.

37. If c is not an eigenvalue of $\pm A$, then $A - cI$ and $A + cI$ are invertible. Write A as the appropriate sum.

39. $C(p) = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}$. The characteristic polynomial is $\lambda^2 + a\lambda + b$.

41. By Exercise 39 it suffices to construct a monic polynomial with roots 4 and -5 and then take its companion matrix. So $p(x) = x^2 + x - 20$, and $C(p) = \begin{bmatrix} 0 & 1 \\ 20 & -1 \end{bmatrix}$.

43. The idea comes from Exercise 40 by looking at the eigenvectors. The companion matrix of $(x-2)(x-3)(x-4) = x^3 - 9x^2 + 26x - 24$ is

$$C(p) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24 & -26 & 9 \end{bmatrix},$$

which has eigenvalues 2, 3, and 4 and corresponding basic eigenvectors $(1,2,4)$, $(1,3,9)$, and $(1,4,16)$.

45. It suffices to compute the eigenvalues and eigenvectors of the standard matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

of the projection p. The eigenvalues are 0, 1 with corresponding bases of eigenvectors $\{(0,0,1)\}$, $\{(1,0,0),(0,1,0)\}$.

47. First, we find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ of T with respect to the standard basis $\{1, x\}$. The eigenvalues are 1 and −1 with corresponding basic eigenvectors $(1,1)$ and $(-1,1)$. Hence T has eigenvalues 1 and −1 with corresponding basic eigenvectors $x+1$ and $-x+1$.

Section 7.2

1.

(a) $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$.

(b) This matrix is not diagonalizable. The only eigenvalue is −2 and has an one-dimensional eigenspace E_{-2} (with $\{(0,1)\}$ as a basis).

3.

(a) $P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 0 & -4 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -7 \end{bmatrix}$.

(b) The matrix is not diagonalizable. The eigenvalues are −2 and 1 with corresponding eigenspace bases $\{(0,1,0)\}$ and $\{(\frac{3}{2}, \frac{5}{3}, 1)\}$.

5. The eigenvalues with their eigenvectors are $\lambda = 2$ with $(0,0,0,1)$, $\lambda = -1$ with $(0,1,1,0)$, $(1,0,0,0)$, and $\lambda = 1$ with $(0,-1,1,0)$.

7. $A(10,0) = (-10,0)$, so $(10,0)$ is an eigenvector with eigenvalue −1. $A(6,5) = (24,20)$, so $(6,5)$ is an eigenvector with eigenvalue 4. Since $(10,0)$ and $(6,5)$ belong to different eigenvalues, they are linearly independent. The matrix A is diagonalizable with $P = \begin{bmatrix} 10 & 6 \\ 0 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$.

9. Let $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -10 & 0 \\ 0 & 12 \end{bmatrix}$. Then $A = PDP^{-1} = \begin{bmatrix} 1 & 11 \\ 11 & 1 \end{bmatrix}$.

11. Let $P = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Then $A = PDP^{-1} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

13. A is diagonalizable, because it has size 3×3 and has three distinct eigenvalues 2, 1, −5. D has these values in its diagonal.

15. $\{(1,0,1),(-2,1,0),(-2,0,1)\}$ (with eigenvalues 3, 0, 0).

17. A is not diagonalizable, because its only eigenvalue 0 has two basic eigenvectors. $\{(-3,2,0),(5,0,2)\}$ is a basis for E_0.

19. A is not diagonalizable, because its only eigenvalue 2 has one basic eigenvector. $\{(1,0,0,0)\}$ is a basis for E_2.

21. A is diagonalizable, so $A = PDP^{-1}$. Hence $A^2 = PD^2P^{-1}$. Thus

$$\operatorname{tr}\left(A^2\right) = \operatorname{tr}\left(PD^2P^{-1}\right) = \operatorname{tr}\left(PP^{-1}D^2\right) = \operatorname{tr}\left(D^2\right) \geq 0,$$

We used the property: $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

23. The matrix is diagonalizable over the reals if and only if $a > 0$.

25. $A^6 = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}^6 \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 4096 & 0 \\ 0 & 4096 \end{bmatrix} = \begin{bmatrix} 2^{12} & 0 \\ 0 & 2^{12} \end{bmatrix}.$

$A^9 = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}^9 \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 524288 \\ 131072 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2^{19} \\ 2^{17} & 0 \end{bmatrix}.$

27. $P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & -1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ diagonalize the matrix. Hence

$$PD^7P^{-1} = \begin{bmatrix} 2 \cdot 5^6 & 2 \cdot 5^6 & 2 \cdot 5^6 \\ 5^6 & 5^6 & 5^6 \\ 2 \cdot 5^6 & 2 \cdot 5^6 & 2 \cdot 5^6 \end{bmatrix} = \begin{bmatrix} 31250 & 31250 & 31250 \\ 15625 & 15625 & 15625 \\ 31250 & 31250 & 31250 \end{bmatrix}.$$

29. By diagonalization we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}^{k+1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2^k & 2^k \\ 2^k & 2^k \end{bmatrix}$$

for $k = 0, 1, 2, \ldots$.

31. *Hint:* The eigenvalues with the corresponding eigenvectors are $\lambda = 24 = 7 + 8 + 9$ with $(1,1,1)$ and $\lambda = 0$ with $\{(-9,0,7), (-8,7,0)\}$. D has $d_{11} = 7 + 8 + 9$ as its only nonzero entry.

Section 7.3

1.

(a) $\begin{bmatrix} 7 & 6 \\ 4 & 5 \end{bmatrix}^5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 193261 \\ 128841 \end{bmatrix}.$

(b) Also, $\mathbf{x}_5 = \frac{1}{5} \cdot 1^5 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{2}{5} \cdot 11^5 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 193261 \\ 128841 \end{bmatrix}.$

3.

(a) $\begin{bmatrix} -6 & 5 \\ 2 & -3 \end{bmatrix}^5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$

(b) Also, $\mathbf{x}_5 = 0 \cdot (-8)^5 \begin{bmatrix} -5 \\ 2 \end{bmatrix} + 1 \cdot (-1)^5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$

5.

(a) $\mathbf{x}_k = \frac{3}{2} \cdot 1^k \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{5}{2} \cdot 5^k \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(b) $A\mathbf{x}_0 = \mathbf{x}_1 = \begin{bmatrix} 11 \\ 14 \end{bmatrix}, A^2\mathbf{x}_0 = \mathbf{x}_2 = \begin{bmatrix} 61 \\ 64 \end{bmatrix}.$

(c) Neither.

7.

(a) $\mathbf{x}_k = \frac{3}{2} \cdot (-7)^k \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{5}{2} \cdot (-1)^k \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(b) $A\mathbf{x}_0 = \mathbf{x}_1 = \begin{bmatrix} 8 \\ -13 \end{bmatrix}, A^2\mathbf{x}_0 = \mathbf{x}_2 = \begin{bmatrix} -71 \\ 76 \end{bmatrix}.$

(c) Neither.

9.

(a) $\mathbf{x}_k = \frac{3}{2} \cdot 2^k \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{5}{2} \cdot 14^k \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(b) $A\mathbf{x}_0 = \mathbf{x}_1 = \begin{bmatrix} 32 \\ 38 \end{bmatrix}, A^2\mathbf{x}_0 = \mathbf{x}_2 = \begin{bmatrix} 484 \\ 496 \end{bmatrix}.$

(c) Repeller.

11.

(a) $\mathbf{x}_k = \frac{3}{2} \cdot \left(-\frac{1}{10}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{5}{2} \cdot \left(\frac{1}{2}\right)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

(b) $A\mathbf{x}_0 = \mathbf{x}_1 = \begin{bmatrix} \frac{7}{5} \\ \frac{11}{10} \end{bmatrix}, \quad A^2\mathbf{x}_0 = \mathbf{x}_2 = \begin{bmatrix} \frac{61}{100} \\ \frac{16}{25} \end{bmatrix}.$

(c) Attractor.

13. Eigenvalues 2, 8, 16. Repeller.

15. Eigenvalues 2, 8, 10. Repeller.

17. *Hint:* D has diagonal entries $-1, -5$.

19. *Hint:* D has diagonal entries $-1, -2, -8$.

21. *Hint:* D has diagonal entries $0, 0, 1, 3$.

Section 7.4

1. $\mathbf{x}_1 = (0, 4)$, $\mathbf{x}_2 = (-8, 8)$, $\mathbf{x}_3 = (-32, 0)$. The origin is a repeller.

3. This is true, because both eigenvalues $\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ are sixth roots of 1. Indeed, we have $(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i)^6 = 1$ and $(\frac{1}{2} \pm \frac{\sqrt{3}}{2} i)^k \neq 1$ for $k < 6$.

5.

(a) The matrix is not regular, because it is lower triangular, and hence all its powers are also lower triangular, so there are always entries that are zero.

(b) The matrix is not regular, because it is upper triangular, and hence all its powers are also upper triangular, so there are always entries that are zero.

7. Let $A = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}$. Solving $[A-I : \mathbf{0}]$ yields $(2r, r)$. We want $2r+r = 1$. So $r = 1/3$. Hence $\mathbf{v} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$ is the steady-state vector of A.

Let $B = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}$. Solving $[B-I : \mathbf{0}]$ yields $(r/2, r)$. We want $r/2+r = 1$. So $r = 2/3$. Hence $\mathbf{v} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ is the steady-state vector of B.

9. $(5/13, 2/13, 6/13)$.

11. The steady-state vector is $\mathbf{v} = (\frac{2}{5}, \frac{2}{5}, \frac{1}{5})$. Hence in the long run, 40 % of the rats will end up in the blue compartment, 40 % will end up in the green compartment, and 20 % will end up in the red compartment. So there is a 40 % probability that a rat in the green compartment will remain there.

13.

(a) After the third generation, there are about 118 females and 282 males.

(b) In the long run, there are 164.67 females and 235.33 males. So the males will eventually dominate the population.

15. M is regular since M^2 has only > 0 entries. The equilibrium of M is $(\frac{5}{13}, \frac{2}{13}, \frac{6}{13})$. So, in the long run, Plan C is the most popular, and Plan B is the least popular.

Section 7.6

1. 5 is an eigenvalue with eigenvector $(1, 1)$.

3. $(2, 1)$ is an eigenvector with corresponding eigenvalue 4.

5. $(2, 1)$ is an eigenvector with corresponding eigenvalue $1/4$.

7. For $\mathbf{x} = (1, 2)$, we have

$$A^3\mathbf{x} = \begin{bmatrix} 61 \\ -64 \end{bmatrix}, \qquad A^4\mathbf{x} = \begin{bmatrix} -311 \\ 314 \end{bmatrix};$$
$$\lambda_{\text{appr}} = -5.0984, \qquad \lambda = -5;$$
$$\mathbf{v}_{\text{appr}} = \begin{bmatrix} -.99 \\ 1.0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

9. For $\mathbf{x} = (1, 2)$, we have

$$A^3\mathbf{x} = \begin{bmatrix} 170 \\ -173 \end{bmatrix}, \qquad A^4\mathbf{x} = \begin{bmatrix} -1199 \\ 1202 \end{bmatrix};$$
$$\lambda_{\text{appr}} = -7.0529, \qquad \lambda = -7;$$
$$\mathbf{v}_{\text{appr}} = \begin{bmatrix} -.997 \\ 1.0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

11. For $\mathbf{x} = (1, 2)$, we have

$$A^3\mathbf{x} = \begin{bmatrix} 363 \\ -366 \end{bmatrix}, \qquad A^4\mathbf{x} = \begin{bmatrix} -3279 \\ 3282 \end{bmatrix};$$
$$\lambda_{\text{appr}} = -9.0331, \qquad \lambda = -9;$$
$$\mathbf{v}_{\text{appr}} = \begin{bmatrix} -.999 \\ 1.0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

13. For $\mathbf{x} = (1,2)$, we have

$$A^3\mathbf{x} = \begin{bmatrix} 364 \\ -148 \end{bmatrix}, \qquad A^4\mathbf{x} = \begin{bmatrix} -2924 \\ 1172 \end{bmatrix};$$

$$\lambda_{\text{appr}} = -8.033, \qquad \lambda = -8;$$

$$\mathbf{v}_{\text{appr}} = \begin{bmatrix} 1.0 \\ -.4008 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ -\frac{2}{5} \end{bmatrix}.$$

15. Starting at $(1,2)$, the first 4 iterations yield -1.4000, -3.9412, -4.9432, -4.9977. So the dominant eigenvalue is -5.

17. Starting at $(1,2)$, the first 4 iterations yield -1.6000, -6.0690, -6.9776, -6.9995. So the dominant eigenvalue is -7.

19. Starting at $(1,2)$, the first 4 iterations yield 0.2800, 0.7882, 0.9886, 0.9995. The true eigenvalue closest to the origin is 1 and is approximated by 0.9995.

21. Starting at $(1,2)$, the first 4 iterations yield 0.2286, 0.8670, 0.9968, 0.9999. The true eigenvalue closest to the origin is 1 and is approximated by 0.9999.

23. Starting at $(1,2)$, the first 4 iterations yield 0.2000, 0.9111, 0.9988, 1.0000. The true eigenvalue closest to the origin is 1 and is approximated to 4 decimal places by 1.0000.

25. Starting at $(1,2)$, the first 4 iterations yield -1.1111, -1.0194, -1.0022, -1.0022. The true eigenvalue closest to the origin is -1 and is approximated by -1.0022.

27. Applying the inverse power method to $\begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}$ starting at $(1,0)$ yields -1.0027 after 4 iterations. So the root nearest to 5 is $5 + (-1.0027)^{-1} = 4.0027$.

Chapter 8

Section 8.1

1. All pairs of dot products are zero. The set is an orthogonal basis of $\mathbf{R}^3$.

3. All pairs of dot products are zero. The set is an orthogonal basis of $\mathbf{R}^4$.

5. Every pair of vectors has dot product zero, and hence the vectors are linearly independent. Hence they form an orthogonal basis of $\mathbf{R}^3$. We have

$$(1,1,1) = \frac{9}{41}\mathbf{v}_1 + \frac{1}{5}\mathbf{v}_2 + \frac{8}{205}\mathbf{v}_3.$$

7. Every pair of vectors has dot product zero, and hence the vectors are linearly independent. Thus they form an orthogonal basis of $\mathbf{R}^3$.

$$(1,1,1) = 0\mathbf{v}_1 + \frac{1}{7}\mathbf{v}_2 + \frac{1}{7}\mathbf{v}_3.$$

9. The vectors are orthogonal but not orthonormal. Dividing each vector by its length yields the orthonormal vectors

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

11. $\mathcal{B}$ is an orthonormal basis for $\mathbf{R}^2$ since both vectors are unit and their dot product is zero.

$$\mathbf{e}_1 = \frac{2}{\sqrt{5}}\mathbf{v}_1 - \frac{1}{\sqrt{5}}\mathbf{v}_2.$$

13. (a), (b), (c), (d) True.

15. Orthogonal. Its inverse is

$$\begin{bmatrix} \frac{3}{\sqrt{14}} & -\frac{2}{\sqrt{14}} & \frac{1}{\sqrt{14}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{21}} & -\frac{1}{\sqrt{21}} & \frac{4}{\sqrt{21}} \end{bmatrix}.$$

17. Perform the multiplication. A is orthogonal only if $a^2 + b^2 + c^2 + d^2 = 1$.

19. $a^2 + c^2 = 1$, $b^2 + d^2 = 1$, and $ab + cd = 0$.

21. Since the columns are orthogonal, they are linearly independent. If $m < n$, then the columns would have to be dependent.

29. The relation $H^T H = nI$ shows that the dot products of all pairs of different columns of H are zero. Hence the columns form an orthogonal set. However, H is not an orthogonal matrix, because the columns are not unit. Each column has length $\sqrt{n}$. Hence $\frac{1}{\sqrt{n}}H$ is an orthogonal matrix.

31. Use the Pythagorean theorem and mathematical induction.

33. The projection does not preserve the lengths of vectors. So it is not orthogonal.

35. The rotation is an orthogonal transformation. It preserves the lengths of vectors and angles between vectors.

Section 8.2

1. $\mathbf{u}_{\text{pr}} = (0, 0)$.

3. (a) $\mathbf{u}_{\text{pr}} = (-\frac{4}{11}, -\frac{4}{11}, \frac{12}{11})$. (b) $\|(\frac{37}{11}, -\frac{7}{11}, \frac{10}{11})\| = \frac{\sqrt{1518}}{11}$.

5. $\mathbf{u} = (-\frac{2}{13}, -\frac{3}{13}) + (-\frac{24}{13}, \frac{16}{13})$.

7. $\mathbf{u} = (-1, 1) + (2, 2)$.

9. $\mathbf{u}_{\text{pr}} = (0, 0, 1)$.

13. Orthogonal:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{4}{3} \\ \frac{7}{3} \\ -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{1}{11} \\ \frac{1}{11} \\ \frac{3}{11} \end{bmatrix} \right\}.$$

Orthonormal:

$$\left\{\begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} \frac{4}{\sqrt{66}} \\ \frac{7}{\sqrt{66}} \\ -\frac{1}{\sqrt{66}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ \frac{3}{\sqrt{11}} \end{bmatrix}\right\}.$$

15. $\left\{\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} \frac{1}{21} \\ \frac{32}{21} \\ \frac{68}{21} \end{bmatrix}\right\}.$

17. A basis for the column space is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subseteq R^4$. This is an orthonormal basis for the column space.

19. A basis for the row space is $\{(1, 1, 1, -1), (0, 1, -1, 0)\}$. An orthonormal basis is

$$\left\{\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right), \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)\right\}.$$

21. First, we apply Gram–Schmidt to get an orthogonal basis for the span: $\{(1, 1, -1, 1), (\frac{5}{4}, -\frac{3}{4}, \frac{3}{4}, \frac{1}{4})\}$. Then we compute the projection $\mathbf{u}_{\text{pr}}$ onto V: $\mathbf{u}_{\text{pr}} = (\frac{27}{11}, -\frac{3}{11}, \frac{3}{11}, \frac{12}{11})$. Thus $\mathbf{u}_c = (-\frac{16}{11}, \frac{14}{11}, -\frac{14}{11}, -\frac{1}{11})$.

23. (a) True. (b) False.

25. $\|\mathbf{u}_c\| = \left\|\begin{bmatrix} \frac{8}{15} \\ -\frac{8}{3} \\ -\frac{16}{15} \end{bmatrix}\right\| = \frac{8}{15}\sqrt{30}.$

27. An orthonormal basis of the null space is

$$\left\{\left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\right\}.$$

$\mathbf{u}_{\text{pr}} = (\frac{3}{2}, 0, 0, \frac{3}{2})$, $\mathbf{u}_c = (-\frac{1}{2}, -1, 0, \frac{1}{2})$. So the distance is $\|\|\mathbf{u}_c\|\| = \sqrt{\frac{3}{2}}$.

Section 8.3

1. $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}.$

3. (a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 0 & -2 \end{bmatrix}$. (b) $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{3\sqrt{2}}{2} \end{bmatrix}.$

5. (a) $\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \end{bmatrix}\begin{bmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}.$

(b) $\begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} \sqrt{5} & 0 & \frac{4}{\sqrt{5}} \\ 0 & 2 & 0 \\ 0 & 0 & \frac{3}{\sqrt{5}} \end{bmatrix}.$

7. The matrix has orthogonal columns. So to get Q, we need to normalize each column. Let $g = \sqrt{a^2 + b^2}$. Then $g \neq 0$. We get

$$QR = \begin{bmatrix} \frac{a}{g} & \frac{b}{g} \\ -\frac{b}{g} & \frac{a}{g} \end{bmatrix} \begin{bmatrix} g & 0 \\ 0 & g \end{bmatrix}.$$

9. $\begin{bmatrix} 2 & 0 \\ 0 & \sqrt{6} \end{bmatrix}$.

11. If A is orthogonal, then $QR = AI$.

13. The matrix has orthogonal columns. So to get Q, we need to normalize each column. Let $g = \sqrt{a^2 + b^2 + c^2 + d^2}$. Then $g \neq 0$. We get $QR = (\frac{1}{g}A)D$, where D is diagonal with diagonal entries equal to g.

15. *Hint:* If $\mathbf{v}$ is a solution of the first system, then $R\mathbf{v}$ is a solution of the second. The converse also holds. A and Q have the same column space.

17. $A_1 = \begin{bmatrix} 10.822 & 7.2177 \\ 0.21792 & 2.1783 \end{bmatrix}$, $A_2 = \begin{bmatrix} 10.966 & 7.0420 \\ 0.042638 & 2.0332 \end{bmatrix}$, $A_3 = \begin{bmatrix} 10.993 & 7.0055 \\ 0.0061491 & 2.0048 \end{bmatrix}$.

The estimated eigenvalues are 10.993, 2.0048. The true eigenvalues are 11, 2. The errors are 0.007, −0.0048.

21. $H = \begin{bmatrix} 0 & \frac{3}{5} & \frac{4}{5} \\ \frac{3}{5} & \frac{16}{25} & -\frac{12}{25} \\ \frac{4}{5} & -\frac{12}{25} & \frac{9}{25} \end{bmatrix}$, $H\mathbf{v} = 5\mathbf{e}_1$.

25. $R = \begin{bmatrix} 2\sqrt{2} & 0 & 2\sqrt{2} \\ 0 & -2 & 0 \\ 0 & 0 & -6\sqrt{2} \end{bmatrix}$.

Section 8.4

1. The normal equations

$$\begin{bmatrix} 3 & -1 \\ -1 & 9 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

yield $\tilde{\mathbf{x}} = \begin{bmatrix} -\frac{6}{13} \\ \frac{8}{13} \end{bmatrix}$.

3. The normal equations

$$\begin{bmatrix} 10 & 5 \\ 5 & 6 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

yield $\tilde{\mathbf{x}} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}$.

5. $y = \frac{7}{2}x + \frac{4}{3}$.

7. $y = \frac{51}{59}x + \frac{14}{59}$.

9. Solving the normal equations

$$\begin{bmatrix} 5 & 1 & -3 \\ 1 & 1 & 1 \\ -3 & 1 & 5 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} -2 \\ 2 \\ 6 \end{bmatrix},$$

we get $\tilde{\mathbf{x}} = (r-1, -2r+3, r)$, $r \in \mathbf{R}$.

11. The least squares solutions are $(-\frac{1}{5} - 2s, s, r)$, where $s, t \in \mathbf{R}$.

13. The least squares spring constant is $k = 3.92$.

15. The system $R\tilde{\mathbf{x}} = Q^T\mathbf{b}$,

$$\begin{bmatrix} 5 & 10 \\ 0 & 5 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} -4 \\ \frac{9}{5} \end{bmatrix},$$

yields $\tilde{\mathbf{x}} = (-\frac{38}{25}, \frac{9}{25})$.

17. From the system $R\tilde{\mathbf{x}} = Q^T\mathbf{b}$ we get

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \tilde{\mathbf{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \Rightarrow \tilde{\mathbf{x}} = \left(\frac{1}{2}, 2\right).$$

19. The denominators are all 1.

21. $(\mathbf{a}_1 \cdot \mathbf{b})/(\mathbf{a}_1 \cdot \mathbf{a}_1) = \frac{1}{2}$ and $(\mathbf{a}_2 \cdot \mathbf{b})/(\mathbf{a}_2 \cdot \mathbf{a}_2) = 2$ to get $(\frac{1}{2}, 2)$.

23. If $\tilde{\mathbf{x}} = (x_1, x_2, x_3)$, then

$$x_1 = \frac{a-b-c-d}{a^2+b^2+c^2+d^2} \quad x_2 = \frac{a+b+c-d}{a^2+b^2+c^2+d^2} \quad x_3 = \frac{a-b+c+d}{a^2+b^2+c^2+d^2}$$

Section 8.5

1.
(a) $-12, \sqrt{19}, 2\sqrt{21}, \sqrt{79}$.
(b) $\|(-3,-5)\| = \sqrt{127}$.
(c) $\sqrt{19} \cdot 2\sqrt{21} \simeq 39.95 \geq |-12| = 12$.
(d) $\sqrt{79} \simeq 8.8882 \leq \sqrt{19} + 2\sqrt{21} \simeq 13.524$.
(e) $\frac{1}{4}(79 - 127) = -12$.

3. $(1, \frac{3}{8})$.

5. A and B are orthogonal. Also, A and C are orthogonal.

7. The orthonormal pairs are (A', B') and (A', C'), where

$$A' = \frac{1}{\|A\|}A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad B' = \frac{1}{\|B\|}B = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad C' = \frac{1}{\|C\|}C = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

9. It is, since A is positive definite.

11. It is not. f is not positive definite, since $f(\mathbf{u}, \mathbf{u}) = 0$ for $\mathbf{u} = (-1, 1)$.

13. It is not. f is not symmetric, since for $\mathbf{u} = (1, 2)$ and $\mathbf{v} = (2, 1)$, we have $f(\mathbf{u}, \mathbf{v}) = 31$, whereas $f(\mathbf{v}, \mathbf{u}) = 49$.

15.
(a) $-2, \sqrt{5}, \sqrt{5}, \sqrt{6}$.
(b) $\|1 + 2x - 3x^2\| = \sqrt{14}$.
(c) $|-2| = 2 \leq \sqrt{5}\sqrt{5} = 5$.
(d) $\sqrt{14} \simeq 3.7417 \leq \sqrt{5} + \sqrt{5} \simeq 4.4721$.

17. $2 + x^2$.

19. Let $w(x) = a + bx + cx^2$ be such that

$$p(-3)w(-3) + p(0)w(0) + p(2)w(2) = 0$$

or $-23a + 37b - 181c = 0$. Hence $w(x) = -144 + 23x + 23x^2$.

21. No, $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = -2$.

27. *Hint:* $T(\mathbf{v}) = \mathbf{0}$ implies $\|T(\mathbf{v})\| = 0$.

31. $(1, 1)_{\text{pr}} = (\frac{2}{3}, 0)$ from the previous exercise. The distance from $(1, 1)$ to l is

$$\left\|(1,1)_{\text{c}}\right\| = \left\|(1,1) - (1,1)_{\text{pr}}\right\| == \left\|(1,1) - \left(\frac{2}{3}, 0\right)\right\| = \left\|\left(\frac{1}{3}, 1\right)\right\| = \sqrt{\frac{7}{3}}.$$

33. $\{1, x, x^2 - \frac{8}{3}\}$ is an orthogonal basis of P_2 for this inner product.

35. Since

$$\int_{-1}^{1} 1^2 dx = 2, \quad \int_{-1}^{1} x^2 dx = \frac{2}{3}, \quad \int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right)^2 dx = \frac{2}{5}, \quad \int_{-1}^{1} \left(\frac{5}{2}x^3 - \frac{3}{2}x\right)^2 dx = \frac{2}{7}.$$

For an orthonormal basis, divide by the square roots of the above to get $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{3}{2}}x$, $\sqrt{\frac{5}{8}}(3x^2 - 1)$, etc.

Section 8.6

1. We have $|\mathbf{u}.\mathbf{v}| = 8\sqrt{5}$, $\|\mathbf{u}\| = \sqrt{6}$, $\|\mathbf{v}\| = \sqrt{62}$, $\|\mathbf{u} + \mathbf{v}\| = 10$.
(a) $8\sqrt{5} \leq \sqrt{6}\sqrt{62}$,
(b) $10 \leq \sqrt{6} + \sqrt{62}$.

3. (b) Not linear: $T(c\mathbf{v}) = \langle \mathbf{u}, c\mathbf{v} \rangle = \overline{c}\langle \mathbf{u}, \mathbf{v} \rangle$, so that $T(c\mathbf{v}) \neq cT(\mathbf{v})$.

5. *Hint:* $\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \overline{\langle \mathbf{u}, \mathbf{v} \rangle} = 2\text{Re}\langle \mathbf{u}, \mathbf{v} \rangle$.

7. (a) True. (b) False. (c) False. (d) True. (e) False.

9. Yes. $A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

11. Yes. $A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

13. Yes. $A^{-1} = \begin{bmatrix} 0 & 0 & -i \\ 0 & -i & 0 \\ -i & 0 & 0 \end{bmatrix}$.

15.
(a) $A^H A = I$.
(b) The eigenvalues are i, $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, $\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$.
(c) They all have absolute value 1.

Section 8.7

1. $q(x) = -\frac{22}{5} + \frac{23}{70}x + \frac{13}{14}x^2$, and $q(6) = 31$.

3. $q(x) = \frac{134}{35} + 2x + \frac{2}{7}x^2$, and $q(6) = \frac{914}{35} \simeq 26.114$.

5. $q(x) = -\frac{34}{35} - \frac{11}{4}x + \frac{11}{14}x^2 + \frac{5}{4}x^3$, $q(3) = \frac{158}{5} \simeq 31.6$.

9. $y = -\frac{13}{6} + 3x$.

11. $y = -\frac{8}{5} + \frac{18}{5}x$.

13. $y = \frac{2}{5} - \frac{12}{5}x + 3x^2$.

15. The least squares quadratic is $4.9t^2 - 2.96t + 4.9$. So $g = 9.8$, $v_0 = -2.6$, $x_0 = 4.9$.

Chapter 9

Section 9.1

1. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$.

3. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$.

5. $Q = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

7. $Q = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$, $D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

9. $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

15. This is because a symmetric matrix is diagonalizable.

17. *Hint:* By the last exercise a skew-symmetric matrix has eigenvalues that are zero or pure imaginary. Thus -1 cannot be an eigenvalue of A.

19. $Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $T = \begin{bmatrix} 1 & -6 \\ 0 & 11 \end{bmatrix}$.

21. $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$.

23. $3 \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} \\ -\frac{3}{\sqrt{10}} \end{bmatrix}^T + 13 \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}^T$.

25. $5 \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}^T + 15 \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}^T$.

Section 9.2

1. $q(\mathbf{x}) = -2x^2 + 3y^2 + 4xy$.

3. $q(\mathbf{x}) = x^2 + 5z^2 - 6xy + 4xz + 2yz$.

5. $\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$.

7. $\begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}$.

9. $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

11. $\begin{bmatrix} 5 & -4 & 0 \\ -4 & 3 & 6 \\ 0 & 6 & 1 \end{bmatrix}$.

13. The matrix of the form is $\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$, which is orthogonally diagonalized by

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Let $\mathbf{y} = (x', y')$. Then

$$\mathbf{y} = Q^T\mathbf{x} = \begin{bmatrix} \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \\ -\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \end{bmatrix},$$

and

$$q(\mathbf{x}) = \mathbf{y}^T \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{y} = 2x'^2 + 4y'^2.$$

15. The matrix of the form is $\begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$, which is orthogonally diagonalized by

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix}$$

Let $\mathbf{y} = (x', y')$. Then

$$\mathbf{y} = Q^T\mathbf{x} = \begin{bmatrix} \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \\ -\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \end{bmatrix}$$

and

$$q(\mathbf{x}) = \mathbf{y}^T \begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix} \mathbf{y} = -2x'^2 - 6y'^2$$

17. The matrix of the form is $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$, which is orthogonally diagonalized by

$$Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let $\mathbf{y} = (x', y', z')$. Then

$$\mathbf{y} = Q^T\mathbf{x} = \begin{bmatrix} y \\ \frac{x}{\sqrt{2}} + \frac{z}{\sqrt{2}} \\ -\frac{x}{\sqrt{2}} + \frac{z}{\sqrt{2}} \end{bmatrix},$$

and

$$q(\mathbf{x}) = \mathbf{y}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y} = y'^2 + 3z'^2.$$

19. The matrix of the form is $\begin{bmatrix} 5 & -4 & 0 \\ -4 & 3 & 4 \\ 0 & 4 & 1 \end{bmatrix}$, which is orthogonally diagonalized by

$$Q = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let $\mathbf{y} = (x', y', z')$. Then

$$\mathbf{y} = Q^T\mathbf{x} = \begin{bmatrix} \frac{1}{3}x + \frac{2}{3}y - \frac{2}{3}z \\ -\frac{2}{3}x + \frac{2}{3}y + \frac{1}{3}z \\ \frac{2}{3}x + \frac{1}{3}y + \frac{2}{3}z \end{bmatrix},$$

and

$$q(\mathbf{x}) = \mathbf{y}^T \begin{bmatrix} -3 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y} = -3x'^2 + 9y'^2 + 3z'^2.$$

$q(\mathbf{x}) = 1$ is an ellipsoid.

33. $q(x, y) = 3X^2 + \frac{8}{3}y^2$, where $X = x - \frac{1}{3}y$.

35. Yes, as follows:

$$q(x, y) = bxy = \frac{b}{4}(x + y)^2 - \frac{b}{4}(x - y)^2 = AX^2 + BY^2,$$

where $A = -B = \frac{b}{4}$, $X = x + y$, and $Y = x - y$.

Section 9.3

1. 3, 2.

3. 2, 1, 0.

5. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^T.$

7. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T.$

9. $\begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}^T.$

11. $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T.$

13. A^TA is messy, but AA^T is diagonal. An SVD of A^T is

$$\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^T.$$

Hence an SVD of A is obtained by transposition

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}^T .$$

17. The verification of the Moore–Penrose properties is straightforward given that A^+ equals

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{5} \end{bmatrix} .$$

19. The verification of the Moore–Penrose properties is straightforward given that

$$A^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

21. A^+ was computed in Exercise 19 and was found to be $\begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$, which is also A^{-1}.

23. $AA^+A = \begin{bmatrix} -3 & 0 \\ 0 & 0 \\ 0 & 4 \end{bmatrix}$, $A^+AA^+ = \begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$, $\left(AA^+\right)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$AA^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\left(A^+A\right)^T = I$, $A^+A = I$.

25. It suffices to verify the Moore–Penrose conditions for (A^+, A), because then A would be the unique inverse of A^+. So $A^{++} = A$. By Exercise 22 we have

$$A^+AA^+ = A^+, \quad AA^+A = A, \quad \left(A^+A\right)^T = A^TA, \quad \left(AA^+\right)^T = AA^+.$$

Hence the conditions hold for the pair (A^+, A).

27. $A^+\mathbf{b} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{5} \end{bmatrix}$.

29. $A^+\mathbf{b} = \begin{bmatrix} -\frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{2}{27} & \frac{2}{27} & \frac{1}{27} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$.

31. $A = PQ = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

33. $A = PQ = \begin{bmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$.

Section 9.5

1. $a_0 = 0, a_n = 0, b_n = \frac{2(1-(-1)^n)}{n\pi}$.

3. $a_0 = \frac{1}{4}, a_n = \frac{\sin\frac{n\pi}{2}}{n\pi}, b_n = \frac{1-\cos\frac{n\pi}{2}}{n\pi}$.

5. *Hint:* You may use $\frac{1}{2}\frac{\sin(n-m)x}{n-m} - \frac{1}{2}\frac{\sin(n+m)x}{n+m}\Big|_0^{\pi} = 0$.

7. *Hint:* You may use $\frac{1}{2}\frac{\sin(n\pi-m\pi)x}{n\pi-m\pi} - \frac{1}{2}\frac{\sin(n\pi+m\pi)x}{n\pi+m\pi}\Big|_{-1}^{1} = 0$.

9. We have

$$\langle 1, \cos m\pi x\rangle = \int_0^2 \cos m\pi x\, dx = \frac{\sin m\pi x}{\pi m}\Big|_0^2 = 0$$

and

$$\langle \cos n\pi x, \cos m\pi x\rangle = \int_0^2 \cos n\pi x \cos m\pi x\, dx = \frac{1}{2}\frac{\sin(n\pi - \pi m)x}{n\pi - \pi m} + \frac{1}{2}\frac{\sin(n\pi + \pi m)x}{n\pi + \pi m}\Big|_0^2 = 0.$$

11. *Hint:* You may use

$$\frac{1}{2}\frac{\sin((n\pi - m\pi)/L)x}{(n\pi - m\pi)/L} - \frac{1}{2}\frac{\sin((n\pi + m\pi)/L)x}{(n\pi + m\pi)/L}\Big|_0^L = 0.$$

Section 9.6

1. Since for $m = 1, 2, \dots,$

$$\psi_{m,0} = \begin{cases} 2^{-m/2}, & 0 \le x \le 2^{m-1}, \\ -2^{-m/2}, & 2^{m-1} < x \le 2^m, \end{cases}$$

we have

$$c_{m,0} = \int_{-\infty}^{\infty} f(x)\psi_{m,0}(x)dx = \int_0^1 (-1)\psi_{m,0}(x)dx = \int_0^1 \left(-2^{-m/2}\right)dx = -2^{-m/2}.$$

Bibliography

[1] A. Tucker. The growing importance of linear algebra in undergraduate mathematics. *The College Mathematics Journal*, 24(1):3–9, 1993.

[2] T. M. Apostol. *Linear Algebra: A First Course with Applications to Differential Equations*. Wiley, 2014.

[3] D. S. Bernstein. *Matrix Mathematics: Theory, Facts, and Formulas*, *Second Edition*. Princeton reference. Princeton University Press, 2009.

[4] F. R. Gantmacher. *The Theory of Matrices. Number v. 1*. Chelsea Publishing Company, 1974.

[5] F. R. Gantmacher. *The Theory of Matrices. Volume 2*. Chelsea Publishing Company, 1960.

[6] F. R. Gantmacher and J. L. Brenner. *Applications of the Theory of Matrices*. Dover Books on Mathematics. Dover Publications, 2005.

[7] G. H. Golub and C. F. Van Loan. *Matrix Computations*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, 1996.

[8] W. H. Greub. *Linear Algebra*. Graduate Texts in Mathematics. Springer New York, 1981.

[9] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, 2012.

[10] A. Jeffrey. *Matrix Operations for Engineers and Scientists: An Essential Guide in Linear Algebra*. SpringerLink: Springer e-Books. Springer Netherlands, 2010.

[11] S. Lang. *Linear Algebra*. Undergraduate Texts in Mathematics. Springer New York, 2013.

[12] W. Nef. *Linear Algebra*. European Mathematics Series. McGraw-Hill, 1967.

[13] G. E. Shilov. *Linear Algebra*. Dover Books on Mathematics. Dover Publications, 2012.

[14] G. Strang. *Linear Algebra and Learning from Data*. Wellesley-Cambridge Press, 2019.

[15] C. B. Boyer and U. C. Merzbach. *A History of Mathematics*. Wiley, 2011.

[16] K. Shen, J. N. Crossley, A. W. C. Lun, and H. Liu. *The Nine Chapters on the Mathematical Art: Companion and Commentary*. Oxford University Press, 1999.

[17] A. Cayley. In *Remarques sur la Notation des Fonctions Algébriques*, Cambridge Library Collection – Mathematics, volume 2, pages 185–188. Cambridge University Press, 2009.

[18] M. F. Barnsley. *Fractals Everywhere*. Dover Books on Mathematics. Dover Publications, 2012.

[19] L. Quan. Invariants of six points and projective reconstruction from three uncalibrated images. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 17(1):34–46, 1995.

[20] P. F. Stiller, C. A. Asmuth, and C. S. Wan. Single-view recognition: the perspective case. In R. A. Melter, A. Y. Wu and L. J. Latecki, editors, *Vision Geometry V*, volume 2826, pages 226–235. International Society for Optics and Photonics, SPIE, 1996.

[21] J. G. Kemeny and J. L. Snell. *Finite Markov Chains: With a New Appendix "Generalization of a Fundamental Matrix"*. Undergraduate Texts in Mathematics. Springer New York, 1983.

[22] C. D. Meyer and I. Stewart. *Matrix Analysis and Applied Linear Algebra*, *Second Edition*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2023.

[23] A. P. Knott. The history of vectors and matrices. *Mathematics in School*, 7(5):32–34, 1978.

[24] P. J. Davis. *The Mathematics of Matrices: A First Book of Matrix Theory and Linear Algebra*. Blaisdell book in the pure and applied sciences. Blaisdell Publishing Company, 1965.

[25] J. T. Sandefur. *Discrete Dynamical Systems: Theory and Applications*. Clarendon Press, 1990.

[26] A. Schwarzenberg-Czerny. On matrix factorization and efficient least squares solution. *Astronomy and Astrophysics Supplement*, 110:405, April 1995.

[27] A. Edelman and G. Strang. Pascal matrices. *The American Mathematical Monthly*, 111(3):189–197, 2004.

[28] G. Strang. The fundamental theorem of linear algebra. 100(9):848–855, November 1993.

https://doi.org/10.1515/9783111331850-013

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